PAC differential fields in positive characteristic
(joint work with Daniel Max Hoffmann)

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Plan of the talk

1. Introduction to PAC fields and PAC structures.
2. General methods for understanding PAC structures.
3. PAC fields of positive characteristic.

PAC fields

- The notion of a PAC (Pseudo Algebraically Closed) field originates from Ax's papers on pseudofinite fields (1960s), since finite fields are "more and more PAC" (Lang-Weil estimates).

- The names "PAC/Pseudo Algebraically Closed" were given by Frey in 1973. A field K is PAC, if each absolutely irreducible variety defined over K has a K-rational point.

- PAC fields show up in different model-theoretic contexts. Our interest comes from the model theory of group actions as explained in the previous talk.

PAC structures

- Let L be a language.
- We fix a stable L-theory T with quantifier elimination.
- Let $T_\forall$ be the theory of L-substructures of models of T and let $P \models T_\forall$.
- We say that P is T-PAC, if any stationary (i.e. "irreducible" in a certain model-theoretic sense) type (consistent collection of formulas) over P is finitely satisfiable in P.
- We say that T-PAC is first-order, if the class of T-PAC L-structures is elementary (axiomatizable by an L-theory).

ACF case

- L: language of fields, $T = \mathrm{ACF}$, $T_\forall$: theory of fields.
- For a field K, stationary types over K are implied by formulas of the form "$x \in V \setminus W$", where V, W are K-varieties and V is absolutely irreducible.
- There is a difference between K-variety ($\{x\}$ is a $\mathbb{F}_p(x^p)$-variety) and variety defined over K ($\{x\}$ is *not* defined over $\mathbb{F}_p(x^p)$) appearing in the definition of the classical PAC.
- Any T-PAC structure is definably closed. In particular, ACF-PAC fields are perfect, which need *not* happen for classical PAC fields. Non-perfect classical PAC fields will be recovered as "SCF-PAC" fields.
- Since any set $V \setminus W$ contains a subset which is K-isomorphic to a K-variety, we get that ACF-PAC fields are exactly perfect PAC fields, and this class is known to be first-order.

Brief history of PAC structures

- Studying PAC structures beyond the case of fields was initiated by Hrushovski in the strongly minimal context.
- Pillay-Polkowska considered PAC in the stable case, there are slight differences with the approach we take here.
- PAC structures also appeared in Afshordel thesis.
- Recently, PAC structures were analized by Hoffmann and also by Dobrowolski-Hoffmann-Lee.

Overview

In this part of the talk, I will discuss two general contexts in which PAC structures are quite well-understood.

1. Totally transcendental theories.
2. "Noetherian theories", that is theories with a built-in topology, which nicely interacts with the definable structure.

Morley rank, Morley irreducibility and PAC

- In this part, we moreover assume that T is totally transcendental that is any formula has ordinal Morley rank and finite Morley degree (with respect to T).
- Main examples: ACF, DCF_0, $\mathrm{DCF}_{0,m}$, CCM (the theory of compact complex manifolds).
- Let $P \models T_\forall$. It is well-known that stationary types over P are implied by formulas of Morley degree 1. Hence we recover Hrushovski's definition (given for a strongly minimal T).
- If $T = \mathrm{ACF}$, then formulas of Morley degree 1 correspond to constructible sets having unique component of maximal dimension which is absolutely irreducible. This recovers the classical definition of PAC as well (modulo perfectness).

DMP and PAC being first-order

- DMP stands for Definable Multiplicity Property, where multiplicity here refers to the Morley degree.
- If T has DMP, then T-PAC is first order.
- ACF (and many other strongly minimal theories) have DMP.
- Freitag showed that DCF_0 does not have DMP.
- It is open whether CCM has DMP (partial results were obtained by Radin).

Uniform topologies

By a Noetherian theory, we mean a pair $(T, \sum)$, where T is an L-theory and $\sum$ consists of L-formulas of the form $\varphi(x; y)$ $(x, y$ vary) s.t. for any $M \models T$ and $A \subseteq M$, we have the following.

- A subset $V \subseteq M^{|x|}$ is A-closed, if and only if there is $a \subset A$ and $\varphi(x; y) \in \sum$ such that $V = \varphi(M; a)$.
- The family of A-closed sets constitutes the family of closed sets of a Noetherian A-topology.
- Constructible sets with respect to the A-topology coincide with A-definable sets (in Cartesian powers of M).

