HOMOLOGICAL ALGEBRA

These are the notes for an informal mini-course on homological algebra given at the Istanbul Bilgi University in Spring Semester 2015. They are based on my Polish notes for a similar (but about twice larger) course given at Wrocław University.

LECTURE 1

Homological algebra is the branch of mathematics that studies homology in a general algebraic setting. Originally, it is a part of the theory of modules which is needed for homology theories in algebraic topology. However, the development of homological algebra was closely connected to category theory. It turned out that homological algebra applies to much wider context than just modules. This context is phrased in the language of categories, so we start with introducing the basic notions from category theory.

Categories

Definition *Category* $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}(\mathcal{C})$, whose elements are called *objects* of $\mathcal{C}$.
- For each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}(X,Y)$, whose elements are called *morphisms* between X and Y.
- For each $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a function

$$\circ : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(Y,Z)$$

  called the *composition* function

such that $\forall X, Y, Z, T \in \mathrm{Ob}(\mathcal{C})$:

(1) $(X,Y) \neq (Z,T) \;\Rightarrow\; \mathrm{Hom}(X,Y) \cap \mathrm{Hom}(Z,T) = \emptyset$.

(2) $\exists!\, \mathrm{id}_X \in \mathrm{Hom}(X,X)$ such that $\forall f \in \mathrm{Hom}(X,Y)\ \forall g \in \mathrm{Hom}(Z,X)$

$$f \circ \mathrm{id}_X = f, \ \mathrm{id}_X \circ g = g.$$

(3) $\forall f \in \mathrm{Hom}(X,Y),\ \forall g \in \mathrm{Hom}(Y,Z),\ \forall h \in \mathrm{Hom}(Z,T)$

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Warning

- $\mathrm{Hom}(X,Y)$ need *not* consist of functions from X to Y!
- The "composition" map $\circ$ need *not* be the composition of functions!

Conventions

- Instead of $\mathrm{Ob}(\mathcal{C})$, we may write $\mathcal{C}$, so "$X \in \mathcal{C}$" means "X is an object in the category $\mathcal{C}$".
- Instead of $f \in \mathrm{Hom}(X,Y)$, we may write $f : X \to Y$.
- Instead of $g \circ f$, we may write gf.
- Instead of $\mathrm{Hom}(X,Y)$ we may write $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ or $\mathcal{C}(X,Y)$.

Examples of categories

(1) **Set**
$\mathrm{Ob}(\mathbf{Set})$ is the class of all sets, and for $X, Y \in \mathbf{Set}$, $\mathrm{Hom}(X,Y) = Y^X$ (the set of all functions from X to Y). (Formally, each function should also carry a "label" specifying its codomain.)

(2) **Top**, $\mathbf{Top}_*$
Ob(**Top**) is the class of all topological spaces, morphisms are the continuous functions.
Ob($\mathbf{Top}_*$) is the class of all topological spaces with a distinguished point, morphisms are the continuous functions which preserve the fixed points.

(3) **Toph**, $\mathbf{Toph}_*$
$\mathrm{Ob}(\mathbf{Toph}) = \mathrm{Ob}(\mathbf{Top})$, morphisms are the homotopy classes of continuous functions (Problem 1).
$\mathrm{Ob}(\mathbf{Toph}_*) = \mathrm{Ob}(\mathbf{Top}_*)$, morphisms are the homotopy classes of continuous functions which preserve the fixed points, homotopies also preserve the fixed points.

(4) **Diff**
Ob(**Diff**) is the class of all smooth manifolds, morphisms are the smooth functions.

(5) $\mathbf{AfVar}_K$, where K is an algebraically closed field K
$\mathrm{Ob}(\mathbf{AfVar}_K)$ is the class of all Zariski closed subsets of K^n (where n varies), morphisms are the restrictions of the polynomial functions.

(6) $\mathbf{Mod}_R$, where R is a ring
$\mathrm{Ob}(\mathbf{Mod}_R)$ is the class of all (left) R-modules, morphisms are the R-modules homomorphisms.
We denote $\mathbf{Mod}_{\mathbb{Z}}$ by **Ab** (Abelian groups), and $\mathbf{Mod}_K$ by $\mathbf{Vect}_K$ (vector spaces over K) if K is a field.

(7) $\mathbf{Alg}_R$, where R is a commutative ring
$\mathrm{Ob}(\mathbf{Alg}_R)$ is the class of all R-algebras (with 1), morphisms are the R-algebra homomorphisms (preserving 1).

(8) **Grp**
Ob(**Grp**) is the class of all groups, morphisms are the group homomorphisms.

(9) $\mathrm{Top}(X)$, where X is a topological space
$\mathrm{Ob}(\mathrm{Top}(X)) = \mathrm{OPEN}(X)$ the set of all open subsets of X, morphisms are the inclusions (Problem 2).

(10) $\mathcal{C}_G$, where $(G, \cdot)$ is a group
$\mathrm{Ob}(\mathcal{C}_G) = \{*\}$ (a singleton), $(\mathrm{Hom}(*, *), \circ) = (G, \cdot)$.

More definitions
Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

- The category $\mathcal{C}$ is *small*, if $\mathrm{Ob}(\mathcal{C})$ is a set (only the categories 5., 9. and 10. above are small).
- $\mathcal{C} \times \mathcal{D}$, the *product* of categories $\mathcal{C}, \mathcal{D}$
$\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and morphisms are the appropriate pairs of morphisms (Problem 2).
- $\mathcal{C}^{\mathrm{op}}$, the *opposite* category to $\mathcal{C}$
$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, $\forall\ X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ (Problem 2).
- A morphism $f : X \to Y$ is an *isomorphism*, if there is $g : Y \to X$ such that: $gf = \mathrm{id}_X,\ fg = \mathrm{id}_Y$. If there is an isomorphism $X \to Y$, then we say that X and Y are *isomorphic* and write $X \cong Y$.
- A category $\mathcal{C}$ is a *subcategory* of the category $\mathcal{D}$, if $\mathrm{Ob}(\mathcal{C})$ is a subclass of $\mathrm{Ob}(\mathcal{D})$ and for all $X, Y \in \mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{D}}(X, Y)$.

- A subcategory $\mathcal{C}$ is a *full* subcategory of the category $\mathcal{D}$, if for all $X, Y \in \mathcal{C}$, we have $\operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathcal{D}}(X,Y)$.

Examples

- **Ab** is a full subcategory of **Grp**.
- **Top** is not a subcategory **Set**.
- Let **Rel** be a category, such that $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and morphisms are relations (note that relations can be composed). Then **Set** is a subcategory of **Rel** and is not full.
- Let U be an open subset of a topological space X. Then $\mathrm{Top}(U)$ is a full subcategory of $\mathrm{Top}(X)$.

Definitions of initial and terminal objects

Let $\mathcal{C}$ be a category. An object $X \in \mathcal{C}$ is an *initial object* in $\mathcal{C}$, if for all $Y \in \mathcal{C}$, we have $|\operatorname{Hom}(X,Y)| = 1$. The definition of a *terminal object* is dual, i.e. a terminal object in $\mathcal{C}$ is an initial object in $\mathcal{C}^{\mathrm{op}}$, i.e. for all $Y \in \mathcal{C}$, we have $|\operatorname{Hom}(Y,X)| = 1$.

Examples

- $\emptyset$ is an initial object in **Set**, any singleton is a terminal object in **Set**.
- $\{e\}$ is both an initial and a terminal object in **Grp**, i.e. it is a *zero object* in **Grp**.
- R is an initial object in $\mathbf{Alg}_R$. The zero ring is a terminal object in $\mathbf{Alg}_R$.
- $\emptyset$ is an initial object in $\mathrm{Top}(X)$ and X is a terminal object in $\mathrm{Top}(X)$.

Remark (Problem 3)

If an initial (resp. a terminal) object exists, it is unique up to an isomorphism.

Functors

Definition

A (*covariant*) *functor* F from a category $\mathcal{C}$ into a category $\mathcal{D}$ (notation $F : \mathcal{C} \to \mathcal{D}$) consists of:

- An assignment
$$\mathcal{C} \ni X \mapsto F(X) \in \mathcal{D}$$
- For all $X, Y \in \mathcal{C}$, a function
$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \ni f \mapsto F(f) \in \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$$
  such that $F(fg) = F(f)F(g)$ (for appropriate f, g), and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

A *contravariant* functor F (notation $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$), is a functor from the opposite category to $\mathcal{C}$ into $\mathcal{D}$, i.e. $F(fg) = F(g)F(f)$ (for appropriate f, g).

Examples

(1) **Cat** is the category of small categories, morphisms are the functors (they can be composed).

(2) **Forgetful functors**

$$\mathbf{Alg}_R \to \mathbf{Mod}_R \to \mathbf{Ab} \to \mathbf{Set}$$

$$\mathbf{Diff} \to \mathbf{Top} \to \mathbf{Set}$$

$$\mathbf{AfVar}_K \to \mathbf{Top} \quad \text{(Zariski topology)}$$

$$\mathbf{AfVar}_{\mathbb{C}} \to \mathbf{Top} \quad \text{(Euclidean topology)}$$

A category $\mathcal{C}$ is called (informally, so far) *concrete*, if there is a forgetful functor $\mathcal{C} \to \mathbf{Set}$.

(3) **Representable functors**.
For each category $\mathcal{C}$, there is a functor

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}.$$

For each $X \in \mathcal{C}$, we have two functors:

$$h_X : \mathcal{C} \to \mathbf{Set}, \quad h_X(Y) = \mathrm{Hom}(X, Y)$$

$$h^X : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}, \quad h^X(Y) = \mathrm{Hom}(Y, X).$$

Both of them act on morphisms by the composition.

Let $F : \mathbf{Top} \to \mathbf{Toph}$, $F_* : \mathbf{Top}_* \to \mathbf{Toph}_*$. Then for all $n \in \mathbb{N}$ we have the functor of

$$\pi_n : \mathbf{Top}_* \to \mathbf{Set}, \quad \pi_n = h_{S^n} \circ F_*$$

the n-th homotopy group (S^n is the n-dimensional sphere). We will see later than $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ i dla $n > 1$, $\pi_n : \mathbf{Top}_* \to \mathbf{Ab}$.

