Strongly minimal sets definable in expansions of RCF

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Peterzil's question

Question (Kobi Peterzil, Norwich Conference 2005)

Let $\mathfrak{M}$ be a strongly minimal structure definable in an o-minimal structure. Assume $\mathfrak{M}$ is not locally modular. Is an algebraically closed field interpretable in $\mathfrak{M}$?

Definability and Interpretability

Let $\mathfrak{M} = (M, f_i, R_j)$ and $\mathfrak{N} = (N, \dots)$ be structures.

Definition

- $\mathfrak{M}$ is definable in $\mathfrak{N}$ if M, f_i's and R_j's are definable in $\mathfrak{N}$.
- $\mathfrak{M}$ is inter-definable with $\mathfrak{N}$ if $\mathfrak{M}$ is definable in $\mathfrak{N}$ and $\mathfrak{N}$ is definable in $\mathfrak{M}$.
- We get interpretable or bi-interpretable, if we replace "definable" with "definable as the quotient by a definable equivalence relation".
- If $\mathfrak{M}$ is definable in $\mathfrak{N}$ and $M = N$, then $\mathfrak{M}$ is a reduct of $\mathfrak{N}$ or $\mathfrak{N}$ is an expansion of $\mathfrak{M}$.

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Strongly minimal structures

Definition

A structure M is strongly minimal if any b-definable set $X_b \subseteq M$ is either finite or cofinite uniformly in b.

Example

- $(\mathbb{C}, +, \cdot)$ is strongly minimal (as any algebraically closed field). $(\mathbb{C}, +, \cdot)$ is definable in $(\mathbb{R}, +, \cdot)$ which is o-minimal.
- If $(K, +, \cdot)$ is a field, then the vector space $(K, +, \cdot\lambda)_{\lambda \in K}$ is strongly minimal and it is a reduct of $(K, +, \cdot)$.
- A set with no structure is strongly minimal.

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Locally modular structures

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For us a locally-modular structure is a strongly minimal structure which is inter-definable with a vector space or has no structure.

Two equivalent "formal" definitions of local-modularity

- No 2-dimensional family of plane curves through a point.
- Any two algebraically closed sets are independent over their intersection.

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Zilber's conjecture

Zilber's dichotomy conjecture

A strongly minimal set is either locally-modular or interprets a field.

Theorem (Hrushovski)

- *There is a strongly minimal set which is not locally-modular and does not interpret even a group.*
- *There is a strongly minimal group which is not locally modular and does not interpret a field.*

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Positive results

Zilber's conjecture holds in:

- Zariski Geometries (Hrushovski-Zilber).
- Differentially closed fields (Hrushovski-Sokolovic).
- Separably closed fields (Hrushovski).
- Algebraically closed fields with a generic automorphism (Chatzidakis-Hrushovski-Peterzil).

Applications

Zilber's Dichotomy for the structures above yields diophantine consequences – Mordell-Lang, Manin-Mumford.

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Zilber's conjecture and Peterzil's Question

Informal Zilber's conjecture

Zilber's Dichotomy holds in structures with "geometric flavor".

Informal statement

O-minimal structures and their reducts have geometric flavor.

Reformulation of Peterzil's question

Does Zilber's Dichotomy hold in reducts of o-minimal structures?

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Reducts of an o-minimal field

Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, +, \cdot)$.

An accessible version of Peterzil's question

Let $\mathfrak{M}$ be a strongly minimal expansion of $(\mathbb{C}, +)$. Assume $\mathfrak{M}$ is definable in $\mathcal{R}$. Does $\mathfrak{M}$ satisfy Zilber's Dichotomy?

This version reduces to:

A reformulation

Assume $X \subset \mathbb{C}^2$ is definable in $\mathcal{R}$ and $\mathbb{C}_X := (\mathbb{C}, +, X)$ is strongly minimal and not locally modular. Does $\mathbb{C}_X$ interpret a field?