Properties of Noetherian theories

- Complete types over A are determined by A-closed A-irreducible sets V in the following way:

$$p_V := \{\text{“}x \in C\text{”} \mid \mathrm{int}_V(C \cap V) \neq \emptyset\}.$$

- In particular, any Noetherian theory is totally transcendental (so also stable).
- Stationary types correspond to absolutely irreducible (irreducible in all A-topologies) closed sets.
- In particular, P is T-PAC iff for any absolutely irreducible P-closed set V and any non-empty relatively P-open $U \subseteq V$, we have that $U(P) \neq \emptyset$.

Examples of Noetherian theories and "Noetherian-PAC"

- ACF (Zariski topology), DCF_0, $\mathrm{DCF}_{0,m}$ (Kolchin topology), CCM (Zariski analytic topology).
- For the theory ACF, we recover again the classical definition of PAC (modulo perfectness).
- In the cases of DCF_0, $\mathrm{DCF}_{0,m}$, CCM, the topological description of definable sets of Morley degree one from the case of ACF does not hold anymore. Hence, we get a different (but equivalent) description of PAC structures here.
- This equivalent description of PAC-$\mathrm{DCF}_{0,m}$ was already given by Sanchez-Tressl.
- The notion of PAC-CCM seems to be new. Does it have any meaningful analytic interpretation?

Definability of irreducibility

- As before: if the topological irreducibility in a Notherian theory T is definable, then T-PAC is first order.
- The topological irreducibility is definable in ACF.
- Topological irreducibility is also definable in CCM (Radin, originally Campana). In particular, PAC-CCM is first-order.
- It is open whether topological irreducibility is definable for DCF_0 (Ritt problem).
- But DCF_0-PAC is still first-order (Pillay-Polkowska). More generally, $\mathrm{DCF}_{0,m}$-PAC is first-order (Sanchez-Tressl).

Three theories

I will discuss the following three stable theories of fields of positive characteristic. Let p be a prime and e be a positive integer.

1. The theory $\mathrm{SCF}_{p,e}$ of separably closed fields of characteristic p and inseparability degree e (that is: $[K : K^p] = p^e$).
2. The theory $\mathrm{SCF}_{p,\infty}$ of separably closed fields of characteristic p and infinite inseparability degree.
3. The theory DCF_p of differentially closed fields of characteristic p.

Set-up for $\mathrm{SCF}_{p,e}$

- $L = L_{\lambda,b}$; the language of fields with constants for a *p*-basis ($b_1, \ldots, b_e$ s.t. after applying p-polynomials, we get a basis of K over K^p) and unary λ-functions (the coefficients with respect to the above basis of K over K^p).
- $T = \mathrm{SCF}_{p,e}$: the $L_{\lambda,b}$-theory of separably closed fields of characteristic p and inseparability degree e.
- T is stable and has elimination of quantifiers and elimination of imaginaries.
- $T_\forall$: the theory of fields of characteristic p and inseparability degree e.
- Side comment: there is a "λ-topology" here, but it is not Noetherian.

Description of PAC-$\mathrm{SCF}_{p,e}$ fields

- Afshordel stated that the notion of PAC-$\mathrm{SCF}_{p,e}$ fields coincide with the notion of (classically) PAC fields of characteristic p and inseparability degree e.
- It is quite easy to show and we immediately get that PAC-$\mathrm{SCF}_{p,e}$ is first-order.

Set-up for $\mathrm{SCF}_{p,\infty}$

- $L = L_\lambda$; the language of fields with multi-variable λ-functions (arguments contain p-independent elements). We can not use constant symbols for p-bases!
- $T = \mathrm{SCF}_{p,\infty}$: the L_λ-theory of separably closed fields of characteristic p and infinite inseparability degree.
- T is stable and has elimination of quantifiers, but no elimination of imaginaries.
- $T_\forall$: the theory of fields of characteristic p.

Description of PAC-$\mathrm{SCF}_{p,\infty}$ fields

- Afshordel: PAC-$\mathrm{SCF}_{p,\infty}$ fields are (classical) PAC fields of characteristic p and infinite inseparability degree.
- Proving that caused some difficulties and the following came to our rescue.

