Jet operators on fields

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Abstract

We classify jet operators in the sense of Buium [Bu] on a field of an arbitrary characteristic. In the positive characteristic we obtain a class of "new" operators: derivations of the Frobenius map.

1 Introduction

In [Bu], Buium introduced jet operators on rings; they are natural generalizations of derivations and endomorphisms (more precisely, difference operators). In the same paper, Buium classifies jet operators on local domains of characteristic 0.

In this paper we classify jet operators on fields of arbitrary characteristic. In characteristic 0, our result is just a part of the Buium's result. In particular we get nothing new; all the jet operators are equivalent to difference operators or derivations. However in the positive characteristic, we obtain a new class of operators: derivations of (powers of) the Frobenius endomorphism. Some of the basic algebraic and logical properties of derivations of the Frobenius map were studied in [Kow1].

In the remaining of this introduction, we recall definitions from [Bu]. Throughout this paper K denotes an infinite field of characteristic $p \geq 0$. Since we are going to work over fields only, we restrict the Buium's definition to the context of fields.

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Definition 1.1 *A map $\delta : K \to K$ is a jet operator on K, if there exist polynomials $P, Q \in K[X, X', Y, Y']$ such that for any $a, b \in K$*

$$\delta(a+b) = P(a, \delta(a), b, \delta(b))$$

$$\delta(a \cdot b) = Q(a, \delta(a), b, \delta(b))$$

$$\delta(0) = 0, \quad \delta(1) = 0$$

and the structure

$$(K \times K, \oplus, *, \mathbf{0}, \mathbf{1})$$

is a commutative ring, where:

$$(a, a') \oplus (b, b') = (a+b, P(a, a', b, b'))$$

$$(a, a') * (b, b') = (a \cdot b, Q(a, a', b, b'))$$

$$\mathbf{0} = (0, 0), \ \mathbf{1} = (1, 0).$$

Examples of operators arise naturally in algebra and geometry. Any derivation of a ring K is an operator:

$$P(X, X', Y, Y') = X' + Y', \quad Q(X, X', Y, Y') = XY' + YX'.$$

The ring structure on $K \times K$ is that of the ring of dual numbers $K[X]/(X^2)$. A difference operator is an operator of the following type:

$$P(X, X', Y, Y') = X' + Y' \quad Q(X, X', Y, Y') = XY' + YX' + X'Y'.$$

It is easy to check that δ is a difference operator if and only if $\delta + \mathrm{id}$ is an endomorphism. The ring structure on $K \times K$ is isomorphic to the cartesian product structure.

It is worth to notice that a jet operator δ does not necessarily determine P and Q, e.g. the 0-map is both a derivation and a difference operator.
Also for a map $\delta : K \to K$ satisfying the "(P, Q)-rule" from the above definition, the ring condition is equivalent to the existence of a "(P, Q)-extension" of δ to a generic map on an extension of K. For the proof see [Bu, Lemma2] or [Kow, 2.1.7].

Let us introduce a new class of jet operators. A derivation of the n-th power of the Frobenius map is an operator δ on K such that:

$$\delta(x+y) = \delta(x) + \delta(y), \quad \delta(xy) = x^q\delta(y) + y^q\delta(x),$$

where $q = p^n$, $p = \mathrm{char}(K)$, $p > 0$. Note that an operator δ on a field of characteristic $p > 0$ is a derivation if and only if Fr^n composed with δ is a derivation of the n-th power of Frobenius.

Definition 1.2 *Two jet operators δ_0, δ_1 on K are equivalent, if there exist $\lambda \in K \setminus \{0\}$ and $F \in K[X]$ such that $F(0) = F(1) = 0$ and*

$$\delta_0(x) = \lambda\delta_1(x) + F(x)$$

for each $x \in K$.

