Independence in positive characteristic

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Schanuel conjectures

Schanuel Conjectures

1. Let $x_1, \ldots, x_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}_{\mathbb{Q}}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geqslant n.$$

2. Let $x_1, \ldots, x_n \in tC[\![t]\!]$ be linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}_{C(t)}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geqslant n.$$

Remark

1. The complex field is Archimedean and the conjecture is very open.
2. The field of Laurent series is non Archimedean (C is any characteristic 0 field) and the conjecture was proved by Ax.

Boris Zilber's suggestion

Question

What about a positive characteristic version of Ax's theorem?

Immediate problem

There is no exponential map in positive characteristic. Why?

- Because $p!$ is not invertible in $\mathbb{F}_p$.
- If a power series F over $\mathbb{F}_p$ satisfies

$$F(X_1 + X_2) = F(X_1)F(X_2),$$

then $F = 0, 1$.

Solution

Try some other power series.

Why does Ax's theorem hold for exp?

Reasons

1. exp is an analytic homomorphism between $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$.
2. exp is (very) non-algebraic.

We should look for such maps.

Example

1. The exponential map to any commutative algebraic group from its Lie algebra (characteristic 0),
2. Raising to powers on algebraic torus (arbitrary characteristic),
3. Formal isomorphisms between algebraic tori and (ordinary) abelian varieties (arbitrary characteristic),
4. Additive power series (positive characteristic).

What is known

Ax theorem for some maps

1. Ax proved power series SC for $\exp_A$, where A is a semi-abelian variety.
   - Differential versions: Brownawell-Kubota, Kirby, Bertrand.
   - A "non-constant" version: Bertrand-Pillay.
2. A power series SC for raising to powers α on an n-dimensional characteristic 0 torus, where $[\mathbb{Q}(\alpha) : \mathbb{Q}] > n$ (K.).
3. A power series SC for additive power series (K., preprint available on my web page). In a way it is similar to the raising to powers case.

Plan of the rest of the talk

1. Statement of the additive version of Ax's theorem.
2. Proof.
3. Discussion of some other cases and the Drinfeld modules situation.

Set-up

- Let us fix a prime number p and let $\mathbb{F}_p[\![\mathrm{Fr}]\!]$ denote the ring of additive power series
$$\sum_{i=0}^{\infty} c_i X^{p^i}$$
with composition. It is commutative.
- Let $\mathbb{F}_p[\mathrm{Fr}]$ be a subring of $\mathbb{F}_p[\![\mathrm{Fr}]\!]$ consisting of additive polynomials. Any ring of characteristic p is also an $\mathbb{F}_p[\mathrm{Fr}]$-module, where X acts as Frobenius.
- Let us fix $F \in \mathbb{F}_p[\![\mathrm{Fr}]\!]$, which has algebraic degree over $\mathbb{F}_p[\mathrm{Fr}]$ greater than n.
- Let t be a variable. The power series F converges on $t\mathbb{F}_p[\![t]\!]$ in the complete non Archimedean field $\mathbb{F}_p((t))$.

The statement

Theorem (Schanuel Conjecture for additive power series)

Let $x_1, \ldots, x_n \in t\mathbb{F}_p[\![t]\!]$. *Assume* $x_1, \ldots, x_n$ *are linearly independent over* $\mathbb{F}_p[\mathrm{Fr}]$ *and*

$$g := (x_1, \ldots, x_n, F(x_1), \ldots, F(x_n)).$$

Then

$$\mathrm{trdeg}_{\mathbb{F}_p(t)}(g) \geqslant n.$$

Outline of the proof

Let us assume that $\mathrm{trdeg}_{\mathbb{F}_p}(g) \leqslant n$ and we want to conclude that $x_1, \ldots, x_n$ are $\mathbb{F}_p[\mathrm{Fr}]$-dependent, i.e. $(x_1, \ldots, x_n) \in N$, where N is a proper algebraic subgroup of $\mathbb{G}_{\mathrm{a}}^n$ over $\mathbb{F}_p$. We proceed as follows:

1. Find (higher) differential forms vanishing on g,
2. Find an additive power series vanishing on g in a certain sense,
3. Find proper algebraic subgroup of $\mathbb{G}_{\mathrm{a}}^{2n}$ over $\mathbb{F}_p$ containing g,
4. Using non-algebraicity of F, find N.

