

Difference algebra and generic rational cohomology

(joint work with Marcin Chałupnik)

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Aim

Our aim is to find a general difference counterpart of the generic cohomology appearing in the following theorem.

Theorem (Cline, Parshall, Scott, van der Kallen)

Let \mathbf{G} be a semisimple simply connected linear algebraic group defined over \mathbb{F}_p and V be a finite dimensional rational \mathbf{G} -module.

- 1 For a fixed n , the groups $H_{\text{dis}}^n(\mathbf{G}(\mathbb{F}_{p^d}), V)$ achieve a stable value as $d \rightarrow \infty$, which is called $H_{\text{gen}}^n(\mathbf{G}, V)$.
- 2 For a fixed n , we have

$$H_{\text{gen}}^n(\mathbf{G}, V) \cong H_{\text{rat}}^n(\mathbf{G}, (\text{Fr}_{\mathbf{G}}^r)^*(V))$$

for sufficiently large r .

Rational Representations: Functors

Fix a base field (or ring) \mathbf{k} and an affine group scheme \mathbf{G} over \mathbf{k} .
Then \mathbf{G} gives a functor (of rational points)

$$F_{\mathbf{G}} : \text{Alg}_{\mathbf{k}} \rightarrow \text{Groups}, \quad F_{\mathbf{G}}(A) = \mathbf{G}(A).$$

Any \mathbf{k} -module M also gives a functor

$$F_M : \text{Alg}_{\mathbf{k}} \rightarrow \text{Mod}_{\mathbf{k}}, \quad F_M(A) = M \otimes A.$$

We call M a **rational \mathbf{G} -module** if there is an action of $F_{\mathbf{G}}$ on F_M by \mathbf{k} -linear maps.

Rational Representations: Comodules

For cohomological reasons, it is convenient to express a rational \mathbf{G} -module as a **comodule** over $\mathbf{k}[\mathbf{G}]$ (Hopf algebra of \mathbf{G}). A $\mathbf{k}[\mathbf{G}]$ -comodule is a pair $(M, \Delta_M : M \rightarrow M \otimes \mathbf{k}[\mathbf{G}])$ such that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\Delta_M} & M \otimes \mathbf{k}[\mathbf{G}] \\ \Delta_M \downarrow & & \downarrow \Delta_M \otimes 1 \\ M \otimes \mathbf{k}[\mathbf{G}] & \xrightarrow{1 \otimes \Delta_{\mathbf{k}[\mathbf{G}]}} & M \otimes \mathbf{k}[\mathbf{G}] \otimes \mathbf{k}[\mathbf{G}] \end{array}$$

Theorem (Hochschild?)

The category of $\mathbf{k}[\mathbf{G}]$ -comodules is equivalent to the category of rational \mathbf{G} -modules. This category is Abelian with enough injectives (and not enough projectives).

Rational Cohomology

- Let M be a rational \mathbf{G} -module. The rational cohomology is defined as follows

$$H_{\text{rat}}^n(\mathbf{G}, M) := \text{Ext}_{\mathbf{G}}^n(\mathbf{k}, M).$$

- The functor H_{rat}^n is the n -th derived functor of the functor $M \mapsto M^{\mathbf{G}}$, where

$$M^{\mathbf{G}} = \ker(\Delta_M - \iota_M).$$

- There is also an equivalent definition of rational cohomology using cocycles, similarly to the discrete (or *definable*) case.

Stable Cohomology

Assume that $\mathbf{k} = \mathbb{F}_p$.

- We have the (relative) Frobenius morphism $\mathrm{Fr}_{\mathbf{G}} : \mathbf{G} \rightarrow \mathbf{G}$.
- For any \mathbf{G} -module M , we have the twisted \mathbf{G} -module

$$M^{(d)} := (\mathrm{Fr}_{\mathbf{G}}^d)^*(M)$$

and the “restriction” maps

$$H_{\mathrm{rat}}^n(\mathbf{G}, M^{(d)}) \rightarrow H_{\mathrm{rat}}^n(\mathbf{G}, M^{(d+1)}).$$

- The **stable cohomology** is defined as

$$H_{\mathrm{sta}}^n(\mathbf{G}, M) := \varinjlim H^n(\mathbf{G}, M^{(d)}).$$

Generic cohomology

Again assume that $\mathbf{k} = \mathbb{F}_p$.

