Unlikely formal intersections

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June 14, 2012

Ax's theorem about formal intersections

Setting

Let us fix:

- G, an algebraic group over $\mathbb{C}$;
- $\mathcal{A}$, a complex analytic subgroup of G;
- V, an irreducible algebraic subvariety of G containing 1;
- $\mathcal{W}$, an analytic subvariety of $\mathcal{A}$ and V.

Ax's Theorem (Amer. J. Math, 1972)

If $\mathcal{W}$ is Zariski dense in V, then there is $\mathcal{B}$, a complex analytic subgroup of G containing V and $\mathcal{A}$ such that

$$\dim(\mathcal{B}) \leqslant \dim(\mathcal{A}) + \dim(V) - \dim(\mathcal{W}).$$

The only reason for unlikely intersections

- Assume that the intersection $\mathcal{W}$ is unlikely, i.e. for some $d > 0$ we have

$$\dim(\mathcal{W}) = \dim(\mathcal{A}) + \dim(V) - \dim(G) + d.$$

- Ax's theorem: there is $\mathcal{B}$ as above of codimension at least d.
- Inside $\mathcal{B}$ the intersection is not unlikely anymore.
- Ax's theorem looks similar to CIT. A differential version of Ax's theorem (closely related to this one) implies weak CIT.

Question

Is there a direct proof of

$$\text{Ax's theorem} \Rightarrow \text{weak CIT}$$

not going through a (logical) compactness argument?

Formal version

Setting (C a field of characteristic 0)

- Let G be an algebraic group over C;
- Let $\mathcal{A}$ be a formal subgroup of $\widehat{G}$;
- Let V be an irreducible algebraic subvariety of G containing 1;
- Let $\mathcal{W}$ be a formal subvariety of $\mathcal{A}$ and $\widehat{V}$.

Ax's Theorem, formal version

If $\mathcal{W}$ is Zariski dense in V, then there is $\mathcal{B}$, a formal subgroup of $\widehat{G}$ containing $\widehat{V}$ and $\mathcal{A}$ such that

$$\dim(\mathcal{B}) \leqslant \dim(\mathcal{A}) + \dim(V) - \dim(\mathcal{W}).$$

Question

Is the above theorem true for C of positive characteristic?

“Formal intersection Ax” implies “Ax-Schanuel”

- Let x be an n-tuple of formal power series in several variables over C without constant terms, linearly independent over $\mathbb{Q}$;
- Let $G = \mathbb{G}_{\mathrm{a}}^n \times \mathbb{G}_{\mathrm{m}}^n$;
- Let V be the algebraic locus of $(x, \exp(x))$ over C;
- Let $\mathcal{A}$ be the graph of $\exp : \widehat{\mathbb{G}_{\mathrm{a}}}^n \to \widehat{\mathbb{G}_{\mathrm{m}}}^n$;
- Let $\mathcal{W}$ be the formal locus of $(x, \exp(x))$ over C.

Formal group $\mathcal{B}$ given by Ax's theorem coincides with $\widehat{G}$ here, so

$$2n \leqslant \dim(\mathcal{A}) + \dim(V) - \dim(\mathcal{W}) = n + \mathrm{trdeg}_C(x, \exp(x)) - \mathrm{rk}(J_x),$$

$$\mathrm{trdeg}_C(x, \exp(x)) \geqslant n + \mathrm{rk}(J_x).$$

The same proof works for A semi-abelian.

"Ax-Lindemann-Weierstrass"

Let A be a semi-abelian variety of dimension n. Similarly as above, Ax's theorem easily implies the following.

Theorem

Assume that

- $Y \subseteq A$ *is an algebraic subvariety;*
- $X \subseteq \mathbb{G}_a^n$ *is a maximal algebraic subvariety such that* $\exp(X) \subseteq Y$*;*
- $Y' := \exp(X)^{\mathrm{Zar}}$.

Then Y' *is an algebraic subgroup of* A *and* $X = \mathrm{Lie}(Y')$.

