Stable groups and algebraic groups

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Abstract

We show that a certain property of some type-definable subgroups of superstable groups with finite U-rank is closely related to the Mordell-Lang conjecture. We discuss this property in the context of algebraic groups.

Let G be a stable, saturated group and p a strong type (over $\emptyset$) of an element of G. We denote by $\langle p \rangle$ the smallest type-definable (over $acl(\emptyset)$) subgroup of G containing p^G (= the set of all realizations of p in G). The following question arises: how to construct generic types of $\langle p \rangle$? By Newelski's theorem ([Ne]) all generic types of $\langle p \rangle$ are limits of sequences of powers of p, where power is taken in the sense of independent product of strong types. Recall that if p, q are strong types of elements of G, then the independent product $p * q$ of p, q is defined as $st(a \cdot b)$, where $a \models p$, $b \models q$ and $a \perp b$, also $p^{-1} = st(a^{-1})$, $p^n = p * \ldots * p$ (n times). But which sequences of powers converge to generic types? It is natural to conjecture that every convergent subsequence of the sequence $(p^n)_{n<\omega}$ converges to a generic type of $\langle p \rangle$. We call this statement the 1-step conjecture. Hrushovski gave a counterexample of Morley rank ω to the 1-step conjecture, but the following statement is true.

Double step theorem ([Ne]) *Suppose q is a limit of subsequence of (p^n) and r is a limit of subsequence of (q^n). Then r is a generic type of $\langle p \rangle$.*

It is easy to see from Newelski's proof that the subsequence (q^{n_i}) converges to generic r even in the following strong sense: for every finite set of formulas

$\triangle$, $q^{n_k}|_\triangle = r|_\triangle$ for k big enough (here $r|_\triangle$ is the set of all $\triangle$ -formulas in p in the sense of [Pi]). This remark implies that for groups of finite U-rank for N large enough, q^N is already a generic of $\langle p \rangle$, so we do not need 2 steps to get generic types, but rather one and a half. This suggests that for groups of finite U-rank 1 step may be enough. However Anand Pillay has pointed out to me a counterexample to the 1-step conjecture, which has Morley rank 2. He and Felipe Voloch also pointed out to me a proof of the 1-step conjecture for algebraic groups in characteristic 0. We will present these results in Section 3.
I would like to thank my advisor Ludomir Newelski for suggesting to me this topic, carefully reading this paper and correcting mistakes. I would like also to thank Anand Pillay for pointing out to me the results of Section 3.

1 Groups of finite U-rank

In this section we prove that the 1-step conjecture for groups of finite U-rank can be reduced to the case of groups generated by an algebraic type. We also formulate a generalization of the 1-step conjecture to the case of a group generated by a finite set of types. In section 2 it will turn out that in the case of semi-abelian varieties in characteristic 0 this generalization is a restatement of Mordell-Lang conjecture.

We work in a monster model G and S denotes the set of strong types (over $\emptyset$) of elements of G. If $p \in S$, then $Stab(p)$ denotes the left stabilizer of p, namely $Stab(p) = \{g \in G : \text{for } a \models p|g,\ g \cdot a \models p|g\}$. The right stabilizer of p is denoted by $Stab_R(p)$. Both stabilizers are type-definable in G. $R(p)$ denotes the sequence $(R_\triangle(p))_\triangle$ of local ranks of p indexed by finite sets $\triangle$ of formulas invariant under translation. Recall that the type p is a generic type of G iff $R(p) = R(G)$. In the following lemma we list simple but useful properties of generic types and stabilizers.

Lemma 1.1 *Let $p, q \in S$.*
i) $Stab(p) = Stab_R(p^{-1})$, $R(p * q) \geq R(p), R(q)$.
ii) If q=st(a), then $a \in Stab(p)$ iff $q * p = p$.
iii) If X is a right coset of Stab(p) containing p^G, then Stab(p) is connected, X is type-definable over $acl(\emptyset)$ and p is the generic type of X.
iv)$R(Stab(p)) \leq R(p)$.
v) If p is the generic type of a right coset of a connected group H, then

$H = Stab(p) = \langle p * p^{-1} \rangle$.
vi) If $\langle p * p^{-1} \rangle \subseteq H$*, where* H *is a type-definable over* $acl(\emptyset)$ *subgroup of* G*, then* p^G *is contained in a single coset of* H *(necessarily* $acl(\emptyset)$*-definable).*

Proof
i), ii) are obvious and iii) is taken from [Pi].
iv) If $a \models p$ and q is a generic type of $Stab(p)$ then $R(Stab(p)) = R(q|a)$. But $R(q|a) = R((q|a)^G \cdot a) \leq R(p)$, because of the invariance of R under translations and the definition of stabilizer.
v) The first equality is a consequence of iii) and the second of iv).
vi) Every two independent realizations of p belong to the same coset. So considering for two arbitrary realizations of p a third one independent from them we see that p^G is contained in a single right coset of H. $\square$

The next two facts give us criteria for checking when a type is generic in an appropriate coset.