For an algebraically closed field K, we have a functor

$$h^K : \mathbf{AfVar}_K \to (\mathbf{Alg}_K)^{\mathrm{op}}, \quad h^K(V) = \mathrm{Hom}_{\mathbf{AfVar}_K}(V, K).$$

We denote $h^K(V)$ by $K[V]$ (the ring of regular functions).

(4) **Presheaves and sheaves**
For a topological space X and a category $\mathcal{C}$, a functor

$$F : \mathrm{Top}(X) \to \mathcal{C}^{\mathrm{op}}$$

is called a *presheaf on X of objects of $\mathcal{C}$*.
For example, if $\mathcal{C} = \mathbf{Ab}$, then we consider *presheaves of Abelian groups*, if $\mathcal{C} = \mathbf{Alg}_K$, then we consider *presheaves of K-algebras*.
Let $U \subseteq V$ be open sets in X and F be a presheaf. The elements of $F(U)$ are called the *sections of F over U*. For $s \in F(V)$ the value on the morphism $F(V) \to F(U)$ on s is denoted by $s|_U$, and is called the *restriction* of s to U.
A presheaf F is called a *sheaf*, if it satisfies:
(a) Locality. For all $U \in \mathrm{Top}(X)$ and each covering $U = \bigcup_{i \in I} U_i$ and each $s, s' \in F(U)$, if for all $i \in I$ we have $s_i|U_i = s_j|U_j$ then $s = s'$.
(b) Gluing. For all $U \in \mathrm{Top}(X)$ and each covering $U = \bigcup_{i \in I} U_i$ together with a choice of $s_i \in F(U_i)$ such that for all $i, j \in I$, $s_i|U_{ij} = s_j|U_{ij}$, there is $s \in F(U)$ such that for all $i \in I$, we have $s|U_i = s_i$,

Examples of pre-sheaves.
(a) For $X \in \mathbf{Diff}$, we have C^∞_X, the sheaf (of $\mathbb{R}$-algebras) of the smooth functions (into $\mathbb{R}$).
(b) For $X \in \mathbf{Top}$, we have C_X, the sheaf (of $\mathbb{R}$-algebras) of the continuous functions (into $\mathbb{R}$).
(c) For $X \in \mathbf{AnfVar}_K$, we have $\mathcal{O}_X$, the *structural sheaf* (of $\mathbb{R}$-algebras): for $U \subseteq X$ open in X, $\mathcal{O}_X(U)$ is the set of the rational functions into K everywhere defined on U.
(d) For $X \in \mathbf{Top}$ and $A \in \mathbf{Ab}$, we have three "constant" presheaves.

- The *constant presheaf* A_X
  For each open set U, $A_X(U) = A$ and the restriction functions are the identities. This presheaf does not satisfy Locality (empty family of open sets covering empty set). Note that if F is a presheaf of Abelian groups satisfying Locality, then $F(\emptyset) = \{0\}$.
- The presheaf A_X^0
  We have $A_X^0(\emptyset) = \{0\}$ and for each open non-empty set $A_X(U) = A$. The restriction functions are either identities or the zero functions. This presheaf does not satisfy Gluing.
- The sheaf $\underline{A}_X$
  This is the sheaf of continuous functions in A considered with the discrete topology. On connected non-empty open sets, the sections of all three sheaves coincide with A.

Definitions

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

- F is *faithful*, if it is "1-1" on each set of morphisms.
- F is *full*, if it is "onto" on each set of morphisms.
- F is *faithfully full*, if it is faithful and full.
- A *concrete category* is a pair $(\mathcal{C}, F)$, where F is a faithful functor from $\mathcal{C}$ into **Set**.

Examples

(1) The categories 1. − 9. above are concrete with the obvious forgetful functors. The category 9. and the category **Rel** are not concrete (in any obvious way).
(2) Assume $\mathcal{C}$ is a subcategory of $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ is the inclusion functor. Then F is a faithful functor and F is faithfully full if and only if $\mathcal{C}$ is a full subcategory of $\mathcal{D}$.
(3) The functor $F : \mathbf{Top} \to \mathbf{Toph}$ is full but not faithful.
(4) The functor $h^K : \mathbf{AfVar}_K \to (\mathbf{Alg}_K)^{\mathrm{op}}$ (of the ring of regular functions) is faithfully full.
(5) The forgetful functors $\mathbf{An} \to \mathbf{Diff} \to \mathbf{Top} \to \mathbf{Set}$ (and the other ones of a similar type) are faithful, but not full.

LECTURE 2

Morphisms of functors

Example

Let $* : \mathbf{Vect}_K \to \mathbf{Vect}_K^{\mathrm{op}}$ be the *dual space* functor:

$$V^* = h^K(V) = \mathrm{Hom}_{\mathbf{Vect}_K}(V, K).$$

For each V, we have a "natural" map:

$$V \ni v \mapsto f_v \in V^{**}, \quad f_v(g) = g(v).$$

This map is "natural", because it does not depend on V, i.e. it is a map between functors:

$$\mathrm{id} : \mathbf{Vect}_K \to \mathbf{Vect}_K, \quad ** : \mathbf{Vect}_K \to \mathbf{Vect}_K.$$

Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then ψ is a *morphism* or a *natural map* (or a *natural*

transformation) between F and G (notation $\psi : F \to G$), if
$$\psi = (\psi_X : F(X) \to G(X))_{X \in \mathcal{C}}$$
such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the following diagram commutes
$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \psi_X}\big\downarrow & & \big\downarrow{\scriptstyle \psi_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array},$$
i.e. $\psi_Y F(f) = G(f)\psi_X$.

Example 1

$$\mathrm{id} : \mathbf{Vect}_K \to \mathbf{Vect}_K, \quad ** : \mathbf{Vect}_K \to \mathbf{Vect}_K.$$
$$\varphi : \mathrm{id} \to **, \quad \varphi_V(v)(v^*) = v^*(v).$$
Take any linear map $A : V \to W$. We need to show that the following diagram commutes:
$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ {\scriptstyle \varphi_V}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_W} \\ V^{**} & \xrightarrow[A^{**}]{} & W^{**} \end{array}.$$
First, we will understand A^* and A^{**}. For $w^* \in W^*, v^{**} \in V^{**}$:
$$A^*(w^*) = w^* \circ A, \quad A^{**}(v^{**}) = v^{**} \circ A^*.$$
Take any $v \in V$ and $w^* \in W^*$.
$$A^{**}(\varphi_V(v))(w^*) = \varphi_V(v) \circ A^*(w^*) = \varphi_V(v)(A^*(w^*)) = \varphi_V(v)(w^* \circ A) = w^* \circ A(v) = w^*(A(v)).$$
$$\varphi_W(A(v))(w^*) = w^*(A(v)).$$
Hence
$$A^{**}(\varphi_V(v)) = \varphi_W(A(v)) \text{ and } A^{**} \circ \varphi_V = \varphi_W \circ A.$$
Example 2
Let $T : \mathbf{Diff} \to \mathbf{Diff}$ be the functor of the tangent bundle. Then we have a natural map (projection):
$$\pi : T \to \mathrm{id}, \quad \pi_X : T(X) \to X, \quad \pi_X(x, v) = x$$
for $(x, v) \in T(X)$. From differential geometry we know that for each smooth function $f : X \to Y$ the following diagram commutes:
$$\begin{array}{ccc} T(X) & \xrightarrow{T(f)} & T(Y) \\ {\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & Y \end{array}.$$
Example 3 (a *non-natural* map)
Take a vector space V and its basis B. Let $(\cdot, \cdot)_B$ be the scalar product on V coming from the basis B. We define:
$$\phi_{V,B} : V \to V^*, \quad \phi_{V,B}(v)(w) = (v, w)_B.$$

This map is not natural mostly because it is between the covariant functor (id) and the contravariant functor $(*)$. Besides, it depends on a choice of basis of V, so it *does* depend on V.

Definition
For categories $\mathcal{C}, \mathcal{D}$, we denote:

- by $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ the class of all functors from $\mathcal{C}$ to $\mathcal{D}$,
- for $F, G \in \mathrm{Func}(\mathcal{C}, \mathcal{D})$, by $\mathrm{Hom}(F, G)$ the class of all natural transformations between F and G.
- if $\mathcal{C}$ is small, then always $\mathrm{Hom}(F, G)$ is a set and $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ denotes the *category* of all natural transformations between F and G.

The main examples are categories of (pre)sheaves and the categories of the form $\mathrm{Func}(\mathcal{C}, \mathbf{Set})$.

Theorem Let $\mathcal{C}$ be a category. Then:

(1) **Yoneda Lemma**
For each functor $T : \mathcal{C} \to \mathbf{Set}$ and $A \in \mathcal{C}$, the map

$$\mathrm{Hom}(h_A, T) \ni \phi \mapsto \phi_A(\mathrm{id}_A) \in T(A)$$

is a bijection.