- For any d , we have the finite group $\mathbf{G}(\mathbb{F}_{p^d})$ and the homomorphism $\mathbf{G}(\mathbb{F}_{p^d}) \rightarrow \mathbf{G}(\mathbb{F}_{p^{d+1}})$. Clearly, M is also a $\mathbf{k}[\mathbf{G}(\mathbb{F}_{p^d})]$ -module.
- Then we have the discrete cohomology groups $H_{\text{dis}}^n(\mathbf{G}(\mathbb{F}_{p^d}), M)$ and the restriction maps

$$H_{\text{dis}}^n(\mathbf{G}(\mathbb{F}_{p^{d+1}}), M) \rightarrow H_{\text{dis}}^n(\mathbf{G}(\mathbb{F}_{p^d}), M).$$

- The **generic cohomology** is defined as

$$H_{\text{gen}}^n(\mathbf{G}, M) := \varprojlim H_{\text{dis}}^n(\mathbf{G}(\mathbb{F}_{p^d}), M).$$

Motivating Theorem again

The theorem of Cline, Parshall, Scott, Van Der Kallen is both about the **stabilization** of the rational and the stable cohomology and about the **isomorphism** between them.

Example

This theorem does not hold for $\mathbf{G} = \mathbb{G}_a \dots$

Our project

- Develop a cohomology theory of difference (algebraic) groups.
- Explain stable cohomology using cohomology of difference algebraic groups.
- More ambitious: stating and proving a difference version of the theorem of Cline, Parshall, Scott, van der Kallen for arbitrary difference algebraic groups, which would include the classical theorem as a special case.

Evaluating rational representation

- A rational representation M of \mathbf{G} can be understood as a “compatible system” of the following situations.

Situation for a given \mathbf{k} -algebra A

The group $\mathbf{G}(A)$ acts on $M \otimes A$ by A -linear maps.

In other words, $M \otimes A$ is an $A[\mathbf{G}(A)]$ -module.

- We want to understand each individual situation in a difference case (such a situation will be an example of a **discrete difference representation**). We fix a difference ring (A, σ_A) and a difference group (G, σ_G) . We define an appropriate ring which will play the role of the ring $A[\mathbf{G}(A)]$.

Twisted polynomials

- For any difference ring (R, σ) (unital, not necessarily commutative), the ring of **twisted polynomials** is defined as

$$R[\sigma] := \left\{ \sum t^i r_i \mid r_i \in R \right\}, \quad t^n r \cdot t^m r' := t^{n+m} \sigma^m(r) r'.$$

- Let M be a left R -module and $\sigma_M : M \rightarrow M$ be additive. (M, σ) is a left $R[\sigma]$ -module if and only if:

$$\sigma_M(\sigma(r).m) = r.\sigma_M(m)$$

E.g. (R, σ^{-1}) is a left $R[\sigma]$ -module (if σ^{-1} exists).

- Right $R[\sigma]$ -modules (e.g. (R, σ)) correspond to the condition

$$\sigma_M(m.r) = \sigma_M(m).\sigma(r)$$

Discrete difference representations

- For a difference group (G, σ_G) and a difference ring (A, σ_A) , we define the difference ring $R := A[G]$ with

$$\sigma \left(\sum \alpha_i g_i \right) := \sum \sigma_A(\alpha_i) \sigma_G(g_i).$$

- A **discrete difference representation** of (G, σ_G) over (A, σ_A) is defined as a left $A[G][\sigma]$ -module.
- Since the category of difference discrete representations is the same as the category of left $A[G][\sigma]$ -modules, it is Abelian with enough injectives and we define (assuming here that σ_A is an automorphism):

$$H_{\sigma}^n((G, \sigma_G), (M, \sigma_M)) := \text{Ext}^n((A, \sigma_A^{-1}), (M, \sigma_M)).$$

Comparison Theorem

- The functor H_σ^n is the n -th derived functor of the functor

$$M \mapsto M^G \cap M^{\sigma_M}.$$

- We have the following spectral sequence (coming from the Grothendieck spectral sequence)

$$E_2^{n,m} = H_\sigma^n(1, H^m(G, M)) \Rightarrow H_\sigma^{n+m}(G, M),$$

where 1 is the trivial group.