Fact 1.2 *The following are equivalent:*
i) p *is a generic type of a right coset of a type-definable over* $acl(\emptyset)$ *group* H.
ii) $R(p * p^{-1}) = R(p)$.
iii) $p * p^{-1} * p = p$.

Proof
i)$\Longrightarrow$ii)
p^G is contained in one right coset of H, so $\langle p * p^{-1} \rangle$ is contained in H and $R(p * p^{-1}) \leq R(H) = R(p) \leq R(p * p^{-1})$.
ii)$\Longrightarrow$iii)
Let $a, b \models p$ and $a \downarrow b$. Then $ab^{-1} \models p * p^{-1}$, so $R(ab^{-1}) = R(a) = R(a/b) = R(ab^{-1}/b)$ (a and ab^{-1} are interalgebraic over b). Equality of ranks implies that $ab^{-1} \downarrow b$, so $a = ab^{-1} \cdot b \models p * p^{-1} * p$ and $p = st(a) = st(ab^{-1} \cdot b) = p * p^{-1} * p$.
iii)$\Longrightarrow$i)
We see that $\langle p * p^{-1} \rangle \subseteq Stab(p)$ (by Lemma 1.1(ii)), $p^G \subseteq$ a right coset of $Stab(p)$ (by Lemma 1.1(vi)) and p is the generic of this coset (by Lemma 1.1(iii)). $\square$

Fact 1.3 *The following are equivalent:*
i) p *is a generic type of a left-and-right coset of a type-definable over* $acl(\emptyset)$ *group* H.

ii) $R(p * p) = R(p)$.
iii) $p * p * p^{-1} = p$.

Proof
i)$\Longrightarrow$ii)
If $p^G \subseteq aH = Ha$ then $(p * p)^G \subseteq aHaH = aaHH = a^2H$, so $R(p * p) \leq R(a^2H) = R(H) = R(p) \leq R(p * p)$.
ii)$\Longrightarrow$iii)
The proof is similar to the proof of the same implication in Fact 1.2 .
iii)$\Longrightarrow$i)
The equality from iii) implies that $R(p * p^{-1}) = R(p)$, so by Fact 1.2 and Lemma 1.1 $R(p^{-1} * p) = R(p)$ and p, p^{-1} are generic types of right cosets of their stabilizers, which have to be connected.
This implies that $R(Stab(p)) = R(p) = R(p^{-1}) = R(Stab(p^{-1})) = R(Stab_R(p))$. But $Stab(p) \subseteq Stab_R(p)$ (since $Stab(p) = \langle p * p^{-1} \rangle$ and $p * p * p^{-1} = p$), so $Stab_R(p) = Stab(p)$. Using the dual version of Fact 1.2 we see that p is also the generic type of a *left* coset of $Stab_R(p) = Stab(p)$. Now it is enough to prove the following Claim
CLAIM
If p is a generic of a left and of a right coset of a connected group H then it is a generic of a left-and-right coset.
Proof
Suppose $p^G \subseteq aH, Ha$ and $a \models p$. Our aim is to show that $aH = Ha$. Define $T = \{x \in H : a^{-1}xa \in H\}$. If $h \in H$ and $h \perp a$ then $ha \models p$, so $ha \in aH$ and $h \in T$. This implies that $R(T) = R(H)$ and $H = T$, since H is connected. $\square$

Remark 1.4 *If $R(p*p) = R(p)$, then $R(p^n) = R(p)$ and $Stab(p)$ is a normal subgroup of $\langle p \rangle$.*

Proof
By Fact 1.3 we have $p*p*p^{-1} = p$ and $p^{n+1} * p^{-1} = p^n$ for $n \geq 1$, which gives us the required equality. For the second statement $p^G \subseteq Norm(Stab(p))$ and then $\langle p \rangle \subseteq Norm(Stab(p))$. $\square$

The next theorem reduces the 1-step conjecture for groups of finite U-rank to the case of algebraic types.