(2) If $\mathcal{C}$ is small, then there is a faithfully full contravariant functor

$$h : \mathcal{C}^{\mathrm{op}} \to \mathrm{Func}(\mathcal{C}, \mathbf{Set}), \quad h(A) = h_A.$$

Proof

(1) We will define an inverse function to the following function:

$$\mathrm{Hom}(h_A, T) \ni \phi \mapsto \phi_A(\mathrm{id}_A) \in T(A).$$

Let $a \in T(A)$. For any $X \in \mathcal{C}$ and $f : A \to X$, we define:

$$\bar{a} : h_A \to T, \quad \bar{a}_X : \mathrm{Hom}(A, X) \to T(X), \quad \bar{a}_X(f) = T(f)(a).$$

We need to check that $\bar{a}$ is a morphism of functors, i.e. whether for all $g : X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, X) \arrow[r, "h_A(g)"] \arrow[d, "\bar{a}_X"'] & \mathrm{Hom}_{\mathcal{C}}(A, Y) \arrow[d, "\bar{a}_Y"] \\
T(X) \arrow[r, "T(g)"] & T(Y)\ .
\end{tikzcd}$$

Take any $f \in \mathrm{Hom}_{\mathcal{C}}(A, X)$.

$$T(g)(\bar{a}_X(f)) = T(g)(T(f)(a)) = T(gf)(a) = \bar{a}_Y(gf) = \bar{a}_Y(h_A(g)(f)).$$

We check whether the maps between $\mathrm{Hom}(h_a, T)$ and $T(A)$ are mutually inversive.
Take $\phi : h_A \to T$. We need to show that $\overline{\phi_A(\mathrm{id}_A)} = \phi$.
Take any $f : A \to X$. We have:

$$\overline{\phi_A(\mathrm{id}_A)}_X(f) = T(f)(\phi_A(\mathrm{id}_A)).$$

But ϕ is a morphism of functors, hence the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "h_A(f)"] \arrow[d, "\phi_A"'] & \mathrm{Hom}_{\mathcal{C}}(A,X) \arrow[d, "\phi_X"] \\ T(A) \arrow[r, "T(F)"] & T(X) . \end{tikzcd}$$

Thus:

$$T(f)(\phi_A(\mathrm{id}_A) = \phi_X(h_A(f)(\mathrm{id}_A)) = \phi_X(f \circ \mathrm{id}_A) = \phi_X(f).$$

therefore $\overline{\phi_A(\mathrm{id}_A)} = \phi$.
For all $a \in T(A)$,

$$\bar{a}_A(\mathrm{id}_A) = T(\mathrm{id}_A)(a) = \mathrm{id}_{T(A)}(a) = a.$$

(2) For $g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, we define:

$$h(g) : h_B \to h_A, \quad h(g)_X = h^X(g) \ \text{ for all } X \in \mathcal{C}.$$

We need to check whether for each $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ the following diagram commutes, which amounts to the associativity of the composition map:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[r, "h_B(f)"] \arrow[d, "(h(g)_X=)h^X(g)"'] & \mathrm{Hom}_{\mathcal{C}}(B,Y) \arrow[d, "h^Y(g)(=h(g)_Y)"] \\ \mathrm{Hom}_{\mathcal{C}}(A,X) \arrow[r, "h_A(f)"] & \mathrm{Hom}_{\mathcal{C}}(A,Y) . \end{tikzcd}$$

From Yoneda Lemma (for $T = h_B$), we have the following bijection:

$$\mathrm{Hom}(B,A) = h_B(A) \leftrightarrow \mathrm{Hom}(h_A, h_B).$$

It is enough to show that it corresponds to the function on morphisms which is induced by the functor h. Take $g \in h_B(A)$ and $f : A \to X$.

$$\bar{g}_X(f) = h_B(f)(g) = fg = h^X(g)(f), \text{ therefore } \bar{g}_X = h^X(g) = h(g)_X.$$

Hence $\bar{g} = h(g)$. □

Using the above contravariant functor, we often (contravariantly) *embed* a given small category $\mathcal{C}$ into the category of functors $\mathrm{Func}(\mathcal{C}, \mathbf{Set})$, where we can perform some operations which may be impossible to perform in $\mathcal{C}$.

LECTURE 3

Fact
A morphism of functors $f : F \to G$ is an isomorphism if and only if for all objects X from the domain of F (and G), $f_X : F(X) \to G(X)$ is an isomorphism.
Proof
$\Rightarrow$ Let $g : G \to F$ be a morphism of functors such that $gf = \mathrm{id}_F$, $fg = \mathrm{id}_G$. Then for all X, $g_X f_X = \mathrm{id}_{F(X)}$, $f_X g_X = \mathrm{id}_{G(X)}$.
$\Leftarrow$ For all X, it is enough to take $g_X = f_X^{-1}$. □

Example
Let $\mathbf{Vect}_K^f$ be the category of finite dimensional vector spaces and $V \in \mathbf{Vect}_K^f$. Then $\varphi : \mathrm{id} \to **$ is an isomorphism, because:

- For each $v \in V \backslash \{0\}$, there is $v^* \in V^*$ such that $v^*(v) \neq 0$, so $\varphi_V(v)(v^*) \neq 0$. Hence $\ker(\varphi_V) = \{0\}$ and φ_V is "1-1".
- Since $\dim(V) = \dim(V^{**}) < \infty$ (V is *non*-naturally isomorphic to V^* and V^{**}!), we get that φ_V is an isomorphism.

Useless definition
Isomorphism between categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there is a functor $G : \mathcal{D} \to \mathcal{C}$, such that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$.

Useful definition

- A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* (of categories $\mathcal{C}$ and $\mathcal{D}$), if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{id}_{\mathcal{C}}$ and $FG \cong \mathrm{id}_{\mathcal{D}}$.
- Such a functor G as above is called a *quasi-inverse* to F.
- Categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, if there is an equivalence $F : \mathcal{C} \to \mathcal{D}$.

Example
The functor $* : (\mathbf{Vect}_K^f)^{\mathrm{op}} \to \mathbf{Vect}_K^f$ is an equivalence, since $** \cong \mathrm{id}$.

Exercise
$\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \cong \mathrm{Func}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})$.

Theorem
A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is faithfully full and for all $Y \in \mathcal{D}$, there is $X \in \mathcal{C}$, such that $Y \cong F(X)$.
Proof
$\Rightarrow$ Let G be a quasi-inverse to F and $l : GF \cong \mathrm{id}$. For each $X, Y \in \mathcal{C}$ and each morphism $GF(X) \to GF(Y)$, using l_X^{-1} and l_Y we get a morphism $X \to Y$. Then the compositions (last arrow is as described above):

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \to \mathrm{Hom}_{\mathcal{C}}(GF(X),GF(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(G(X),G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(FG(X),FG(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

are identities, so F and G are faithful.
Take any $f : F(X) \to F(Y)$ and let

$$g := l_Y G(f) l_X^{-1} : X \to Y.$$

Then we have

$$l_Y GF(g) l_X^{-1} = g = l_Y G(f) l_X^{-1}.$$

hence $GF(g) = G(f)$, and since G is faithful, we get $f = F(g)$, so F is full.
For all $Y \in \mathcal{D}$, $F(G(Y)) \cong Y$, so we can take $X = G(Y)$.

$\Leftarrow$ For each $Y \in \mathcal{D}$, we choose $\alpha_Y : F(X) \cong Y$ such that for $Y = F(X)$, we have $\alpha_Y = \mathrm{id}_Y$). For $f : Y \to Y'$, we define:

$$G : \mathcal{D} \to \mathcal{C}, \quad G(Y) = X, \quad G(f) = F^{-1}(\alpha_{Y'}^{-1} f \alpha_Y).$$

We check that G is a functor. It is clear that F^{-1} preserves the compositions as well. So for $f : Y \to Y', g : Y' \to Y''$, we have:

$$G(gf) = F^{-1}(\alpha_{Y''}^{-1} g f \alpha_Y) = F^{-1}(\alpha_{Y''}^{-1} g \alpha_{Y'} \alpha_{Y'}^{-1} f \alpha_Y = F^{-1}({}_{Y''}^{-1} g \alpha_{Y'}) F^{-1}(\alpha_{Y'}^{-1} f \alpha_Y) = G(g)G(f).$$

From our construction we get $GF = \mathrm{id}_{\mathcal{C}}$.
Let

$$\alpha : FG \to \mathrm{id}_{\mathcal{D}}, \quad \alpha_Y : FG(Y) \cong Y.$$

We know that for all $f : Y \to Y'$, $G(f) = F^{-1}(\alpha_Y f \alpha_{Y'}^{-1})$. Hence

$$FG(f) = \alpha_Y f \alpha_{Y'}^{-1},$$

and the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\alpha_Y"'] & Y' \arrow[d, "\alpha_{Y'}"] \\ FG(Y) \arrow[r, "FG(f)"] & FG(Y') \end{tikzcd} .$$

Thus α is a morphism of functors. α is an isomorphism, since it is an isomorphism on objects. $\square$

Remark
We do not have a "natural" (quasi-)inverse functor, but from the last theorem we know that *there is* a quasi-inverse functor given by Axiom of (Global) Choice. So it is a bit surprising example of a "non-natural" (the common sense meaning of the word) functor.

Example
The functor of regular functions $V \mapsto K[V]$ has a quasi-inverse.

Representable functors

Definition
A functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* by $A \in \mathcal{C}$, if F is isomorphic to the functor h_A, that is for all $B \in \mathcal{C}$ there is a natural bijection

$$F(B) \longleftrightarrow \mathrm{Hom}(A, B).$$

Remark
By Yoneda Lemma, if F is representable by A and F is representable by A', then $A \cong A'$. Let $\mathrm{Repr}(\mathcal{C}, \mathcal{D})$ denote the subclass of $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ consisting of representable functors.
By Yoneda Lemma again, $\mathrm{Repr}(\mathcal{C}, \mathcal{D})$ is a category and the functors

$$\mathcal{C} \ni X \mapsto h^X \in \mathrm{Repr}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$
$$\mathcal{C}^{\mathrm{op}} \ni X \mapsto h_X \in \mathrm{Repr}(\mathcal{C}, \mathcal{D})$$

are equivalences of categories.
Let $\imath$ be a quasi-inverse functor to the first of them, so for any representable functor $F : \mathcal{C} \to \mathbf{Set}$, there is an isomorphism of functors $F \cong h_{\imath(F)}$.

Example (the first definition of product: representability)
Let $X, Y \in \mathcal{C}$. If there is an object representing the functor:

$$F_{X,Y} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, \quad F_{X,Y}(A) = \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, Y),$$

then we call this object a *product* (in the category $\mathcal{C}$) of X and Y and denote it by $X \times Y$. By Yoneda Lemma, a product of X and Y is unique up to an isomorphism (if it exists). We say that *category has products*, if for each pair of objects, their product exists.
Notice that

$$\mathcal{C} \times \mathcal{C} \ni (X, Y) \mapsto F_{X,Y} \in \mathrm{Func}(\mathcal{C}, \mathbf{Set})$$

is a functor. Hence, if $\mathcal{C}$ has products, then

$$\mathcal{C} \times \mathcal{C} \ni (X, Y) \mapsto X \times Y \in \mathcal{C}$$

is a functor, as the composition of the upper functor with the quasi-inverse $\imath$ from the last remark. We will similarly argue that other natural constructions which are functorially representable are in fact functors.