Dense formal subvarieties (characteristic 0)

The above applications suggested me a more general statement which also looks better for positive characteristic generalizations.

Setting

- Let A be a commutative algebraic group over C;
- Let V be an algebraic variety over C and $v \in V(C)$;
- Let $\mathcal{F} : \widehat{V} \to \widehat{A}$ be a "special" formal map;
- Let $\mathcal{W}$ be a formal subvariety of $\widehat{V}$ such that $\mathcal{F}(\mathcal{W}) = 0$.

Theorem (easily following from Ax's proof)

If $\mathcal{W}$ is Zariski dense in V, then there is $\mathcal{A}$, a formal subgroup of $\widehat{A}$ such that $\mathcal{F}(\widehat{V}) \subseteq \mathcal{A}$ and

$$\dim(\mathcal{A}) \leqslant \dim(V) - \dim(\mathcal{W}).$$

Remarks

- A continuous map between Hausdorff spaces which is constant on a dense set is constant everywhere.
- The same principle applies to an algebraic map between algebraic varieties and the Zariski topology.
- In the Ax's theorem situation the categories are mixed: a formal map is constant on a Zariski dense set. The theorem says that the above principle can be saved at the cost of quotiening out by a subgroup of a controlled dimension.

Special formal maps

I call a formal map $\mathcal{F} : \widehat{V} \to \widehat{G}$ "special" if it has certain properties of formal homomorphisms (even when V is not a group!)

Definition

$\mathcal{F}$ is special if it takes invariant differential forms on G into algebraic differential forms on V.

Example

$$\exp^*\left(\frac{\mathrm{d}X}{X}\right) = \frac{\exp(X)\mathrm{d}X}{\exp(X)} = \mathrm{d}X.$$

Formalizations of algebraic maps are special.
Formal homomorphisms are special.

Additive and multiplicative power series

Assume that $\mathrm{char}(C) = p > 0$. There is no exponential map anymore. But there are other interesting formal homomorphisms.

Example

- Additive power series.

$$\mathcal{F} : \widehat{\mathbb{G}_{\mathrm{a}}} \to \widehat{\mathbb{G}_{\mathrm{a}}}, \quad \mathcal{F} = \sum c_i X^{p^i}.$$

- Multiplicative power series. For $\gamma = \sum a_i p^i \in \mathbb{Z}_p$

$$\mathcal{F} : \widehat{\mathbb{G}_{\mathrm{m}}} \to \widehat{\mathbb{G}_{\mathrm{m}}}, \quad \mathcal{F} = X^{\gamma}.$$

  $\mathcal{F}$ corresponds to $\prod (X^{p^i} + 1)^{a_i} - 1$.

- A formal isomorphism between $\mathbb{G}_{\mathrm{m}}$ and an ordinary elliptic curve (defined over C^{alg}).

Towards positive characteristic Ax

Setting

Let us fix:

- C, a perfect field of characteristic $p > 0$;
- A, a commutative algebraic group over C;
- V, an algebraic variety over C and $v \in V(C)$;
- $\mathcal{F} : \widehat{V} \to \widehat{A}$, a formal map;
- $\mathcal{W}$, a formal subvariety of $\widehat{V}$ such that $\mathcal{F}(\mathcal{W}) = 0$.

I will describe a positive characteristic variant of Ax's theorem. Unfortunately, I have to put extra assumptions on the formal map $\mathcal{F}$ and the algebraic group A.

Limit maps

Definition (positive characteristic)

I call a formal map $\mathcal{F} : \widehat{V} \to \widehat{A}$ an *A-limit* if there is a sequence of rational maps $(f_m : V \to A)_m$ such that $f_m(v) = 0$ and $\mathcal{F}$ is the limit of $(f_m)_m$ in a certain strong sense,
i.e.