It is clear from the definitions (since K is infinite!) that jet operators correspond to certain objects in algebraic geometry. By an *algebraic ring* $\mathbf{R} = (V, +, \cdot)$ over K, we mean an algebraic variety V over K, with morphisms $+, \cdot : V^2 \to V$ over K such that $V(K)$ becomes a ring. Equivalently $V(\bar{K})$ becomes a ring, since K is infinite (where $\bar{K}$ is the algebraic closure of K). We also sometimes say that the above $(+, \cdot)$ is an algebraic ring structure on V and write $\mathbf{R}(L)$ instead of $V(L)$ for a K-algebra L.
If $\mathbf{R} = (\mathbb{A}^2, \oplus, *)$ is the algebraic ring corresponding to a pair of polynomials (P, Q) as in Definition 1.1, then a jet operator on K gives a ring homomorphism section of the projection $\mathbf{R}(K) \to K$, i.e. a map of the form

$$K \ni x \mapsto (x, \delta(x)) \in \mathbf{R}(K).$$

For example the algebraic ring of dual numbers corresponds to derivations. If we have two algebraic rings $\mathbf{R}_1$, $\mathbf{R}_2$ as above, it is clear that a homomorphism between them of the form

$$\mathbf{R}_1 \ni (x, y) \mapsto (x, \lambda y + F(x)) \in \mathbf{R}_2,$$

for $\lambda \in K \setminus \{0\}$ and $F \in K[X]$ gives rise to an equivalence of the corresponding jet operators.

We take a closer look at the algebraic ring $\mathbf{R}_n$ corresponding to derivations of the n-th power of the Frobenius map. Let $q = p^n$ and $\mathbf{R}_0$ be the algebraic ring of dual numbers. Then the map:

$$(\mathrm{id}, \mathrm{Fr}^n) : \mathbf{R}_0 \to \mathbf{R}_n$$

is a homomorphism of ring, and actually $\mathbf{R}_n$ is obtained from $\mathbf{R}_0$ as a push-out of the algebraic ring structure of $\mathbf{R}_0$ by $(\mathrm{id}, \mathrm{Fr}^n)$.

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2 Classification Theorem

Theorem 2.1 *Suppose δ is a jet operator on an infinite field K. Then δ is equivalent to a difference operator or a derivation or a derivation of a power of the Frobenius map.*

Before the proof let us look at the possible types of some additive operators. Let δ be an endomorphism, F be an additive polynomial with coefficients in K such that $F(0) = 0, F(1) = 1$ (note that $\delta(1) = 1$) and $\delta' = \delta - F$. We will call such a δ' an F-difference operator. It is equivalent to a difference operator. Its corresponding algebraic ring structure on $\mathbb{A}^2_K$ comes from hitting the product structure by the morphism $(x, y) \mapsto (x, y + F(x))$. Therefore, the multiplication is given by:

$$(*) \qquad Q(X, X', Y, Y') = X'F(Y) + F(X)Y' + X'Y' + F(X)F(Y) - F(XY).$$

For $F(X) = X$ an F-difference operator is a usual difference operator.

Now let δ be a derivation of $x \mapsto x^q$, where $q = 1$ if $\mathrm{char}(K) = 0$ and $q = p^n$, if $\mathrm{char}(K) = p > 0$. So, δ is a derivation of the n-th power of Frobenius, provided $\mathrm{char}(K) \neq 0$. Let s be an additive polynomial over K such that $s(0) = s(1) = 0$ and let $\delta' = \delta + s$. We will call such a δ' an s-derivation of $x \mapsto x^q$. Here we have the following multiplicative rule

$$(**) \qquad Q(X, X', Y, Y') = X'Y^q + X^qY' - Y^q s(X) - X^q s(Y) + s(XY).$$

We need an easy, well-known fact on algebraic group structures on the affine line.