How a power series may vanish

- A power series is a limit of a Cauchy sequence from $\mathbb{F}_p[X]$ in the topology given by $(X^m\mathbb{F}_p[X])_m$.
- However an additive power series $\sum c_i X^{p^i}$ is also a limit of

$$(\sum_{i=0}^{m} c_i X^{p^i})_m$$

in the topology given by $(\mathbb{F}_p[X]^{p^m})_m$.
- Such a topology may be considered on any $\mathbb{F}_p$-algebra T. Let $\phi : \mathbb{F}_p[X] \to T$ be a $\mathbb{F}_p$-algebra homomorphism.

Definition

Let $h = \lim h_m$ (second sense!) be an additive power series. We say that h vanishes on T if for each m, we have $\phi(h_m) \in T^{p^{m+1}}$.

Power series vanishing on $\mathbb{F}_p((t))$

Let us set $\bar{X} = (X_1, \ldots, X_n)$, $\bar{Y} = (Y_1, \ldots, Y_n)$ and we have

$$\mathbb{F}_p[\bar{X}, \bar{Y}] \ni W \mapsto W(g) \in \mathbb{F}_p((t))$$

Example

Since $g = (x_1, \ldots, x_n, F(x_1), \ldots, F(x_n))$, each series $Y_i - F(X_i)$ vanishes on $\mathbb{F}_p((t))$.

Linear dependence of differential forms

A usage of the Lie derivative is crucial in Ax's proof to obtain C-dependence of certain differential forms. Here we use:

Proposition

Let $\mathbb{F}_p \subseteq L \subseteq K$ be a tower of fields and $\mathbb{F}_p[\bar{X}, \bar{Y}] \to L$ an $\mathbb{F}_p$-algebra homomorphism. Assume $f_1, \ldots, f_n$ are additive power series in variables $\bar{X}, \bar{Y}$ and:

- $K^{p^\infty} = \mathbb{F}_p$,
- $\mathrm{trdeg}_{\mathbb{F}_p}(L) \leqslant n$,
- $L \not\subseteq K^p$,
- $f_1, \ldots, f_n$ vanish on K.

Then $\mathrm{d}(f_1), \ldots, \mathrm{d}(f_n)$ are $\mathbb{F}_p$-dependent in $\Omega_{L/\mathbb{F}_p}$. An appropriate version for higher forms is also true.

Vanishing additive power series

We set $L = \mathbb{F}_p(g)$ and $K = \mathbb{F}_p((t))$.

Proposition

There is a non-zero tuple $h_1, \ldots, h_n$ of additive power series s. t.

$$h := h_1 \circ (Y_1 - F(X_1)) + \ldots + h_n \circ (Y_n - F(X_n))$$

vanishes on L.

Idea of the proof

By the linear dependence result we get $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_p$ such that

$$\alpha_1 \mathrm{d}(F(x_1) - x_1) + \ldots + \alpha_n \mathrm{d}(F(x_n) - x_n) = 0 \in \Omega_{L/\mathbb{F}_p}.$$

Each α_i is (almost) the constant term of h_i. Other coefficients are obtained using higher differential forms.

Vanishing power series and formal subvarieties

Let $A = \mathbb{G}_a^{2n}$ and $W \subseteq A$ be an algebraic subvariety containing 0 as a smooth point. The series h may vanish on W in two ways:

"Strong" vanishing

Using the restriction map $C[\bar{X}, \bar{Y}] \to C(W)$ it makes sense to say that h vanishes on $C(W)$.

Vanishing on $\widehat{W}$

Let $\widehat{\mathcal{O}}_W = \varprojlim(\mathcal{O}_{W,0}/\mathfrak{m}_{W,0}^{p^{m+1}})$ and $\pi : \widehat{\mathcal{O}}_A \to \widehat{\mathcal{O}}_W$ be the restriction map. We say that h vanishes on $\widehat{W}$ if $\pi(h) = 0$.

Easy to see that strong vanishing implies vanishing on $\widehat{W}$.

Algebraic subgroup

Let V be the locus of g over $\mathbb{F}_p^{\mathrm{alg}}$ and H be the coset generated by V (Chevalley-Zilber).

- From the form of g, H is an algebraic subgroup over $\mathbb{F}_p$.
- h vanishes on $\mathbb{F}_p(g)$.
- h vanishes on $\widehat{V}$ (perhaps after translating V).
- h vanishes on $\widehat{H}$.