$$f_{m+1} - f_m \in A((\mathcal{O}_{V,v})^{p^{m+1}}).$$

Example

- For $\mathcal{F} = \sum c_i X^{p^i}$, $\mathcal{F}$ is the limit of $(\sum^m c_i X^{p^i})_m$.
- For $\mathcal{F} = X^{\gamma}$, $\mathcal{F}$ is the limit of $(X^{\sum^m a_i p^i})_m$, where

$$\gamma = \sum a_i p^i \in \mathbb{Z}_p.$$

Questions about limit maps

1. Any A-limit map $\mathcal{F} : \widehat{V} \to \widehat{A}$ is special (in the proper sense involving higher differential forms).
2. The converse is true for A affine.
3. More generally, the converse is true for A such that

$$\ker(H^1(K^p, A) \to H^1(K, A)) = 0,$$

where $C \subseteq K$ is a finitely generated field extension.
4. Is the above map on cohomology always injective?
5. Formal homomorphisms are special. Are they A-limits?

Integrable groups

Definition

Let D be a 1-dimensional algebraic group. I call D integrable if for any $c \in C$ there is an algebraic endomorphism $\varphi : D \to D$ such that φ^* (the map induced on differential forms) is the multiplication by c.

Example

The following algebraic groups are integrable:

- Any 1-dimensional D over $\mathbb{F}_p$.
- $\mathbb{G}_{\mathrm{a}}$ over any C.

Main Theorem

Setting

Let us fix:

- D, an integrable algebraic group;
- $A := D^n$;
- V, an algebraic variety over C and $v \in V(C)$;
- $\mathcal{F} : \widehat{V} \to \widehat{A}$, an A-limit map;
- $\mathcal{W}$, a formal subvariety of $\widehat{V}$ such that $\mathcal{F}(\mathcal{W}) = 0$.

Theorem (K.)

If $\mathcal{W}$ is Zariski dense in V, then there is $\mathcal{A}$, a formal subgroup of $\widehat{A}$ such that $\mathcal{F}(\widehat{V}) \subseteq \mathcal{A}$ and

$$\dim(\mathcal{A}) \leqslant \dim(V) - \dim(\mathcal{W}).$$

Remarks

- For $D = \mathbb{G}_{\mathrm{a}}$ or $D = \mathbb{G}_{\mathrm{m}}$, we can replace "$A$-limit" with "special".
- In the characteristic 0 case any commutative algebraic group is isomorphic to $\mathbb{G}_{\mathrm{a}}^n$ as a formal group, so our theorem may be thought of as a generalization of Ax's theorem to the arbitrary characteristic case. However it is not satisfactory, since it does not fully answer the original question.

Application I: Additive transcendence

We assume that:

- F is an additive power series over $\mathbb{F}_p$ which is transcendental over the ring of additive polynomials.
- $x_1, \ldots, x_n$ are power series over $\mathbb{F}_p$ without a constant term.

Theorem (K.)

If $x_1, \ldots, x_n$ are linearly independent over the ring of additive polynomials, then

$$\mathrm{trdeg}_{\mathbb{F}_p}(x_1, F(x_1), \ldots, x_n, F(x_n)) \geqslant n + 1.$$

Application II: Multiplicative transcendence

We assume that:

- $\gamma \in \mathbb{Z}_p$ is transcendental over $\mathbb{Q}$.
- F is the multiplicative power series corresponding to γ.
- $x_1, \ldots, x_n$ are power series over $\mathbb{F}_p$ with constant term 1.

Theorem (K.)

If $x_1, \ldots, x_n$ are multiplicatively independent, then

$$\mathrm{trdeg}_{\mathbb{F}_p}(x_1, F(x_1), \ldots, x_n, F(x_n)) \geqslant n + 1.$$

Application III: Ax-Lindemann-Weierstrass for $p > 0$

Let us fix

- D, an integrable algebraic group;
- γ, a formal endomorphism of D which is not algebraic;
- Y, an algebraic subvariety of D^n containing 0;

Theorem (K. proof to be checked)

If X is an algebraic subvariety containing 0 and maximal such $\gamma(\widehat{X}) \subseteq \widehat{Y}$ and $Y' := \exp(X)^{\mathrm{Zar}}$, then both X and Y' are algebraic subgroups of D^n.