We give the positive answer when X is the graph of a function.

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Our theorem

Theorem (Hasson, K.)

Assume $f : \mathbb{C} \to \mathbb{C}$ is definable in $\mathcal{R}$ and $\mathbb{C}_f := (\mathbb{C}, +, f)$ is strongly minimal and not locally-modular. Then, there is $A \in \mathrm{GL}_2(\mathbb{R})$ such that $\mathbb{C}_{AfA^{-1}}$ is bi-interpretable with $(\mathbb{C}, +, \cdot)$.

Although our assumptions are much stronger than Kobi's, the conclusions are also stronger, since:

- We identify a definable field – complex field twisted by A.
- There is nothing more than the field structure on $\mathbb{C}_f$.
- AfA^{-1} is rational on a cofinite set.

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The idea of the proof

1. Using topological arguments show that f extends to a continuous ramified covering of the Riemann sphere.
2. Prove that for some special $a \in \mathbb{C}$

$$\det f'(a) = 0 \quad \Rightarrow \quad f'(a) = 0$$

   (a weak version of Cauchy-Riemann).
3. Using the theory of Lie groups, find an open $U \subseteq \mathbb{C}$ such that $f|_U$ is holomorphic.
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Frontier of a strongly minimal set is finite

The first step (based on a paper of Peterzil-Starchenko) is:

Fact

Let $X \subset \mathbb{C}^2$ be $\mathbb{C}_f$-definable and strongly minimal. Then $\mathrm{cl}(X) \setminus X$, called the frontier of X, is finite.

A few words about the proof.

Peterzil-Starchenko look how complex lines intersect with X. We do not have enough lines, so we use the sets

$$l_a^b = \mathrm{graph}(f(x+a)+b).$$

The main problem is to show that enough of these l_a^b meet X transversally, and in particular that enough of the curves l_a^b are smooth at all the intersection points with X.

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Without loss $f : S^2 \longrightarrow S^2$ is continuous and open

Let $S^2 = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere.
Using finiteness of the frontier (mostly with graph(f)), we show that f has all the topological properties of rational functions:

Fact

- *f is continuous outside a finite set F.*
- *Resetting, if needed, the values of f on F (to possibly $\infty \in S^2$), we can assume that $f : \mathbb{C} \rightarrow S^2$ is continuous.*
- *f is open.*
- *f is proper, i.e. continuously extends to $f : S^2 \rightarrow S^2$.*

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$f : S^2 \to S^2$ is a ramified covering

We use the following topological theorem:

Theorem

If f is as in our case, then f is a ramified covering, i.e. it is locally topologically equivalent to $z \mapsto z^k$ on $|z| \leqslant 1$ (k may vary).

Definition

1. If $k > 1$ at c, then c is a branch point of degree k (of f), e.g. 0 is a branch point of degree 3 of $g(z) = z^3 + 7$.
2. If c is a branch point, then $f(c)$ is a ramification point.

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Jacobian matrix of f

Remark

Since f is definable in an o-minimal structure, f is C^1 on a codimension 1 subset of $\mathbb{C}$.

Definition

Let $f'(c)$ denote the Jacobian matrix of f at c (if defined). It is an element of $M_2(\mathbb{R})$.

Our aim

We want to show that f is holomorphic on some open $U \subseteq \mathbb{C}$, i.e. for each $c \in U$, $f'(c) \in M_1(\mathbb{C})$ ($M_1(\mathbb{C}) \hookrightarrow M_2(\mathbb{R})$).

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We want to show that f is holomorphic on some open $U \subseteq \mathbb{C}$, i.e. for each $c \in U$, $f'(c) \in M_1(\mathbb{C})$ ($M_1(\mathbb{C}) \hookrightarrow M_2(\mathbb{R})$).

Jacobian matrix of f

Remark

Since f is definable in an o-minimal structure, f is C^1 on a codimension 1 subset of $\mathbb{C}$.