Lemma 2.2 *Let G be an algebraic group such that the underlying algebraic variety of G is $\mathbb{A}^1$ and 0 is the identity of G. Then $G = \mathbb{G}_a$.*

Proof Let ∇ be the comultiplication map. We need to show that $\nabla(X) = X \otimes 1 + 1 \otimes X$. Denote $X \otimes 1$ by X_1, $1 \otimes X$ by X_2 and $\nabla(X)$ by g.
Since the augmentation map is the 0-map, we get:

$$g(X_1, 0) = X_1, \quad g(0, X_2) = X_2.$$

Hence there are polynomials g_1, g_2 such that

$$g = X_1 + g_2X_2, \quad g = X_2 + g_1X_1.$$

But $X_1 + g_2X_2 = X_2 + g_1X_1$ implies $g_1 = g_2 = 1$, so $\nabla(X) = X_1 + X_2$. $\square$

Proof of Theorem 2.1

Step 1: δ is equivalent to an additive operator.

Let $\bar{K}$ denotes the algebraic closure of K.
Fix a pair of polynomials (P, Q) corresponding to δ and consider $\mathbf{R}$, the algebraic ring (with underlying variety being the affine plane) corresponding to (P, Q). Denote by G the algebraic group $\mathbf{R}_+$ (the additive group of $\mathbf{R}$). The map

$$K \ni x \mapsto (x, \delta(x)) \in \mathbf{R}(K)$$

gives $\mathbf{R}(K)$ a K-algebra structure. Therefore $G(K)$ is a vector space over K. In particular, if $\mathrm{char}(K) = p > 0$, then each nonzero element of $G(K)$ has order p.

The projection on the first coordinate gives an epimorphism of algebraic groups $F : G \to \mathbb{G}_a$. Let N denote the kernel of $G \to \mathbb{G}_a$. Since the underlying variety of N is $\{0\} \times \mathbb{A}^1$ and the identity of N is $(0,0)$, we get by 2.2 that $N = \{0\} \times \mathbb{G}_a$. One easily sees (as in the proof of [Se, Proposition 4.a), page 168]) that

$$P(X, X', Y, Y') = X' + Y' + f(X, Y)$$

for an $f \in H^2_{\mathrm{reg}}(\mathbb{G}_a, \mathbb{G}_a)$.
We know that G is an extension of $\mathbb{G}_a$ by $\mathbb{G}_a$. If $\mathrm{char}(K) = 0$, then any such extension splits (see [Se, Corollaire on p. 172]). If $\mathrm{char}(K) = p > 0$, then there are nontrivial extensions called Witt groups (see [Se, Proposition 8, p. 172]). But any nontrivial extension of $\mathbb{G}_a$ by $\mathbb{G}_a$ has $\bar{K}$-rational points of order bigger than p (e.g. Exercise 8(c) on p. 67 in [Wat] applies to any non-split extension). Since $G(K)$ is Zariski dense in $G(\bar{K})$, G would have K-rational points of order bigger than p, if G was a non-splitting extension of $\mathbb{G}_a$ by $\mathbb{G}_a$. Hence $G \to \mathbb{G}_a$ splits, so there is $w \in H^1_{\mathrm{reg}}(\mathbb{G}_a, \mathbb{G}_a)$ such that

$$f(X, Y) = w(X + Y) - w(X) - w(Y).$$

It is enough to show that $w \in K[X]$ (then $\delta + w$ is additive).
Any additive monomial in w does not change $w(X+Y) - w(X) - w(Y)$, so we can assume that w does not have such monomials. Write $w(X) = \sum_{i \leq n} \alpha_i X^i$.

Take any $i \leq n$ such that $\alpha_i \neq 0$. By our non-additivity assumption, p does not divide i. Therefore

$$f(X, Y) = i\alpha_i X^{i-1}Y + \text{other terms},$$

so $\alpha_i \in K$.

From now on we assume that $G = \mathbb{G}_a \times \mathbb{G}_a$.