Remark
The set $\mathrm{Hom}(A, X)$ can be understood as the set of "A-points" of X, which is sometimes denoted by $X(A)$. Yoneda Lemma tells us that to know the object X, it is enough to look at the *all possible* sets $X(A)$.

Example
The name "A-points" and the intuitions behind this name come from algebraic geometry. Let $V \in \mathbf{AfVar}_K$, $R \in \mathbf{Alg}_K$ and $V(R)$ be the set of R-rational points of V (the solutions over R of the polynomial system of equations defining V). Then we have a natural bijection:

$$\mathrm{Hom}_{\mathbf{Alg}_K}(K[V], R) \longleftrightarrow V(R).$$

Fact (the second definition of product: a universal property)
An object Z is a product of objects X and Y if and only if the following holds:

- there are morphisms $\pi_X : Z \to X$, $\pi_Y : Z \to Y$,
- such that for each morphisms $f_X : Z' \to X$, $f_Y : Z' \to Y$,
- there is a unique morphism $f : Z' \to Z$,
- such that $f_X = f\pi_X$, $f_Y = f\pi_Y$.

Proof
$\Longrightarrow$
For each $A \in \mathcal{C}$ we have a natural equivalence

$$\mathrm{Hom}(A, Z) \longleftrightarrow \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, Y).$$

Plugging $A = Z$, we get (π_X, π_Y) corresponding to id_Z.
Plugging $A = Z'$, we get f corresponding to (f_X, f_Y).
Exercise: complete the proof.
$\Longleftarrow$
Let F denote the functor $F_{X,Y}$, hence $F(A) = \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, Y)$. Yoneda Lemma implies that the morphisms between h^Z and F corresponds to $F(Z)$. The element $(\pi_X, \pi_Y) \in F(Z)$ corresponds to a functor isomorphism: exercise. $\square$

We will usually introduce different natural constructions using a universal property (as in the second definition) rather than a representable functor (as in the first definition). However, thanks to the functorial representability, we will always know that our construction gives a functor (and it is the right one!).

Definition
The notion of *coproduct* is a notion which is *dual* to the notion of a product (we reverse all the arrows in the second definition of product). A coproduct of X, Y is denoted by $X \coprod Y$.

We have natural morphisms

$$\Delta : X \to X \times X \text{ (diagonal)}$$

$$\nabla : X \coprod X \to X \text{ (codiagonal).}$$

For $f : X \to Y,\ g : X \to Z$, the composition

$$(f \times g) \circ \Delta : X \to Y \times Z$$

is denoted by (f, g).

Yoneda Lemma also gives a functorial isomorphism

$$(X \times Y) \times Z \cong X \times (Y \times Z),$$

so we can skip the parenthesis and write $X \times Y \times Z$ (similarly for $\coprod$).

Example

(1) **Set**, **Top**, **Diff**, $\mathbf{AfVar}_K$.
Product is the Cartesian product, coproduct is the disjoint sum.
Hence generally, coproduct represents the functor:

$$\mathcal{C}^{\mathrm{op}} \ni A \mapsto \mathrm{Hom}(A, X) \uplus \mathrm{Hom}(A, Y) \in \mathbf{Set}.$$

(2) $\mathbf{Alg}_R$. Product is the Cartesian product, coproduct is the tensor product.
(3) **Grp**. Product is the Cartesian product, coproduct is the free product.
(4) $\mathbf{Mod}_R$. Both product and coproduct are Cartesian product.
(5) $\mathrm{Top}(X)$. Product is the intersection, coproduct is the union.
(6) In the category of fields there are no products and no coproducts.
(7) In the category $\mathcal{C}_G$ there are no products and no coproducts.

Definition
Assume that there are products in the categories $\mathcal{C}$ and $\mathcal{D}$. A functor $F : \mathcal{C} \to \mathcal{D}$ *preserves products*, if the following functors are isomorphic:

$$(X, Y) \mapsto F(X) \times F(Y), \qquad (X, Y) \mapsto F(X \times Y).$$

Fact
Representable functors preserve products and terminal objects.
Proof
Directly from the first definition of a product and from the definition of a terminal object. $\square$

LECTURE 4

Adjoint functors

We start with an example. Let R be a ring, X be a set and $R[X]$ be the ring of polynomials, where the variables come from the set X. We obtain the following functor

$$F : \mathbf{Set} \to \mathbf{Alg}_R, \quad F(X) = R[X].$$

Let $G : \mathbf{Alg}_R \to \mathbf{Set}$ be the forgetful functor. For each R-algebra S and each set X, we have a bijection:

$$\mathrm{Hom}_{\mathbf{Alg}_R}(R[X], S) \longleftrightarrow \mathrm{Hom}_{\mathbf{Set}}(X, S), \qquad \phi \mapsto \phi|_X,$$

which is natural with respect to S and X (since the restriction of functions commutes with the composition of functions). Hence the following functors are isomorphic:

$$\mathbf{Set}^{\mathrm{op}} \times \mathbf{Alg}_R \ni (X, S) \mapsto \mathrm{Hom}_{\mathbf{Alg}_R}(F(X), S) \in \mathbf{Set};$$

$$\mathbf{Set}^{\mathrm{op}} \times \mathbf{Alg}_R \ni (X,S) \mapsto \mathrm{Hom}_{\mathbf{Set}}(X, G(S)) \in \mathbf{Set}.$$

Definition
Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. F is *left-adjoint* to G (or G is *right-adjoint* to F), if the following functors are isomorphic

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \ni (X,Y) \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \in \mathbf{Set};$$

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \ni (X,Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \in \mathbf{Set}.$$

Examples (left-adjoint to a forgetful functor G)

(1) $G : \mathbf{Grp} \to \mathbf{Set}$, $F(X) = F_X$ – free group functor.
(2) $G : \mathbf{Ab} \to \mathbf{Grp}$, $F(H) = H^{\mathrm{ab}} = H/[H,H]$ – abelianization functor.
(3) S is an R-algebra, $G : \mathbf{Mod}_S \to \mathbf{Mod}_R$, $F(M) = M \otimes_R S$ – extension of scalars functor.
(4) $G : \mathbf{Field} \to \mathbf{Domain}$, $G(R) = R_0$ – fraction field.

Other examples

(1) For $(X,x) \in \mathbf{Top}_*$ let

$$\Omega(X,x) = \mathrm{Hom}_{\mathbf{Top}_*}((S^1,*),(X,x))$$

(loops at x) with compact-open topology.

$$\Sigma(X,x) = X \times [0,1]/(X \times \{0,1\} \cup \{x\} \times [0,1])$$

(distinguished point for $\Omega(X,x)$ is the constant loop and for $\Sigma(X,x)$, it is $(x,0)$).
Σ is left-adjoint to Ω, where Σ and Ω are functors from $\mathbf{Toph}_*$ to $\mathbf{Toph}_*$.

(2) For a commutative ring R and an R-module M, the functor

$$\mathbf{Mod}_R \ni N \mapsto M \otimes_R N \in \mathbf{Mod}_R$$

is left-adjoint (Problem 2.4) to

$$\mathbf{Mod}_R \ni N \mapsto \mathrm{Hom}_{\mathbf{Mod}_R}(M,N) \in \mathbf{Mod}_R.$$

Theorem
Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that for all $Y \in \mathcal{D}$, there is $X_Y \in \mathcal{C}$ representing the functor $h^Y F$, i.e. the functor

$$\mathcal{C} \ni X \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \in \mathbf{Set}.$$

Then there is a unique functor $G : \mathcal{D} \to \mathcal{C}$ such that:

- For all $Y \in \mathcal{D}$, $G(Y) = X_Y$
- G is right-adjoint to F.

Going to the opposite category, we get the dual theorem where we start from G and obtain a left-adjoint functor F. (We skip the proof.)

Example
Let K be a field. We consider

$$G : \mathbf{Alg}_K \to \mathbf{Alg}_K, \quad G(R) = R[x]/(x^2)$$

(the functor of *dual numbers*).
For each K-algebra R, we fix a K-algebra isomorphism $R \cong K[X]/I$, for a set of variables X and an ideal I. Let $\delta : R[X] \to R[X,X']$ ($X' := \{x' : x \in X\}$) be a derivation such that $\delta(K) = 0$ and for all $x \in X$, $\delta(x) = x'$. We define

$$R' = K[X,X']/(I, \delta(I)).$$

Then R' represents the functor

$$\mathbf{Alg}_K \ni S \mapsto \mathrm{Hom}_{\mathbf{Alg}_K}(S, G(R)) \in \mathbf{Set}.$$

Hence (by the last theorem), the assignment $R \mapsto R'$ extends uniquely to a functor F, which is left-adjoint to G.
For K algebraically closed and V an affine algebraic variety over K, if we set $R = K[V]$ (the ring of regular functions), then we get $F(R) \cong K[TV]$, where TV is the tangent bundle. Hence F corresponds to the functor of the tangent bundle.

LECTURE 5

Limits

Let $(I, \leq)$ be a *poset* (*p*artially *o*rdered *set*) and $\mathcal{C}$ a category. For each $i \in I$, we have $X_i \in \mathcal{C}$ and for each $i \leq j$ from I, we have $f_{ij} : X_i \to X_j$ such that for all $i \leq j \leq k$, we have

$$f_{jk} f_{ij} = f_{ik}, \quad f_{ii} = \mathrm{id}_{X_i}.$$

We will write $(X_i, f_{jk})_{i \in I, j<k}$ or $(X_i)_{i \in I}$ or $(f_{jk})_{j<k}$.

Remark
Having such a choice as above is equivalent to having a functor

$$F : (I, \leq) \to \mathcal{C},$$

where $(I, \leq)$ is regarded as a category.

Definition of limit
Inverse limit of a functor $F : (I, \leq) \to \mathcal{C}$ (or *projective limit* or just *limit*) is an object $X \in \mathcal{C}$ together with a collection of morphisms $(f_i : X \to X_i)_{i \in I}$ such that the following holds.