Reduction Theorem 1.5 *The 1-step conjecture for algebraic types implies the 1-step conjecture for groups of finite U-rank.*

Proof

Suppose that $p \in S$ and (p^{n_i}) converges to q, where (n_i) is an increasing sequence of integers. Our aim is to show that q is a generic type of $\langle p \rangle$ (assuming the 1-step conjecture is true for algebraic types), so without loss we can assume $G = \langle p \rangle$. The U-rank of p^{n_i} grows with i, so it has to stabilize at some power p^{n_k}. It implies that $R(p^{n_k}) = R(p^{n_k+1}) = \ldots$ and by Fact 1.3, p^{n_k} is a generic type of a left-and-right coset of its stabilizer, which is a normal subgroup of $\langle p^{n_k} \rangle$, by Remark 2.4 . But for all $n > n_k$, p^n is a generic of a left-and-right coset of $Stab(p^n)$, which is a connected group of the same rank as $Stab(p^{n_k})$ and contains it. Then clearly $Stab(p^{n_k}) = Stab(p^{n_k+1}) = \ldots$, and $(p^{n_k})^G, (p^{n_k+1})^G \subseteq Norm(Stab(p^{n_k}))$, which implies that $Stab(p^{n_k})$ is a normal subgroup of G.

Let $H = Stab(p^{n_k})$. The group G/H cannot be treated as an interpretable group — we have to quotient out by an infinitely definable relation. But $H = \bigcap_n H_n$, where H_n are definable, normal subgroups of G ([Pi]) and the group G/H can be well approximated by groups G/H_n. Let π_n denote the quotient map $G \to G/H_n$ and for $r \in S$ let r_n be the corresponding type of elements of G/H_n.

CLAIM 1

$If\ r, s \in S,\ H \subseteq Stab(r), Stab(s)\ and\ r^G/H = s^G/H,\ then\ r = s.$

Proof

Let $a \models r$ and take $h \in H$ such that $ha \models s$. If $g \in G$ is generic over $\{h, a\}$ then $gh \perp h$ and $gh \perp a(h)$. By transitivity of forking we have $gh \perp a$, so $gh \cdot a \models r$. But also $g \perp ha$ hence $g \cdot ha \models s$ and $r = s$.

CLAIM 2

$Stab_R(q) = \bigcap_n (\pi_n)^{-1}(Stab_R(q_n))$.

Proof

$\subseteq$

obvious

$\supseteq$

Take $r \in S$ such that $q_n * r_n = q_n$ for all n. $Stab(p^{n_k}) \subseteq Stab(q), Stab(q * r)$, since q is a limit of sequence of powers of p and independent multiplication is continuous coordinate-wise ([Ne]) by the open mappig theorem. By Claim 1 it suffices to show that

$$(\star) \qquad (q * r)^G/H = q^G/H$$

We know that $(q * r)^G/H_n = q^G/H_n$ for all n, so if $a \models q * r$ then by compactness q is realized in $\bigcap_n aH_n$, which witnesses $\supseteq$ in $(\star)$. By symmetry we

get also $\subseteq$.
All types p_n are algebraic and generate groups G/H_n, so by assumption q_n is a generic type of G/H_n (as a limit of a sequence of powers of p_n). This implies that $Stab_R(q_n) = (G/H_n)^0$ and $(\pi_n)^{-1}(Stab_R(q_n)) \supseteq G^0$. By Claim 2 $Stab_R(q) = G^0$, so q is a generic type of G. $\square$

In fact, the proof of Theorem 1.5 works in a more general setting. Specifically, assume C is a class of type-definable subgroups of finite U-rank, which is closed under taking subgroups and quotients by definable normal subgroups. Then the 1-step conjecture for C reduces to the case of algebraic type (in a group from C). Newelski ([Ne]) asked about possible generalizations of the double step theorem and the 1-step conjecture for arbitrary generating set of types. We formulate such a generalization, which will also include the Mordell-Lang conjecture (see section 2).

1-step conjecture for several types 1.6 *Let P be a finite subset of S. If $q \in cl(*P) - (*P)$, then q is a generic type of a left-and-right coset of a type-definable over $acl(\emptyset)$ subgroup of $\langle P \rangle$ (here $*P = \{p_1 * \ldots * p_n : p_i \in P, n < \omega\}$).*

It makes no sense to state this conjecture for an infinite set of types. For instance if P is the set of all isolated types then we can get as q arbitrary type (if G is ω-stable). The 1-step conjecture for several types is not true for many groups of finite Morley rank (e.g. for the groups for which Mordell-Lang conjecture is not true — the simplest example is $G_a \times G_a$). It would be very profitable to distinguish using model-theoretical notions the class of groups for which the 1-step conjecture for several types *is* true.
Using the double step theorem we can see that the 1-step conjecture for several types implies the 1-step conjecture — if q is a limit of some sequence of powers of p, then the sequence of powers (q^{n_i}) strongly converges to generic type of $\langle p \rangle$, but by Fact 1.3 $R(q) = R(q^2) = \ldots$, so $R(q) = R(\langle p \rangle)$ and q is already a generic type of $\langle p \rangle$.