Step 2: $Q(X, X', Y, Y')$ is either of the form $(*)$ or of the form $(**)$

We look closer at the axiom of associativity of multiplication in $\mathbf{R}$. For all $a, a', b, b', c, c' \in K$ we have:

$$(a, a') * [(b, b') * (c, c')] = [(a, a') * (b, b')] * (c, c') \tag{1}$$

By the obvious computation (1) yields:

$$Q(ab, Q(a, a', b, b'), c, c') = Q(a, a', bc, Q(b, b', c, c')) \tag{2}$$

Since K is infinite (2) implies the following equality of polynomials in variables X, X', Y, Y', Z, Z':

$$Q(XY, Q(X, X', Y, Y'), Z, Z') = Q(X, X', YZ, Q(Y, Y', Z, Z')) \tag{3}$$

Looking at X' in (3), we get:

$$\deg_{X'}(Q) = (\deg_{X'}(Q))^2,$$

so $\deg_{X'}(Q) \leq 1$.
Since

$$Q(X, X', Y, Y') = Q(Y, Y', X, X')$$

(by commutativity of $\mathbf{R}$), we get $\deg_{Y'}(Q) \leq 1$.
Considering the distributivity axiom in $\mathbf{R}$:

$$(a, a') * [(b, b') \oplus (c, c')] = (a, a') * (b, b') \oplus (a, a') * (c, c') \tag{4}$$

we get the polynomial equalities:

$$Q(X, X', Y + Z, Y' + Z') = Q(X, X', Y, Y') + Q(X, X', Z, Z')$$

$$Q(X+Y, X'+Y', Z, Z') = Q(X, X', Z, Z') + Q(Y, Y', Z, Z')$$

Hence Q, regarded as a function from $K^2 \times K^2$ into K, is biadditive. So the variables X and X' (and similarly Y, Y') are separated in Q.
Summarizing what we have got so far, Q is of the form:

$$Q(X, X', Y, Y') = X'F(Y) + Y'F(X) + \alpha X'Y' + G(X, Y), \qquad (5)$$

where F is additive and G is biadditive and symmetric.
Since $(0,0) * (y_0, y_1) = (0,0)$ for any $y_1, y_2 \in K$, we get:

$$0 = Q(0, 0, Y, Y') = 0F(Y) + Y'F(0) + \alpha 0 Y' + G(0, Y) = Y'F(0) + G(0, Y).$$

Hence $F(0) = 0$ and $G(0, Y) = G(Y, 0) = 0$.
Similarly using that $(1,0) * (y_0, y_1) = (y_0, y_1)$ we get:

$$Y' = Q(1, 0, Y, Y') = 0F(Y) + Y'F(1) + \alpha 0 Y' + G(1, Y) = Y'F(1) + G(1, Y).$$

Hence $F(1) = 1$ and $G(1, Y) = G(Y, 1) = 0$.
Multiplying δ by a scalar from K, we can assume that $\alpha = 0, 1$, which will correspond to either the ("Frobenius"-)differential or the difference case.

Case 1: $\alpha = 1$
Plugging $X = 0, X' = 1, Y' = Z' = 0$ in (3) we get:

$$Q(0, Q(0, 1, Y, 0), Z, 0) = Q(0, 1, YZ, Q(Y, 0, Z, 0))$$

Applying (5), we get $G(Y, Z) = F(Y)F(Z) - F(YZ)$.
Hence Q is of the form $(*)$. We know that $F(0) = 0, F(1) = 1$ and F is additive, so δ is an F-difference operator.