(1) For each $i, j \in I$, we have $f_{ij} f_i = f_j$.
(2) For any other collection $(f'_i : X' \to X_i)_{i \in I}$ satisfying condition (1), there is a unique morphism $f : X' \to X$ such that for each $i \in I$, we have $f_i f = f'_i$.

The limit object X is denoted by $\varprojlim F$ or just $\lim F$.

In the above definition, the category coming from $(I, \leq)$ can be replaced with an arbitrary category $\mathcal{I}$ (understood as an index category).

As usual, one can also give an equivalent definition of the inverse limit using representable functors.

Examples (Let $\mathcal{C}$ be a category)

(1) Product
$X \times Y = \varprojlim(X, Y)$, i.e. $I = \{0, 1\}$ with the discrete order and $F(0) = X, F(1) = Y$.
If I is a discrete poset and for $i \in I$, $F(i) = X_i$, then we have

$$\prod_{i \in I} X_i := \varprojlim F.$$

(2) Fiber product (pull-back)
For $f : X \to Z, g : Y \to Z$, we define the *pull-back* f and g as

$$X \times_Z Y := \varprojlim(f : X \to Z, g : Y \to Z).$$

Our poset is $I = \{0, 1, 2\}$ with the ordering $0 \le 1, 1 \le 2$ and $F(0) = X, F(1) = Y, F(2) = Z, F(0 \le 2) = 0, F(1 \le 2) = f$.
In other words, the morphism $X \times_Z Y \to X, Y$ are universal among those for which the following diagram is commutative

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow f \\ Y & \xrightarrow{g} & Z \end{array} .$$

W **Set**:

$$X \times_Z Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$$

(3) Kernel
Assume that the category $\mathcal{C}$ has a zero object 0 (i.e. 0 is an initial object and a terminal object at the same time). Then for each $X, Y \in \mathcal{C}$ there is the *zero morphism* $0 : X \to Y$.
For each morphism $f : X \to Y$, $\ker(f)$ is a morphism $k : K \to X$ which is universal with respect to morphisms $s : Z \to X$ such that $fs = 0$.
Equivalently:

$$\ker(f) = \varprojlim(f : X \to Y, 0 : 0 \to Y) = X \times_Y 0.$$

(a) In the category of groups $\ker(f : G \to H)$ is the inclusion of $f^{-1}(e)$ in G.
(b) In $\mathbf{Set}_*$, $0 = *$ and $\ker(f : X \to Y)$ is the inclusion of $f^{-1}(*)$ in X.

(4) If $(I, \le)$ has a least element i_0 (more generally, if the index category has an initial object), then $\varprojlim F = F(i_0)$.

Definition of colimit
Direct limit of a functor $F : (I, \le) \to \mathcal{C}$ (*inductive limit* or *colimit*) is an object $X \in \mathcal{C}$ together with a collection o morphisms $(f_i : X_i \to X)_{i \in I}$ such that

- for each $i, j \in I$, $f_j f_{ij} = f_i$.
- For any other collection $(f_i' : X' \to X_i)_{i \in I}$ satisfying condition (1), there is a unique morphism $f : X \to X'$ such that for each $i \in I$, $ff_i = f_i'$.

A colimit X is denoted by $\varinjlim F$ or $\operatorname{colim} F$.
Examples ($\mathcal{C}$ is a category)

(1) Coproduct
$X \coprod Y = \varinjlim(X, Y)$ and for any discrete poset I,

$$\coprod_{i \in I} X_i := \varinjlim F.$$

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$$\coprod_{i \in I} G_i = \bigoplus_{i \in I} G_i.$$

(2) Fibered coproduct (push-out)
For $f : Z \to X, g : Z \to Y$, we define a *push-out* f and g as

$$X \coprod_Z Y := \varinjlim(f : Z \to X, g : Z \to Y).$$

In other words, the morphism $X, Y \to X \times_Z Y$ are universal among those for which the following diagram is commutative

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r] & X \coprod_Z Y \end{tikzcd} .$$

In **Sets**,

$$X \coprod_Z Y = (X \coprod Y)//f(z) \sim g(z).$$

In $\mathbf{Alg}_K$,

$$S_1 \coprod_R S_2 = S_1 \otimes_R S_2.$$

(3) Cokernel
Assume that $\mathcal{C}$ has a zero object 0. For each morphism $f : X \to Y$, coker(f) is a morphism $k : Y \to C$ which is universal w.r.t. morphisms $s : Y \to Z$ such that $sf = 0$.
Equivalently

$$\operatorname{coker}(f) = \varinjlim(f : X \to Y, 0 : X \to 0) = Y \coprod_X 0.$$

(a) In $\mathbf{Mod}_R$, $\operatorname{coker}(f : M \to N) = N \to N/f(M)$.
(b) In $\mathbf{Set}_*$, $\operatorname{coker}(f : X \to Y) = Y \to Y/f(X)$ (where $f(X)$ is contracted to the distinguished point).

(4) If $(I, \le)$ has a greatest element i_∞ (more generally the index category has a terminal object), then $\varinjlim F = F(i_\infty)$.

Definition
Assume that a poset $(I, \le)$ is *directed*, i.e. for each $i, j \in I$ there is $k \in I$ such that $i, j \le k$. Then a functor $F : (I, \le) \to \mathcal{C}$ is called *direct system* and $F : (I, \le) \to \mathcal{C}^{\text{op}}$ is called *inverse system*.
$\varprojlim F$ is called *inverse limit* and $\varinjlim F$ is called *direct limit*.

Remark (exercise)
Let $(I, \le)$ be a directed poset, $i \in I$ and $I_i = \{j \in I | j \ge i\}$. Then for each inverse system $(X_i)_{i \in I}$, we have

$$\varprojlim(X_j)_{j \in I} = \varprojlim(X_j)_{j \in I_i}.$$

Similarly for the direct limit.

Examples

(1) In $\mathbf{Mod}_R$ direct limit of $(M_i)_{i \in I}$, is (exercise)

$$\varinjlim(M_i)_{i \in I} = \bigoplus_{i \in I} M_i/\langle (w_j f_{ij} - w_i)(M_i) | \forall i \le j \rangle,$$

where $w_i : M_i \to \bigoplus_{i\in I} M_i$ is the embedding into the direct sum.
Inverse limit $(M_i)_{i\in I}$ is (exercise)

$$\varprojlim (M_i)_{i\in I} = \{(a_i)_{i\in I} \in \prod_{i\in I} M_i | (\forall i \le j) f_{ij}(a_j) = a_i\}.$$

The above construction of inverse limit works also in the categories of groups, rings and topological spaces.

(2) take a ring R and an ideal $I \subset R$. We obtain an inverse system $(R/I^n)_{n\in\mathbb{N}}$. Its limit $\varprojlim(R//I^n)$ is called *completion* of R with respect to I. E.g, we get the ring of p-adic integers:

$$\mathbb{Z}_p := \varprojlim(\mathbb{Z}/p^n\mathbb{Z})_{n\in\mathbb{N}}.$$

(3) Let us take the following directed poset $(\mathbb{N}, |)$ (the relation of divisibility). Our direct system consists of finite fields of characteristic p, $(\mathbb{F}_{p^n})_{n\in\mathbb{N}}$ where the maps are field extensions. Then we have:

$$\varinjlim(\mathbb{F}_{p^n})_{n\in\mathbb{N}} = \overline{\mathbb{F}_p},$$

the algebraic closure of $\mathbb{F}_p$.
Much more generally: each structure is the direct limit of its finitely generated substructures.

(4) A *profinite group* is the inverse limit of finite groups in the category of topological groups.

Definition
Functor $G : \mathcal{C} \to \mathcal{D}$ *preserves inverse limits*, if
for any $F : (I, \le) \to \mathcal{C}$, if $X = \varprojlim F$, then $G(X) = \varprojlim(G \circ F)$.
There is an analogous (obvious) notion of a functor which *preserves direct limits.*

Theorem
If F is a left-adjoint functor to a functor G, then F preserves direct limits and G preserves inverse limits.

Examples

(1) G – forgetful functor, F – left-adjoint to G

(a) $G : \mathbf{Grp} \to \mathbf{Set}$, $F : \mathbf{Set} \to \mathbf{Grp}$ – free group functor.
G preserves inverse limits e.g. products:

$$G(H_1 \times H_2) = G(H_1) \times G(H_2).$$

G does not preserve direct limits, e.g. coproducts:

$$G(H_1 * H_2) \ncong G(H_1) \cup G(H_2).$$

F preserves direct limits, e.g. coproducts:

$$F_{X\cup Y} \cong F_X * F_Y.$$

But G does not preserve inverse limits, e.g. products:

$$F_{X\times Y} \ncong F_X \oplus F_Y.$$

(b) $G : \mathbf{Ab} \to \mathbf{Grp}$, $F : \mathbf{Grp} \to \mathbf{Ab}$ – Abelianization functor.
G preserve products and does not preserve coproducts:

$$H_1 * H_2 \ncong H_1 \oplus H_2.$$

F preserves coprodukts

$$(H_1 * H_2)^{\mathrm{ab}} \cong H_1^{\mathrm{ab}} \oplus H_2^{\mathrm{ab}},$$

and, just by chance, also products

$$(H_1 \oplus H_2)^{\mathrm{ab}} \cong H_1^{\mathrm{ab}} \oplus H_2^{\mathrm{ab}}.$$

However, F does not preserve kernels (which are also inverse limits). Take e.g.

$$\mathbb{Z}/3\mathbb{Z} \xrightarrow{\ f\ } S_3 \xrightarrow{\ g\ } \mathbb{Z}/2\mathbb{Z},$$

where $f = \ker(g)$, $g = \mathrm{coker}(f)$. After applying F, we get

$$\mathbb{Z}/3\mathbb{Z} \xrightarrow{f^{\mathrm{ab}}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{g^{\mathrm{ab}}=\mathrm{id}} \mathbb{Z}/2\mathbb{Z}\ .$$