Main point behind

Let $\mathcal{H}$ be a formal subgroup ("zeroes of power series") of A. Then

$$\widehat{V} \subseteq \mathcal{H} \implies \widehat{\langle V \rangle} \subseteq \mathcal{H}.$$

Conclusion of the proof I

We have:

- $g = (x, F(x))$.
- $g \in H(\mathbb{F}_p((t)))$.
- h vanishes on $\widehat{H}$.

We want:

- A proper algebraic $N < \mathbb{G}_{\mathrm{a}}^n$ such that $x \in N(\mathbb{F}_p((t)))$.

Conclusion of the proof II

- We know that h vanishes on $\widehat{H}$ and

$$h := h_1 \circ (Y_1 - F(X_1)) + \ldots + h_n \circ (Y_n - F(X_n)).$$

- If the projection of H to $\mathbb{G}_a^n$ is proper we are done. Assume not. Then we get $M = (t_{ij}) \in M_n(\mathbb{F}_p[\mathrm{Fr}])$ such that

$$h_1 \circ t_1 + \ldots + h_n \circ t_n = h_k \circ F$$

for each $1 \leqslant k \leqslant n$, so F is a characteristic value of M.

- By Cayley-Hamilton, F is algebraic over $\mathbb{F}_p[\mathrm{Fr}]$ of degree $\leqslant n$.

Coefficients other than $\mathbb{F}_p$

- If we replace $\mathbb{F}_p$ with an arbitrary perfect field C, then the proof goes smoothly till the very last sentence – the usage of Cayley-Hamilton.
- If $C \ncong \mathbb{F}_p$, then $C[\mathrm{Fr}]$ is not commutative, so Cayley-Hamilton can not be applied. Proceeding "by hand" one can still obtain that F is "algebraic of degree at most n" over $C[\mathrm{Fr}]$, i.e. there are $\alpha_{i,j} \in C[\mathrm{Fr}]$ such that

$$\alpha_{0,n}^{\pm 1} \circ F \circ \alpha_{1,n}^{\pm 1} \circ F \circ \ldots \circ F \circ \alpha_{n,n}^{\pm 1} + \ldots + \alpha_{0,1}^{\pm 1} \circ F \circ \alpha_{1,1}^{\pm 1} + \alpha_{0,0}^{\pm 1} = 0.$$

Drinfeld modules

Definition

Let $A = \mathbb{F}_p[t]$ and $K = \mathbb{F}_p((\frac{1}{t}))$. A Drinfeld A-module (over K) is a (nontrivial) homomorphism

$$\varphi : A \longrightarrow \mathrm{End}_K(\mathbb{G}_{\mathrm{a}}) = K[\mathrm{Fr}].$$

- An additive power series over K is attached to each Drinfeld module, which "formally trivializes" it. This series plays the role of the exponential (Weierstrass) map.
- Many transcendence results were obtained for such "exponential maps". A couple of them are on the next slide.

Carlitz exponential and logarithm

- The Carlitz module is a Drinfeld module where

$$\varphi(t) = tX + X^p$$

  and the corresponding "exponential map" is denoted $\exp_C$.

- It has the following form

$$\exp_C = X + \sum_{i=1}^{\infty} \frac{X^{p^i}}{(t^{p^i} - t)(t^{p^i} - t^p)\dots(t^{p^i} - t^{p^{i-1}})}$$

- Denis obtained some Schanuel-type results for $\exp_C$.
- Papanikolas proved a Carlitz version of the (still open) conjecture on algebraic independence of logarithms of algebraic numbers.

Our case vs Drinfeld modules

- The power series considered here do not fit in the Drinfeld modules framework, since they have constant coefficients, i.e. there is no transcendental element present.
- The Carlitz exponential $\exp_C$ is "algebraic" in our terminology since it satisfies the following functional equation:

$$\exp_C \circ \theta X = \theta X \circ \exp_C + X^p \circ \exp_C .$$

- A Drinfeld (or even Carlitz) version of the *full* Schanuel conjecture is still open.

Non-additive power series

- Our transcendence statement was obtained for certain additive power series, i.e. for sufficiently non-algebraic formal maps between vector groups.
- It is natural to extend this result to the context of an arbitrary "sufficiently non-algebraic" formal map between algebraic groups.
- An example of such a map is a formal isomorphism between an ordinary elliptic curve and the multiplicative group.
- This is work in progress.