Berkovich spaces, Problem List 4

Let $(k, |\cdot|)$ be an algebraically closed complete non-Archimedean non-trivial normed field and let $\mathbb{A}^1_{\mathrm{Berk}}$ denote the Berkovich affine line over k.

1. The image of the natural map $k \to \mathbb{A}^1_{\mathrm{Berk}}$ is dense.
2. Let $x \in \mathbb{A}^1_{\mathrm{Berk}}$. Show the following.
   (a) If x is of type (2), then the connected components of $\mathbb{A}^1_{\mathrm{Berk}} \setminus \{x\}$ are in a bijection with $\mathbb{P}^1(k^{\mathrm{res}})$ (in particular, x is a branch point).
   (b) If x is of type (3), then x is an ordinary point.
3. For $x \in \mathbb{A}^1_{\mathrm{Berk}}$, let

   $$\mathcal{O}_x := \{f \in k[T] \mid |f|_x \leqslant 1\},\ \ \mathfrak{m}_x := \{f \in k[T] \mid |f|_x < 1\},\ \ k^{\mathrm{res}}_x := \mathcal{O}_x/\mathfrak{m}_x$$

   (we consider k^{res} as a subfield of k^{res}_x),

   $$\Gamma := |k \setminus \{0\}|, \quad \Gamma_x := |k[T] \setminus \{0\}|_x.$$

   We define

   $$E_x := \dim_{\mathbb{Q}}(\Gamma_x/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad F_x := \mathrm{trdeg}_{k^{\mathrm{res}}} k^{\mathrm{res}}_x.$$

   Show the following.
   (a) The group Γ_x/Γ is cyclic.
   (b) $E_x + F_x \leqslant 1$ (a special case of *Abhyankar's inequality*).
   (c) x is of type (1) if and only if $k^{\mathrm{res}} = k^{\mathrm{res}}_x$.
   (d) x is of type (2) if and only if $F_x = 1$.
   (e) x is of type (3) if and only if $E_x = 1$.
   (f) x is of type (4) if and only if $E_x = 0 = F_x$ and x is not of type (1).
4. We define

   $$\mathcal{D}(0,1) := \left\{x \in \mathbb{A}^1_{\mathrm{Berk}} \mid x \leqslant \zeta_{0,1}\right\}.$$

   Show that

   $$\mathcal{O}^{\mathrm{an}}_{\mathcal{D}(0,1)} \cong \left\{\sum a_i T^i \in k[\![T]\!] \mid |a_i| \to 0\right\}.$$

5. Formulate and prove a generalization of the previous problem from $\mathbb{A}^1_{\mathrm{Berk}}$ to $\mathbb{A}^n_{\mathrm{Berk}}$.