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Jacobian matrix vanishes at branch points

Fact (weak Cauchy-Riemann)

If f is C^1 at c and c is a branch point, then $f'(c) = 0$.

Idea of the proof.

Let $f = (f_1, f_2)$. We can assume $f(c) = 0$. It is enough to show that for almost all directions $\alpha \in S^1$,

$$\frac{\partial f_i}{\partial \alpha}(c) = 0, \quad i = 1, 2.$$

Since f is equivalent locally at c to $z \mapsto z^k$ and $k > 1$, $f^{-1}([-1, 1]) \setminus \{c\}$ has $2k$ connected components X_j.
Since $f_2(X_j) = 0$, it is enough (for f_2) to take $\alpha \neq \alpha_j$, where

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Fact

1. *We can assume f is not 1-to-1.*
2. *There is a branch point of f.*

Proof.

1. If f is 1-1 (e.g. when $f(x) = 1/x$), we replace f with $f(x+1) - f(x)$.
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There is a $\mathbb{C}_f$-definable $g : S^2 \to S^2$ having a C^1 branch point.

Idea of the proof.

- By the theory of local degrees (winding numbers), we can control the way branch points move in families.
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Multiplication of Jacobian matrices

Our aim again

- We want to show that for some open $U \subseteq \mathbb{C}$, we have:

$$f'(U) \subseteq \mathrm{GL}_1(\mathbb{C}).$$

  So, $f'(U)$ is a subset of a 2-dim. Lie subgroup of $\mathrm{GL}_2(\mathbb{R})$.

- In particular, for any $U_1, \ldots, U_n \subseteq U$, we should have

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There are $U_1, \ldots, U_n \subseteq \mathbb{C}$ open such that f is C^1 on each U_i and $f'(U_1) \cdot \ldots \cdot f'(U_n) \subseteq f'(\mathbb{C})$, so $\dim(f'(U_1) \cdot \ldots \cdot f'(U_n)) \leqslant 2$.

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Consider $f_a^b(x) = f(a + f(x)) - f(x + b)$. Then, for small enough $|a|, |b|$, f_a^b has a C^1 branch point c_a^b. Hence $(f_a^b)'(c_a^b) = 0$, so:

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Take $U_1 = \text{locus}(a + f(c_a^b))$, $U_2 := \text{locus}(c_a^b)$ (for generic a, b). It works for $n = 2$. For $n > 2$, we take a more complicated f_a^b. □

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There is a local Lie subgroup of $GL_2(\mathbb{R})$ around

Definition

For a Lie group G, $A \subset G$ is a local Lie subgroup, if there is a relatively open $B \subset A$ such that $1 \in B$, $B = B^{-1}$ and $B \cdot B \subseteq A$.

Taking $n = 9$ in the last fact we obtain:

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There is an open $U \subseteq \mathbb{C}$ such that $f'(U)$ is a subset of a local Lie subgroup $A \subset GL_2(\mathbb{R})$ and $\dim A \leqslant 2$.

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Definition

For a Lie group G, a virtual Lie subgroup of G is a smooth injective homomorphism of Lie groups $\phi : H \to G$.

Virtual Lie subgroups of G correspond exactly to Lie subalgebras of $\mathrm{Lie}(G)$ and the following is well-known:

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If A is a local Lie subgroup of G, then there is a virtual Lie subgroup $\phi : H \to G$ such that $\dim H = \dim A$ and $\phi(H) \cap A$ is open in A.

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The image of a virtual Lie subgroup need not be a Lie subgroup as the "non-commutative torus" example shows.

Example

Let a be an irrational number, $T = S^1 \times S^1$ a 2-dimensional torus and take:

$$\mathbb{R} \ni r \mapsto \phi(r) = (r, ar) + \mathbb{Z}^2 \in \mathbb{R}^2/\mathbb{Z}^2 = T$$

Then $\phi(\mathbb{R})$ is dense in T, so it is not a Lie subgroup.
The quotient $T/\phi(\mathbb{R})$ is called a non-commutative torus.