2 Algebraic groups

Algebraic groups are the most natural example of groups of finite U-rank. In this section we will deal with the 1-step conjecture in the case of algebraic groups, translating it into the language of algebraic geometry. All algebraic varieties will be defined over a fixed algebraically closed field K and for simplicity we expand the language by naming elements of K. We will work in a big algebraically closed field M extending K. A subvariety will mean an irreducible Zariski closed subset. In this chapter cl means Zariski closure.

Model-theoretical notions have natural interpretations in algebraic geometry. Types of elements of a variety V are in one-to-one correspondence with subvarieties of V (more precisely, a type p corresponds to the subvariety $V(p) = cl(p^M)$). The set of subvarieties of V with Zariski topology is the underlying space of a scheme associated with V. Associating this scheme with V is a functor — if f is a morphism of varieties then $V \mapsto cl(f(V))$ is the associated morphism of schemes. If G is an algebraic group, then on the set of subvarieties we have independent product coming from independent product of types. The next proposition shows that this product is the image of the group operation by the above functor.

Proposition 2.1 *Let V and W be subvarieties of G, $p, q \in S$ and define $V * W = cl(V \cdot W)$. Then $V(p) * V(q) = V(p * q)$.*

Proof
It is easy to see that $V(p \otimes q) = V(p) \times V(q)$, so $(p \otimes q)^M$ is Zariski dense in $V(p) \times V(q)$. This implies that $(p * q)^M$ is Zariski dense in $V(p) \cdot V(q)$ and $V(p) * V(q)$ (as an image of a dense set by a continuous surjection). □

We can equip the set of subvarieties of V with the Stone topology coming from the topology of the space of types. The next fact describes convergence in Stone topology in terms of Zariski topology.

Fact 2.2 *(V_n) converges to V in the Stone topology iff for some natural number N, for every increasing sequence of integers (n_i), $cl(\bigcup_{n_i > N} V_{n_i}) = V$.*

Proof
Easy. □

Suppose G is an algebraic group and V a subvariety of G. Denote by $\langle V \rangle$ the algebraic subgroup of G generated by V. We give an easy description of $\langle V \rangle$.

Proposition 2.3 $\langle V \rangle = cl(\bigcup_{n<\omega} V^n)$ *($V^n = V * \ldots * V$ n times).*

Proof
$\supseteq$
Obvious.
$\subseteq$
$cl(\bigcup_{n<\omega} V^n)$ is a Zariski closed subset of $\langle V\rangle$ and is closed under the group operation of G, since it is continous coordinate-wise. By [Po] a definable subset of a stable group which is closed under group operation is a group. □

The class of algebraic groups is closed under definable subgroups and quotients, so we can use the reduction theorem (1.5), which says that the 1-step conjecture for algebraic groups can be reduced to the case, when the generating subvariety is a point (corresponding to an algebraic type).
Using this reduction we can formulate equivalent statements of the 1-step conjecture for algebraic groups, which emphasize its geometric nature.

Proposition 2.4 *The following are equivalent:*
i) The 1-step conjecture for algebraic groups is true.
ii) If $a \in G$, X is a Zariski closed subset of G and Γ is a cyclic group generated by a such that $\Gamma \cap X$ is Zariski dense in X, then X is a union of cosets of algebraic subgroups of G.
iii) The components of X are cosets of $\langle a\rangle^0$ or points, (X and a are as in ii)).

Proof
i)$\Longrightarrow$iii)
If a component of X is a closure of an infinite subset of Γ, then this component contains an infinite sequence of powers of a or a^{-1}. The 1-step conjecture implies that such a component is a coset of $\langle a\rangle^0 = \langle a^{-1}\rangle^0$.
iii)$\Longrightarrow$ii)
Obvious.
ii)$\Longrightarrow$i)
Suppose that (V^{n_i}) converges to W in the Stone topology, where (n_i) is an increasing sequence of integers. By the reduction theorem we can assume $V = \{a\}$. By Fact 2.2 W is the closure of an infinite set of powers of a, so by ii) W is a coset of a connected algebraic subgroup of G. This subgroup contains a subgroup of Γ of finite index, so it contains a subgroup of $\langle a\rangle$ of finite index. It follows that W is a coset of $\langle a\rangle^0$, since W is irreducible. □

In 1965 Lang formulated the Mordell-Lang conjecture ([La1], [La2]), which

generalizes the Mordell conjecture about rational points on curves and the Munin-Mumford conjecture about torsion points on abelian varieties. By Proposition 2.4
the 1-step conjecture for semi-abelian varieties is also a part of Mordell-Lang conjecture, which we will recall now.

