Berkovich spaces, Problem List 5

Let $(\mathcal{A}, |\cdot|), (\mathcal{B}, |\cdot|)$ be Banach rings and k be a field.

1. Show that
$$\mathcal{M}(\mathcal{A}) = \{x \in \mathcal{M}(\mathcal{A}) \mid (\forall a \in \mathcal{A})\ |a|_x \leqslant |a|\}.$$
2. Let $C > 0$ and $\|\cdot\|_1, \|\cdot\|_2$ be norms on k such that for all $x \in k$ we have $\|x\|_1 \leqslant C\|x\|_2$. Show that $\|\cdot\|_1 = \|\cdot\|_2$.
3. Show that any bounded homomorphism $\mathcal{A} \to \mathcal{B}$ induces a continuous map $\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$.
4. Show that the set $\mathcal{A}^*$ (invertible elements of $\mathcal{A}$) is open.
5. Show that any maximal ideal of $\mathcal{A}$ is closed.
6. Let I be a closed ideal of $\mathcal{A}$ and for $a \in \mathcal{A}$ define:
$$|a + I|_I := \inf\{|a + x| \mid x \in I\}.$$
Show that $(\mathcal{A}/I, \|\cdot\|_I)$ is a Banach ring.
7. Let $(\mathcal{A}_i, |\cdot|_i)_{i\in I}$ be a collection of Banach rings. We define
$$\prod_{i\in I} \mathcal{A}_i := \{(a_i)_{i\in I} \mid (\exists C > 0)\ (\forall i \in I)\ |f_i|_i < C\}, \quad |(a_i)_{i\in I}| := \sup_{i\in I} |f_i|_i.$$
Show that $(\prod_{i\in I} \mathcal{A}_i, |\cdot|)$ is a Banach ring.
8. Show that the maps $\mathbf{a} \mapsto \Phi_{\mathbf{a}},\ \Phi \mapsto \mathbf{a}_\Phi$ defined during the lecture are mutually inversive bijections.