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But in our case we still obtain:

Fact

There is a solvable Lie subgroup $\bar{H} < \mathrm{GL}_2(\mathbb{R})$ containing $f'(U)$.

Proof.

- We have $f : H \to \mathrm{GL}_2(\mathbb{R})$ and $\dim H \leqslant 2$, hence H is solvable.
- Therefore $f(H)$ is solvable.
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Fact

$f'(U)$ is contained in a conjugate of $\mathrm{GL}_1(\mathbb{C})$.

Proof.

- $f'(U)$ is contained in a solvable Lie subgroup $\bar{H}$.
- By classification of such, $\bar{H}$ (possibly after conjugation) is a subgroup of the triangular group or $\mathrm{GL}_1(\mathbb{C})$.
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Remark

This how we find the matrix A from the statement of our theorem. It is the conjugation matrix above.

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The fact that for $a \in U$, $f'(a) \in \mathrm{GL}_1(\mathbb{C})$ means exactly that f satisfies Cauchy-Riemann at a, so f is holomorphic at a. □

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If U is dense in $\mathbb{C}$, we can easily show that f is rational and we are done. But we do not know it at this stage.

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Group configuration

Fact

There is a field interpretable in $\mathbb{C}_f$.

Proof.

- Take U such that f is holomorphic on U.
- Then, for $c \in U$, $f'(c) = 0$ implies c is a branch point.
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There is a field interpretable in $\mathbb{C}_f$.

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- Let $\mathbb{K}$ be a field interpretable in $\mathbb{C}_f$. By a result of Peterzil-Starchenko and Hrushovski's internality theory, $\mathbb{K}$ is bi-interpretable with $\mathbb{C}_f$.
- Hence, $(\mathbb{C},+)$ is a 1-dimensional $\mathbb{K}$-algebraic group.
- Since $(\mathbb{C},+)$ is torsion-free, it is $\mathbb{C}_f$-definably isomorphic to $\mathbb{G}_a(\mathbb{K})$.
- Using the above isomorphism, we get a $\mathbb{C}_f$-definable operation $\star : \mathbb{C}^2 \to \mathbb{C}$ such that $(\mathbb{C},+,\star)$ is a field.
- Then, it is easy to find $A \in \mathrm{GL}_2(\mathbb{R})$ such that

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Let $\mathcal{R}$ be an o-minimal expansion of an arbitrary real closed field.

Remark

The proof of our theorem generalizes from $\mathbb{C}$ to any $\mathcal{K} = \mathcal{R}[i]$.

About the proof

- The theory of winding numbers, differentiable/analityc manifolds etc. was developed in this context by Berarducci, Otero, Peterzil, Pillay, Starchenko and others.
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Not much of o-minimality was used in the proof.

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- Can we assume that $\mathcal{R}$ is e.g. just weakly o-minimal?
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Question

Can we replace $(\mathbb{C}, +, f)$ with any strongly minimal expansion of $(\mathbb{C}, +)$ definable in $\mathcal{R}$?

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We know it is enough to consider $\mathbb{C}_X = (\mathbb{C}, +, X)$ with a relation (so "multi-function") X replacing f. We do not know if our proof still works. It should be still possible to prove the finiteness of frontier of strongly minimal $\mathbb{C}_X$-definable subsets of $\mathbb{C}^2$.

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Open Question

Let $(A, +)$ be a strongly minimal group which is not locally modular. Is there a definable function $f : A \to A$ such that $(A, +, f)$ is not locally-modular?

Remarks

- Positive answer to the above question extends our theorem to any strongly minimal expansion of $(\mathbb{C}, +)$.
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Other algebraic groups

Remark

Most likely our argument still works when we replace $(\mathbb{C}, +)$ with another one-dimensional algebraic group, i.e. the multiplicative group or an elliptic curve.