Mordell-Lang conjecture 2.5 *Suppose char(K)=0. Let A be a semi-abelian variety, X a Zariski closed subset of A and Γ an (abstract) subgroup of A such that $\Gamma \otimes Q$ is finitely generated (as a Q-module) and $\Gamma \cap X$ is Zariski dense in X. Then X is a union of cosets of semi-abelian subvarieties of A.*

3 Characteristic 0 and positive characteristic

In this chapter we discuss the truth of the 1-step conjecture in the algebraic groups context. It turned out in Section 2 that for semi-abelian varieties the 1-step conjecture is a part of the Mordell-Lang conjecture. However the Mordell-Lang conjecture stated as in Section 2 is not true in positive characteristic, so one may expect that the 1-step conjecture also fails. It is indeed the case, which was pointed out to me by Anand Pillay. In this section G, X, Γ and a are as in Proposition 2.4 .

Theorem 3.1 *The 1-step conjecture is not true in positive characteristic.*

Proof
Take $G = G_m \times G_m$ (G_m is the multiplicative group of the field K). Let X be defined by the equation $y = x + 1$. Every algebraic subgroup of G is defined by equation $x^n y^m = 1$ for some integers m, n [Hu] so it is easy to see that X is not a translate of an algebraic subgroup of G. Take $a = (x, x + 1)$, where x is transcendental over F_p. Then X contains infinitely many powers of a namely $a^{p^n} = (x^{p^n}, x^{p^n} + 1)$, so $\Gamma \cap X$ is Zariski dense in X as X is a one-dimensional subvariety. □

The Mordell-Lang conjecture in characteristic 0 was proved by McQuillan ([Mc]). Aiming to prove the 1-step conjecture we can restrict ourselves to the groups generated (as algebraic groups) by a single element. Such a group is commutative, but it need not to be a semi-abelian variety, so we cannot use the McQuillan's result. The simplest example of a group generated by one element which is not a semiabelian variety (for which also the Mordell-Lang fails) is $G_a \times G_m$ (G_a stands for the additive group). However Anand Pillay and Felipe Voloch pointed out a proof of the 1-step conjecture for all algebraic groups in characteristic 0. This seems to be a folklore. Singer has a related proof for linear algebraic groups in characteristic 0 which uses difference equations and the Skolem-Mahler-Lech theorem [Si].

Theorem 3.2 *The 1-step conjecture is true for algebraic groups in characteristic 0.*

Proof
Without loss we can assume that X is irreducible and G is the Zariski closure of Γ, hence a commutative group. We can find a tuple c of elements of K over which X, G (with the group operation and inverse), a, and a^{-1} are defined. Let L be the field extension of Q generated by c. By a result from [Ca] (chapter V, theorem 1.1) we can embed L in Q_p (for infinitely many p) in such a way that the p-adic norm of elements of c equals 1. Then G, X and elements from Γ are defined over Z_p. So we can consider Γ as a subgroup of the compact p-adic analytic group $H = G(Z_p)$. By [DduSMS] H contains an open subgroup H_0 which is analytically isomorphic to $(Z_p)^n$ ($n = dim(H)$). H_0 is of finite index (H is compact), so H_0 contains an infinite subgroup of Γ. Denote this subgroup by Γ_0 and by Γ_1 its p-adic closure, which is analytically isomorphic to Z_p. Γ is a finite union of cosets of Γ_0, which implies that some coset of $X \cap \Gamma_0$ is Zariski dense in X, since X is irreducible. Without loss we can assume that $X \cap \Gamma_0$ is Zariski dense in X. $X \cap \Gamma_1$ is an infinite analytic subset of the compact one-dimensional group Γ_1, so it has to contain an open subset of Γ_1. But Γ_1 is a profinite topological group, hence X contains a coset of some open subgroup Γ_2 of Γ_1 of finite index. It follows that Γ_2 contains an infinite subgroup of Γ_0 (and Γ), so G^0 is contained in X. $\square$

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