But $f^{\mathrm{ab}} = 0 : \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not the kernel of $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$.
In particular, F does not take monomorphisms to monomorphisms: f goes to 0.
F preserves cokernels: $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} = \mathrm{coker}(0 : \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z})$.
Preserving cokernels implies the following:

$$H_1 \leqslant H_2 \implies (H_1/H_2)^{\mathrm{ab}} \cong H_1^{\mathrm{ab}}/H_2^{\mathrm{ab}}.$$

(2) For an R-module M, the functor

$$\cdot \otimes_R M : \mathbf{Mod}_R \to \mathbf{Mod}_R$$

is left-adjoint to h_M (understood as a functor into $\mathbf{Mod}_R$).
In particular, the functor $\cdot \otimes_R M$ preserves cokernels, so also preserves epimorphisms, i.e.

$$N \twoheadrightarrow N' \implies N \otimes_R M \twoheadrightarrow N' \otimes_R M.$$

The functor $\cdot \otimes_R M$ does not preserve monomorphisms – e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$, but

$$\mathbb{Z} \otimes_Z \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q} \otimes_Z \mathbb{Z}/2\mathbb{Z}$$

is not a monomorphism, since

$$\mathbb{Z} \otimes_Z \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \ \text{ i } \ \mathbb{Q} \otimes_Z \mathbb{Z}/2\mathbb{Z} = 0.$$

Theorem
Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. Then the representable functor

$$h_A : \mathcal{C} \to \mathbf{Set}$$

preserves (inverse) limits. Hence (applying the above to the category $\mathcal{C}^{\mathrm{op}}$), h^A takes direct limits to inverse limits.
Proof
Let

$$(X, f_i) = \varprojlim(X_i, f_{jk}).$$

We want to show that (in the category **Set**)

$$(\mathrm{Hom}(A, X), h_A(f_i)) = \varprojlim(\mathrm{Hom}(A, X_i), h_A(f_{jk})).$$

Let us take (Y, g_i) which is a possible "candidate" for $\varprojlim(\mathrm{Hom}(A, X_i), h_A(f_{jk}))$, i.e. $g_i : Y \to \mathrm{Hom}(A, X_i)$ and these maps commute with all the maps $h_A(f_{jk})$.
For any $y \in Y$, we have

$$f_{jk} \circ g_j(y) = h_A(f_{jk})(g_j(y)) = g_k(y).$$

Hence $(A, g_i(y))$ is a "candidate" for $\varprojlim(X_i, f_{jk})$. Since

$$(X, f_i) = \varprojlim(X_i, f_{jk}),$$

the morphism $g(y) : A \to X$ is a unique morphism $A \to X$ commuting with all the $g_i(y)$. Hence $g : Y \to \mathrm{Hom}(A, X)$ is a unique map commuting with all the $h_A(g_i)$. $\square$

LECTURE 5

We want to generalize the notions of monomorphism and epimorphism to an arbitrary category.

Definition
A morphism $f : X \to Y$ is a *monomorphism*, if for all $g, h : Z \to X$, $fg = fh$ implies $g = h$. A morphism f is an *epimorphism*, if for all $g, h : Y \to Z$, $gf = hf$ implies $g = h$.
A monomorphism is denoted by $X \hookrightarrow Y$ and an epimorphism is denoted by $X \twoheadrightarrow Y$.

Examples

(1) Any isomorphism is a monomorphism and an epimorphism, but the opposite implication does not hold.
(2) Monomorphism in **Set** are one-to-one functions. Epimorphisms in **Set** are onto functions. Similarly in **Top** or $\mathbf{Mod}_R$. But there are bijective continuous functions which are not homeomorphisms (e.g. $[0, 2\pi) \to S^1$), so indeed we do not have the opposite implication as mentioned above.
(3) Monomorphism in **Haus** (topological Hausdorff spaces) are continuous one-to-one functions. However, epimorphisms in **Haus** are the *dominant* functions, i.e. continuous functions $f : X \to Y$ such that $f(X)$ is *dense* in Y. Similarly in the category $\mathbf{Var}_K$ (we need *Zariski dense* here).

Remark
In terms of representable functors the above definitions can be phrased as follows.

(1) The morphism $f : X \to Y$ is a monomorphism if and only if for all $Z \in \mathcal{C}$ the map
$$h_Z(f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$$
is one-to-one.
(2) The morphism $f : X \to Y$ is an epimorphism if and only if for all $Z \in \mathcal{C}$ the map
$$h^Z(f) : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$
is one-to-one.

So, both the notions are defined in terms of *one-to-one* functions of sets (of morphisms). They can *not* be phrased in terms of *onto* functions of sets (of morphisms).

Abelian categories

We will restrict now our attention to some special kind of categories in which one can do *homological algebra*.

Definition (Ab-categories and additive categories)
Let $\mathcal{C}$ be a category.

A1 (The definition of *Ab-category*)
For all $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}(X, Y)$ has a structure of an *Abelian group* such that the composition of morphisms is distributive with respect to this Abelian group operation.

A2 There is $0 \in \mathcal{C}$, the zero object (terminal and initial).

Remark
In any category $\mathcal{C}$ satisfying A2 and for any $X, Y \in \mathcal{C}$, we have a unique *zero morphism* $0_{XY} : X \to Y$. If $\mathcal{C}$ satisfies also A1, then it easily follows (using distributivity) that 0_{XY} is the neutral element in the group $\mathrm{Hom}(X, Y)$.

A3 (we assume A1 and A2)
For all $X_1, X_2 \in \mathcal{C}$ there is $Y \in \mathcal{C}$ together with the following morphisms:

$$i_1 : X_1 \to Y,\ i_2 : X_2 \to Y,\ b_1 : Y \to X_1,\ b_2 : Y \to X_2$$

such that

$$b_1 i_1 = \mathrm{id}_{X_1},\ b_2 i_2 = \mathrm{id}_{X_2},\ i_1 b_1 + i_2 b_2 = \mathrm{id}_Y,\quad b_1 i_2 = b_2 i_1 = 0.$$

The object Y is called *direct sum* (or *biproduct*) of X_1 and X_2 and denoted $X_1 \oplus X_2$.
The category $\mathcal{C}$ is *additive*, if it satisfies A1, A2 i A3.

Fact
A direct sum in an additive category is both product and coproduct.
Proof
We will show that $X_1 \oplus X_2$ together with b_1 and b_2 is a product of X_1 and X_2. The argument for coproduct is similar using the morphisms i_1 and i_2 (actually, it is exactly a dual argument).
Take $f : Z \to X_1, g : Z \to X_2$. We will show that $\phi := i_1 f + i_2 g$ is a unique morphism $\phi : Z \to X_1 \oplus X_2$ such that the appropriate diagrams commute.
Commuting:

$$b_1 \phi = b_1 i_1 f + b_1 i_2 g = \mathrm{id}_{X_1} f + 0g = f.$$

Similarly we get that $b_2 \phi = g$.
Uniqueness: Assume that we have a morphism $\phi : Z \to X_1 \oplus X_2$ such that $b_1 \phi = f,\ b_2 \phi = g$. Then we have

$$i_1 b_1 \phi = i_1 f,\ i_2 b_2 \phi = i_2 g,$$

$$i_1 f + i_2 g = (i_1 b_1 + i_2 b_2)\phi = \mathrm{id}_Y \phi = \phi. \quad \square$$

Definition (of Abelian categories)
A4 For any morphism $f : X \to Y$ there is a sequence of morphisms (*canonical decomposition* of f):

$$K \xrightarrow{\ k\ } X \xrightarrow{\ i\ } I \xrightarrow{\ j\ } Y \xrightarrow{\ c\ } C$$

such that

(1) $ji = f$.
(2) $\ker(f) = (K, k),\ \mathrm{coker}(f) = (c, C)$.
(3) $\ker(c) = (I, j),\ \mathrm{coker}(k) = (i, I)$ (I is called the *image* of f).

A category is *Abelian*, if it is additive and if it satisfies A4.

Remark
In particular, if $\mathcal{C}$ is an Abelian category, then $\mathcal{C}$ is additive, and kernels and cokernels exist in $\mathcal{C}$.
On the other hand, in any additive category (assuming that $\mathcal{C}$ has a zero object would be enough) with kernels and cokernels, for any $f : X \to Y$ we have:

$$K \xrightarrow{k} X \xrightarrow{i} I, \qquad I' \xrightarrow{j} Y \xrightarrow{c} C,$$

where $k = \ker(f),\ i = \operatorname{coker}(k),\ c = \operatorname{coker}(f),\ j = \ker(c)$.
I' is called the *image* of f and I is called the *coimage* of f.
Then we have a unique (in a proper sense) morphism $l : I \to I'$ such that $\ker(l) = \operatorname{coker}(l) = 0$. A category is Abelian if and only if l is an *isomorphism* (in particular image is isomorphic with coimage). Hence A4 corresponds to the (first) *Isomorphism Theorem* (e.g. for groups: $G/\ker(f) \cong \operatorname{im}(f)$).

Examples

(1) $\mathbf{Mod}_R$ is an Abelian category.

(2) Commutative algebraic groups: additive category, *not* an Abelian category. Consider the Frobenius homomorphism

$$\mathrm{Fr} : (K, +) \to (K, +),$$

where $\mathrm{Fr}(a) = a^p$ and K is a field of characteristic $p > 0$.
Then Fr is a monomorphism and an epimorphism, but not an isomorphism. It is not possible in an Abelian category (we will see it soon).
In a bigger category of *group schemes* Fr is *not* a monomorphism, since $\ker(\mathrm{Fr})$ is not the zero object bur corresponds to the ring (actually a *Hopf algebra*) $K[X]/(X^p)$.
Over a field of characteristic 0, the category of commutative algebraic groups *is* an Abelian category.

(3) For any small category $\mathcal{C}$ and an Abelian category $\mathcal{A}$, the category $\mathrm{Func}(\mathcal{C}, \mathcal{A})$ is an Abelian category. It follows directly, e.g.

$$(G \oplus F)(X) := G(X) \oplus F(X).$$

In particular, the category of presheaves of Abelian groups over a space X, denoted $\mathbf{Psh}_X$, is an Abelian category.

