Berkovich spaces, Problem List 7

Let $(\mathbb{U}, v, \Gamma)$ be a "monster model" of ACVF, $(k, v|_k, \Gamma_k) \prec \mathbb{U}$ and $V \subseteq \mathbb{U}^n$ be an irreducible affine algebraic variety defined over k.

1. Show that for any semi-norm v' on $k[V]$ extending $v|_k$ there is a type $p \in S_V(k)$ such that $v_p = v'/\sim$.

2. Assume that $F \subset k$ is a subfield such that v restricted to F is trivial. Show that v restricted to F^{alg} is trivial as well.

3. Let $p \in S_V(k)$. Show that if for all finite-dimensional k-vector subspaces $E < k[V]$ the following sets are k-definable

$$\{e \in E \mid v_p(e) = 0\}, \quad \{e \in E \mid v_p(e) \leqslant 1\};$$

then the type p is definable.

4. Let $p \in S_V(k)$ and $f : V \to \Gamma$ be a definable function. Show that the following conditions are equivalent.

   (a) The function f is generically constant on p.
   (b) The function f is constant on $p^{\mathbb{U}}$.
   (c) $f(p^{\mathbb{U}}) \subseteq \Gamma_k$.
   (d) $f(p^{\mathbb{U}}) \cap \Gamma_k \neq \emptyset$.

5. Assume that $v(K) \subseteq \mathbb{R}_{\geqslant 0}$. Show that the map (defined during the lecture)
$$\pi : V^{\mathrm{an}} \to S_V(k)$$
is continuous.

6. Assume that $v(K) \subseteq \mathbb{R}_{\geqslant 0}$ and assume that $p \in S_{\mathbb{A}^1}(k)$ is in the image of the map
$$\pi : \mathbb{A}^1_{\mathrm{Berk}} \to S_{\mathbb{A}^1}(k).$$
Show that the following conditions are equivalent.

   - p is definable,
   - p is of type (1) or p is of type (2).