Theorem (Tamagawa's Theorem)

Let V be an absolutely irreducible affine variety over a PAC field K of characteristic $p > 0$. Suppose that $f_1, \ldots, f_m \in K[V]$ are p-independent in $K(V)$ and $p^m \leqslant [K : K^p]$. Then, there is $a \in V(K)$ such that $f_1(a), \ldots, f_m(a)$ are p-independent in K.

- Using Tamagawa's Theorem, we can prove Afshordel's statement above.

Set-up for DCF_p

- $L = L_{\lambda_0,D}$; the language of differential fields with an extra unary function symbol λ_0 interpreted on K as the inverse of Frobenius on K^p and 0 elsewhere.
- $T = \mathrm{DCF}_p$: the theory of differentially closed fields of characteristic p (Shelah, Wood).
- T is stable and has elimination of quantifiers, but no elimination of imaginaries.
- $T_\forall$: the theory of differential fields of characteristic p.

Geometric axioms

Geometric axioms of DCF_0 (Pierce-Pillay)

(K, D) is a differential field of characteristic 0 and for each pair of affine K-varieties (V, W) such that $W \subseteq \tau^D(V)$ ("D-twisted" tangent bundle) and the projection $\pi : W \to V$ is dominant, there is $a \in V(K)$ such that $D_V(a) \in W(K)$.

- Using these axioms, Pillay and Polkowska showed that DCF_0-PAC is first-order.
- Our strategy is similar here. We use geometric axioms of DCF_p, I proposed almost 20 years ago.
- There is one extra difficulty here: admissible tuples. A tuple $(V; f_1, \ldots, f_n)$ is admissible, if V is a K-variety and $f_1, \ldots, f_n \in K(V) \setminus K(V)^p$.

Axioms for DCF_p-PAC

Theorem (Hoffmann, K.)

A differential field (K, D) of characteristic p is DCF_p-PAC iff for each affine K-varieties (V, W) and each $f_1, \ldots, f_n \in K(V)$ s.t.

- *W is absolutely irreducible,*
- *$W \subseteq \tau^D(V)$,*
- *the projection $\pi : W \to V$ is dominant,*
- *an extra "equalizer condition" corresponding to separability,*
- *the tuple $(W; f_1 \circ \pi, \ldots, f_n \circ \pi)$ is admissible;*

there is $x \in V(K)$ such that $f_1(x), \ldots, f_k(x)$ are not p-th powers in K and $D_V(x) \in W(K)$.

Admissibility is first-order by Tamagawa, so DCF_p-PAC is first-order.

Model theory of finite group actions

- Assume now that G is a group and let $G\text{-}T_\forall$ be the theory of actions of G (by L-automorphisms) on L-substructures of T.
- If $G\text{-}T_\forall$ has a model companion, then we call it $G\text{-}T$ and say that $G\text{-}T$ exists.
- Previous talk: G finite, T-PAC first-order implies $G\text{-}T$ exists.
- Hence, for a finite G, we get:
  1. $G\text{-}\mathrm{CCM}$: supersimple of finite SU-rank.
  2. $G\text{-}\mathrm{SCF}_{p,e}$: analyzed with Hoffmann already (strictly simple) in the context of model theory of finite group scheme actions.
  3. $G\text{-}\mathrm{SCF}_{p,\infty}$: to be analyzed.
  4. $G\text{-}\mathrm{DCF}_p$: to be analyzed.

Question

DCF_p results above can be widely generalized to "local" Moosa-Scanlon operators, or, more generally, $\mathcal{B}$-operators introduced by Gogolok and myself.

Question

Assume that T is stable and has quantifier elimination. Is the class of T-PAC structures (always) elementary?

- Positive answer to this question implies that for a finite group G and T as above, the theory G-T exists. This is a question of Hoffmann.
- So far, we can not cover the theories $\mathrm{DCF}_{p,m}$ (several commuting derivations in positive characteristic).

Beyond stability

- If T is not stable, it is not clear (at least to me) what a "right" definition of T-PAC should be.
- One should test some simple and NIP theories, possibly with finite group actions.
- A theory of particular interest: ACVF, that is the theory of algebraically closed valued fields.
- It is planned further research.