(4) However, we are mostly interested in the category of *sheaves*: $\mathbf{Sh}_X$.
Clearly, $\mathbf{Sh}_X$ is an additive category. However, the cokernel (in the category of presheaves) of a morphism of sheaves *need not* be a sheaf. Let $\mathcal{O}_{\mathbb{C}}$ be the sheaf of holomorphic functions into $\mathbb{C}$ (with addition of functions) and $\mathcal{O}^*_{\mathbb{C}}$ be the sheaf of holomorphic functions into $\mathbb{C} \setminus \{0\}$ (with multiplication of functions). Consider the morphism

$$\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}^*_{\mathbb{C}}.$$

Let $\mathcal{F}$ be the cokernel of exp in the category of presheaves (i.e. for each open V, $\mathcal{F}(V) = \mathcal{O}^*_{\mathbb{C}}(V)/\exp(\mathcal{O}_{\mathbb{C}}(V))$) and take $U = \mathbb{C} \setminus \{0\}$.
Then $s := [\mathrm{id}_U] \in \mathcal{F}(U) \setminus \{0\}$ (since there is no global logarithm!), but for a covering U by disks (or any open simple-connected sets) $U = \bigcup U_i$, we have $s|_{U_i} = 0$ (since there is a logarithm on a disk). Hence $\mathcal{F}$ is not a sheaf.
However the category $\mathbf{Sh}_X$ still *is* Abelian, which we will see below.

Theorem
There is a left-adjoint functor to the forgetful functor

$$\mathbf{Sh}_X \to \mathbf{Psh}_X,$$

called the *sheafification* functor

$${}^+ : \mathbf{Psh}_X \to \mathbf{Sh}_X.$$

For a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$, we have

$$\ker_{\mathbf{Sh}_X}(f) = \ker_{\mathbf{Psh}_X}(f), \quad \mathrm{coker}_{\mathbf{Sh}_X}(f) = \mathrm{coker}_{\mathbf{Psh}_X}(f)^+$$

and $\mathbf{Sh}_X$ is an Abelian category.
Proof
We will not show the existence of $^+$, the idea is to construct $\mathcal{F}^+$ using functions into the groups of *germs* of $\mathcal{F}$.

For any morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$, the kernel of f in the category of presheaves $\mathcal{K}$ is a sheaf, since for any open covering $U = \bigcup_i U_i$ and an appropriate collection $s_i \in \mathcal{K}(U_i)$, the collection (s_i) extends uniquely to $s \in \mathcal{F}(U)$ (since $\mathcal{F}$ is a sheaf) and $f(s)|_{U_i} = f(s_i) = 0$, hence $f(s) = 0$ (since $\mathcal{G}$ is a sheaf) and $s \in \mathcal{K}(U)$. Then $\mathcal{K}$ is also the kernel of f in the category of sheaves.
Let $\mathcal{H}$ be the cokernel of f in the category of presheaves. We will show that $\mathcal{H}^+$ is the cokernel of f in the category of sheaves. Take any morphism of sheaves $g : \mathcal{G} \to \mathcal{S}$ such that $gf = 0$. Then g factors through a unique $l : \mathcal{H} \to \mathcal{S}$ which (using the adjointness) factors through a unique $k : \mathcal{H}^+ \to \mathcal{S}$. Hence $\mathcal{H}^+$ is the cokernel of f in the category of sheaves.
We will show now that the category $\mathbf{Sh}_X$ is Abelian. Let

$$\mathcal{K} \xrightarrow{\quad k \quad} \mathcal{F} \xrightarrow{\quad i \quad} \mathcal{I} \xrightarrow{\quad j \quad} \mathcal{G} \xrightarrow{\quad c \quad} \mathcal{K}'$$

be the canonical decomposition of f in the (Abelian) category of presheaves. Then the sequence

$$\mathcal{K} \xrightarrow{\quad k \quad} \mathcal{F} \xrightarrow{\quad i^+ \quad} \mathcal{I}^+ \xrightarrow{\quad j^+ \quad} \mathcal{G} \xrightarrow{\quad c^+ \quad} \mathcal{K}'^+$$

is a canonical decomposition of f in the category of sheaves (since the sheafification functor preserves kernels and cokernels). □

In Abelian categories monomorphisms and epimorphisms have a nice characterization.

Fact
Let $\mathcal{A}$ be an Abelian category and $f : X \to Y$ be a morphism. Then we have:

(1) f is a monomorphism if and only if $\ker(f) = 0$;
(2) f is an epimorphism if and only if $\mathrm{coker}(f) = 0$;
(3) f is an isomorphism if and only if $\ker(f) = \mathrm{coker}(f) = 0$.

In particular, f is an isomorphism if and only if f is both an epimorphism and a monomorphism.
Proof

(1) $\Rightarrow$ Let $k : K \to X$ be the kernel of f. Then $fk = 0$. But also $f0 = 0$, hence $k = 0$, since f is a monomorphism. Using the universal property of kernel, we get $\mathrm{id}_K = 0$, hence (easy) $K = 0$.

$\Leftarrow$ Assume that $fg = fh$. Then $f(g-h) = 0$. Hence $g-h$ factors through the kernel, that is $g - h = 0$, since $\ker(f) = 0$.

(2) Similarly as for monomorphisms.

(3) $\Rightarrow$ Follows from 1. and 2.

$\Leftarrow$ We look at the canonical decomposition of f from the property A4

$$0 \xrightarrow{0} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{0} 0,$$

However, we have $\operatorname{coker}(0 \to X) = \mathrm{id}_X$ and $\ker(Y \to 0) = \mathrm{id}_Y$. Hence $I \cong X$, $J \cong Y$ and f is an isomorphism as the composition of two isomorphisms. □

Remark/Notation

For a monomorphism, the object $\operatorname{coker}(X \hookrightarrow Y)$ can be understood as *quotient object*, and denoted by Y/X.

Let us notice one more good property of Abelian categories.

Fact

In an Abelian category any kernel is a monomorphism and any cokernel is an epimorphism. In particular, the decomposition of a morphism $f : X \to Y$ from A4 of the form $f = ij$ is a decomposition into the epimorphism j composed with the monomorphism i.

Proof (in the case of kernel)

It is enough to notice that $\ker(\ker f) = 0$ (it holds in any category with a zero object) and use the fact that in an Abelian category, a morphism t is a monomorphism if and only if $\ker(t) = 0$. □

LECTURE 7

Definition (complex and cohomology)

(1) *Complex* (of cochains), in an Abelian category $\mathcal{A}$, is any infinite sequence of objects and morphisms (called *boundary operators*)

$$X^* : \quad \dots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \dots$$

such that for all n, we have $d^{n+1}d^n = 0$.

In the category $\mathbf{Mod}_R$, the condition $d^{n+1}d^n = 0$ is equivalent to $\operatorname{im}(d^n) \subseteq \ker(d^{n+1})$. There is a similar interpretation in any Abelian category: $d^{n+1}d^n = 0$ iff d_n factors through $\alpha_n : X^n \to \ker(d^{n+1})$ iff $X^n \to \operatorname{im}(d^n)$ factors through a monomorphism $\operatorname{im}(d^n) \to \ker(d^{n+1})$.

(2) For a complex X^*, we define $H^n(X^*)$ (n-th *cohomology* of X^*) as $\operatorname{coker}(\alpha_{n-1})$. Equivalently

$$H^n(X^*) = \operatorname{coker}(\operatorname{im}(d^{n-1}) \hookrightarrow \ker(d^n)),$$

so $H^n(X^*)$ can be understood as $\frac{\ker(d^n)}{\operatorname{im}(d^{n-1})}$.

(3) Complex X^* is *acyclic* at object X^n, if $H^n(X^*) = 0$. A complex acyclic at each object is called an *exact sequence*. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a *short exact sequence*.

Example

For direct sum $A \oplus B$,

$$0 \longrightarrow A \xrightarrow{i_1} A \oplus B \xrightarrow{b_2} B \longrightarrow 0,$$

is a short exact sequence.

(4) We obtain the category of complexes (of cochains) $\mathrm{Com}(\mathcal{A})$ (morphisms should preserve the derivations).
After defining kernels/cokernels/direct sums "object by object", we get that $\mathrm{Com}(\mathcal{A})$ is an Abelian category.

Definition
Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between Abelian categories.

(1) F is *additive*, if it induces homomorphisms on the Abelian groups of morphisms.
An additive functor takes complexes to complexes.
(2) An additive functor is *exact*, if it takes exact sequences to exact sequences.
(3) An additive functor F is *right-exact*, if it takes an exact sequence $0 \to A \to B \to C \to 0$, to a complex $0 \to F(A) \to F(B) \to F(C) \to 0$, which is acyclic in $F(B)$ and $F(C)$.
(4) An additive functor F is *left-exact*, if it takes an exact sequence $0 \to A \to B \to C \to 0$, to a complex $0 \to F(A) \to F(B) \to F(C) \to 0$, which is acyclic in $F(B)$ and $F(A)$.

Fact

$$\text{A sequence} \quad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \quad \text{is}$$

(1) acyclic in X and Y iff $f = \ker(g)$.
(2) acyclic in Y and Z iff $g = \mathrm{coker}(f)$.

Fact
Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between Abelian categories. We list below some properties F may enjoy (or not).

(1) F is right-adjoint (resp. left-adjoint).
(2) F preserves inverse limits (resp. direct limits).
(3) F preserves kernels (resp. cokernels).
(4) F is left-exact (resp. right-exact).
(5) F preserves monomorphisms (resp. epimorphism).

Then we have: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).
Proof
Collecting the things we know ((3) $\Rightarrow$ (4) follows from the previous fact). $\square$

Motivating remark
We want to know $F(C)$. If F is exact and there is a short exact sequence

$$0 \to A \to C \to B \to 0,$$

then by knowing $F(A)$ and $F(B)$, we know something about $F(C)$.
If F is *not* exact, then, thanks to homological algebra, we will measure "how far" is $F(C)$ from $F(A)$ and $F(B)$.