Case 2: $\alpha = 0$
Denote the multiplicative group of $\mathbf{R}$ by H. By (5), for any field extension $K \subseteq L$, we have:

$$H(L) = \{(a, b) \in L \times L \mid a \neq 0, \exists x \;\; bF(1/a) + xF(a) + G(a, 1/a) = 0\}$$

$$H(L) = \{(a, b) \in L \times L \mid aF(a) \neq 0\} = \{(a, b) \in L \times L \mid F(a) \neq 0\},$$

since $F(0) = 0$.
Therefore, $\dim(H) = 2$ and the projection homomorphism $G \to \mathbb{G}_m$ is onto. Hence, F vanishes at 0 only (over any field extension of K!), and since

$F(1) = 1$, $F(X) = X^q$ for q a power of p ($q = 1$, if $p = 0$).
Thus we obtain:

$$Q(X, X', Y, Y') = X'Y^q + Y'X^q + G(X, Y). \tag{6}$$

Denote the kernel of $H \to \mathbb{G}_m$ by N'. Since the underlying algebraic variety of N is $\{1\} \times \mathbb{A}^1$, we get by 2.2 that $N = \{1\} \times \mathbb{G}_a$.
By the classification of commutative linear groups (see [Wat, 9.3]) any extension of $\mathbb{G}_m$ by $\mathbb{G}_a$ splits. Hence, there is a homomorphism of the form

$$\mathbb{G}_m \ni x \mapsto (x, s(x)) \in H$$

for an $s \in \bar{K}(X)$.
Therefore

$$G(X, Y) = s(XY) - Y^q s(X) - X^q s(Y),$$

and since $G(X, 1) = 0$, we get $-X^q s(1) = 0$, so $s(1) = 0$. Similarly $G(0, Y) = 0$ implies $s(0) = 0$. If $s \in K[X]$, then Q is of the form $(**)$, so it is enough to show that $s \in K[X]$.
If s has a pole of order n at 0 (s is regular on $\mathbb{A}^1 \setminus \{0\}$), then $H(X) := G(X, X)$ has a pole of order $2n$ at 0. But H is a polynomial, so s has no poles.
Let $s(X) = \sum_i \alpha_i X^i$. Then the polynomial:

$$G(X, Y) = \sum_i \alpha_i (XY)^i - Y^q \sum_i \alpha_i X^i - X^q \sum_i \alpha_i Y^i$$

has coefficients in K. If $i \neq q$, then automatically $\alpha_i \in K$. For $i = q$, we get that $-\alpha_q X^q Y^q \in K[X, Y]$, so $\alpha_q \in K$, too. Hence $s \in K[X]$. $\square$

There is an essential difference between the case of characteristic 0 and the positive characteristic case. In characteristic 0, up to isomorphism, the algebraic ring structures on $\mathbb{A}^2$ are the product structure and the ring of dual numbers. These rings correspond to difference operators (the first case) and to derivations (the second case). So to each possible algebraic ring structure on $\mathbb{A}^2$ is associated a jet operator.

In characteristic $p > 0$, there are more algebraic ring structures on $\mathbb{A}^2$ given by rings of Witt vectors. However there are no operators corresponding to these algebraic rings, which was used in Step 1 of our proof. It was also noted in [Sc].

One could wonder, if derivations of Frobenius exist. But it is easy to check that δ is a derivation if and only if Fr composed with δ is a derivation of Frobenius. However, there are derivations of Frobenius, which can not be obtained in such a way [Kow1].

It should be remarked that a derivation of Fr^n is not equivalent to a derivation of Fr^m if $n \neq m$. To see this we need to know that that an equivalence of jet operators implies existence of a homomorphism of corresponding algebraic rings of the form $(x, y) \mapsto (x, cy + F(y))$. It is implicit in [Bu] and shown in [Kow]. The point is that this map is a homomorphism on the graph of δ and that a jet operator can be extended to a jet operator (corresponding to the same algebraic ring) on a bigger field such that the graph of the new jet operator is Zariski dense in the affine plane. If $\mathbf{R}_k$ is the algebraic ring corresponding to a derivation of Fr^k, then it is easy to see that for any infinite field K and $c \in K$, $F \in K[X]$ the map

$$\mathbf{R}_n(K) \ni (x, y) \mapsto (x, cy + F(y)) \in \mathbf{R}_m(K)$$

is not a homomorphism of rings.

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