Examples

(1) Representable functors h_A are left-exact.
(Co)representable functors h^A are also left-exact as functors from $\mathcal{C}^{\mathrm{op}}$ to **Ab**, i.e. an exact sequence

$$0 \leftarrow X \leftarrow Y \leftarrow Z \leftarrow 0$$

goes to an exact sequence outside of $\mathrm{Hom}(Z, A)$,

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(Y, A) \to \mathrm{Hom}(Z, A) \to 0.$$

(2) The functor

$$\cdot \otimes_R M : \mathbf{Mod}_R \to \mathbf{Mod}_R$$

is right-exact (as left-adjoint to h_M).
(3) The functor

$$\Gamma : \mathbf{Sh}_X \to \mathbf{Ab}, \quad \Gamma(\mathcal{F}) = \mathcal{F}(X)$$

is right-adjoint to the *constant sheaf functor*

$$\mathbf{Ab} \ni A \mapsto A_X \in \mathbf{Sh}_X.$$

Hence it is left-exact (another reason is that it preserves kernels, since kernels in the category of sheaves are the same as kernels in the category of presheaves).
The following sequence of sheaves

$$0 \longrightarrow \mathbb{Z}_{\mathbb{C}^*} \xrightarrow{\cdot 2\pi i} \mathcal{O}_{\mathbb{C}^*} \xrightarrow{\exp} \mathcal{O}^*_{\mathbb{C}^*} \longrightarrow 0$$

is exact, since:
(a) Clearly, we have

$$\mathbb{Z}_{\mathbb{C}^*} \hookrightarrow \mathcal{O}_{\mathbb{C}^*}.$$

(b) For each function $g : U \to \mathbb{C}$ and a connected U, if $\exp(g) = 1$, then $g(U) \subset 2\pi i\mathbb{Z}$. But $g(U)$ is also connected, so g is the constant function $2\pi n i$ for some $n \in \mathbb{Z}$. Hence $\mathrm{im}(\cdot 2\pi n i) = \ker(\exp)$.
(c) We also know that for a simply-connected U, the map

$$\exp|_U : \mathcal{O}_{\mathbb{C}^*}(U) \to \mathcal{O}^*_{\mathbb{C}^*}(U)$$

is an epimorphism.
However the sequence of global sections

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathbb{C}^*}(\mathbb{C}^*) \xrightarrow{\exp} \mathcal{O}^*_{\mathbb{C}^*}(\mathbb{C}^*) \longrightarrow 0,$$

is not exact at $\mathcal{O}^*_{\mathbb{C}^*}(\mathbb{C}^*)$, since for $\mathrm{id}_{\mathbb{C}^*}$ there is no global logarithm.

Fact
We say that a short exact sequence *splits*, if it is isomorphic to the short exact sequence coming from a direct sum. The following conditions are equivalent.
(1) A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0,$$

splits.
(2) There is a morphism $g' : B \to C$ such that $gg' = \mathrm{id}_C$.
(3) There is a morphism $f' : A \to B$ such that $f'f = \mathrm{id}_A$.

Definition
(1) An object A is *projective*, if the functor h_A is exact.
(2) An object A is *injective*, if the functor h^A is exact.

Fact (categorical definition of projective and injective objects)

(1) An object Y is projective if and only if for all morphisms
$$f : X \twoheadrightarrow X', \quad g : Y \to X'$$
there is a morphism $h : Y \to X$ such that $fh = g$.
(2) An object Y is injective if and only if for all morphisms
$$f : X' \hookrightarrow X, \quad g : X' \to Y$$
there is a morphism $h : X \to Y$ such that $hf = g$.

Examples

(1) A free module is projective. More precisely, a module is projective if and only if it is a direct summand of a free module.
(2) An Abelian group is injective if and only if it is divisible. Hence e.g. $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Fact
Assume that $0 \to A \to B \to C \to 0$ is an exact sequence. If C is projective or A is injective, then this sequence splits.

Derived functors

We fix an Abelian category $\mathcal{A}$.
A complex of *cycles* is a complex of the form
$$X_* : \qquad \ldots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \ldots .$$
For each complex of cycles X_*, we define $H_n(X_*)$, *the n-th homology of* X_*, similarly as for complexes of cochains.
The following arrow $X_* \to M$ means that we have a complex of cycles of the form
$$\ldots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} M \longrightarrow 0,$$
and $M \to X^*$ means that we have aa complex of cocycles of the form
$$0 \longrightarrow M \xrightarrow{d^0} X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \ldots \quad .$$

Hilbert's Syzygy Theorem

In astronomy, a *syzygy* (from the Ancient Greek *suzugos* meaning, "yoked together") is a straight-line configuration of three celestial bodies in a gravitational system. The word is often used in reference to the Sun, Earth and either the Moon or a planet, where the latter is in conjunction or opposition. Solar and lunar eclipses occur at times of syzygy. The term is often applied when the Sun and Moon are in conjunction (new moon) or opposition (full moon).

Here, in our homological algebra context, for any finitely generated (by n_0 elements) R-module M, we create an exact sequence consisting of free modules. We assume that R is a Noetherian ring. We find first an epimorphism $d_0 : F_0 \to M$, (where $F_0 = R^{n_0}$) which kernel, that is R-dependencies ("syzygies") between generators of M, has finitely many generators, as a submodule of a finitely generated module over a Noetherian ring, say n_1 generators.

We take now an epimorphism $d_1 : F_1 \to \ker(d_0)$ (where $F_1 = R^{n_1}$), so $\ker(d_1)$ corresponds to R-dependencies ("syzygies") between generators of $\ker(d_0)$ (so R-dependencies between R-dependencies), and so on...

Hilbert's syzygy theorem says that there are choices of generators for which this sequence becomes 0 after $r+1$ steps, that is we obtain an acyclic complex of the form

$$0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to M \to 0,$$

where F_i are free (in particular projective). This complex can be understood as "full information" about M.

Definition
Projective resolution of X is an acyclic complex (of chains) of the form $P_* \to X$, where for all i, the object P_i is projective.
Injective resolution of X is an acyclic complex (of cochains) of the form $X \to I^*$, where for all i, the object I^i in injective.

Definition
An Abelian category $\mathcal{A}$ is said *to have enough projectives*, if for each object $X \in \mathcal{A}$ there is a projective object $P \in \mathcal{A}$ and an epimorphism $P \twoheadrightarrow X$.
An Abelian category $\mathcal{A}$ is said *to have enough injectives*, if for each object $X \in \mathcal{A}$ there is an injective object $I \in \mathcal{A}$ and a monomorphism $X \hookrightarrow I$.

Fact
If $\mathcal{A}$ has enough projectives (resp. injectives), then each object has a projective (resp. injective) resolution.

Definition (derived functors)
Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between Abelian categories ($\mathcal{B}$ is often the category of R-modules for some ring R).
$L_nF : \mathcal{A} \to \mathcal{B}$, *the n-th left derived functor of F*, is defined as follows

$$L_nF(X) := H_n(F(P_* \to 0)),$$

where $P_* \to X$ is a projective resolution of X.
We define in an analogous way *the n-th right derived functor of F* as

$$R^nF(X) := H^n(F(0 \to I^*)),$$

for an injective resolution $X \to I^*$.

Theorem 1 (existence and uniqueness of L_nF, R^nF)
If $\mathcal{A}$ has enough projectives (resp. injectives) and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then the functor $L_nF : \mathcal{A} \to \mathcal{B}$ (resp. $R^nF : \mathcal{A} \to \mathcal{B}$) exists and does not depend on a choice of resolution.

In Theorems 2. and 3. below we assume that there are enough projectives or injectives (depending on the context) in the category $\mathcal{A}$ and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an additive functor between Abelian categories.

Theorem 2 (derived functors and exactness)
$L_0F = F$ if and only if F is right-exact.
$R_0F = F$ if and only if F is left-exact.

Theorem 3 (axioms of homology theory)
If F is right-exact, then for each short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0,$$

there is a long exact sequence

$$\dots \xrightarrow{L_2F(g)} L_2F(B) \xrightarrow{\delta_2} L_1F(A) \xrightarrow{L_1F(f)} L_1F(C) \xrightarrow{L_1F(g)} L_1F(B)$$

$$L_1F(B) \xrightarrow{\delta_1} F(A) \xrightarrow{F(f)} F(C) \xrightarrow{F(g)} F(B) \longrightarrow 0,$$

where morphisms δ_n are functorial.
A sequence of functors and natural morphisms (L_nF, δ_n) satisfying the above and vanishing on projective objects is unique up to an isomorphism.
An analogous theorem holds for right-exact functors F and the corresponding right-derived functors R^nF.

Theorem 4
The category $\mathbf{Mod}_R$ has enough projectives and injectives.
The category $\mathbf{Sh}_X$ has enough injectives.

Definitions

(1) $A, B \in \mathcal{A}$, $\mathcal{A}$ is an Abelian categories.

$$\mathrm{Ext}^n(A, B) := R^n h^B(A) \ (\cong R^n h_A(B)).$$

Interpretation of $\mathrm{Ext}^1(A, B)$ as the group of *extensions*.

(2) $A, B \in \mathbf{Mod}_R$

$$\mathrm{Tor}_n(A, B) := L_n(A \otimes_R \cdot)(B) \ (\cong L_n(\cdot \otimes_R B)(A)).$$

(3) Cohomology of sheaves $\mathcal{F} \in \mathbf{Sh}_X$

$$H^n(X, \mathcal{F}) := R^n\Gamma(\mathcal{F}),$$

where Γ is the functor of global sections.
Singular cohomology of topological spaces as a special case.
Cohomology of sheaves in algebraic geometry.

(4) Cohomology of groups: Let G be a group acting (by automorphisms) on an Abelian group A. Then A is a $\mathbb{Z}G$-module. Let $\mathbb{Z}$ be the trivial $\mathbb{Z}G$-module (G acts by identity).
We define *the n-th cohomology of G with coefficients from A* and *the n-th homology of G with coefficients from A* as

$$H^n(G, A) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A), \quad H_n(G, A) := \mathrm{Tor}^{\mathbb{Z}G}_n(\mathbb{Z}, A).$$

Interpretation of H^0, H_0, H^1, H_1, H^2.