- After an easy computation of the difference cohomology of the trivial group, this spectral sequence yields the following short exact sequence

$$0 \rightarrow H^{n-1}(G, M)_\sigma \rightarrow H_\sigma^n(G, M) \rightarrow H^n(G, M)^\sigma \rightarrow 0.$$

What should be a rational difference module

Let us fix a base inversive difference field (\mathbf{k}, σ) and an *affine difference algebraic group* (precise definition later) \mathcal{G} . A rational difference \mathcal{G} -module should satisfy the following conditions.

- 1 For each difference (\mathbf{k}, σ) -algebra (A, σ_A) , we should have a “compatible system” of discrete difference representations.
- 2 The category of rational difference \mathcal{G} -modules should be Abelian with enough injectives.
- 3 We should have spectral sequences connecting rational difference cohomology with rational cohomology.
- 4 Rational difference cohomology should explain stable cohomology.

What is a difference algebraic group

- An **affine difference algebraic group** is a representable functor from the category of difference (\mathbf{k}, σ) -algebras to the category of groups.
- It is represented by a **difference Hopf algebra** which may be defined as (H, σ_H) , where H is a Hopf algebra over \mathbf{k} and $\sigma_H : \sigma^*(H) \rightarrow H$ is a Hopf algebra morphism.
- Dualizing, we see that a difference algebraic group \mathcal{G} is the same as a pair $(\mathbf{G}, \sigma_{\mathbf{G}})$ where \mathbf{G} is an affine group scheme over \mathbf{k} and $\sigma_{\mathbf{G}} : \mathbf{G} \rightarrow \sigma^*(\mathbf{G})$ is a group scheme morphism.
- It fits to the general set-up from Tom's first lecture: group objects in the category $\sigma\mathcal{C}$ may be identified with pairs (G, σ_G) ... Here, the situation is a bit twisted, since we consider objects *over* a fixed difference object (\mathbf{k}, σ) .

Twisted rational representations

We fix a difference algebraic group $\mathcal{G} = (\mathbf{G}, \sigma_{\mathbf{G}})$ and for simplicity we assume that \mathbf{G} is defined over \mathbf{k}^{σ} . Let (M, σ_M) be a left $\mathbf{k}[\sigma]$ -module. In this “Attempt 1 case”, we have the following.

Definition

A **rational difference \mathcal{G} -module** is a pair (M, σ_M) as above, together with a rational \mathbf{G} -module structure on M such that

$$\sigma_M : \sigma_{\mathbf{G}}^*(M) \rightarrow M$$

is a rational \mathbf{G} -module morphism.

We will see that this definition satisfies the conditions (2), (3), (4). However, (to our taste) it does not satisfy the condition (1).

Difference comodules

In the comodule terms, a rational difference \mathcal{G} -module is a triple (M, σ_M, Δ_M) such that (M, Δ_M) is a comodule over $\mathbf{k}[\mathbf{G}]$ and the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\sigma_M} & M \\ \downarrow \Delta_M & & \downarrow \Delta_M \\ \mathbf{k}[\mathbf{G}] \otimes M & \xrightarrow{\sigma_{\mathbf{k}[\mathbf{G}]} \otimes \sigma_M} & \mathbf{k}[\mathbf{G}] \otimes M, \end{array}$$

where $\mathbf{k}[\mathbf{G}]$ is the Hopf algebra of \mathbf{G} .

Good properties

The category of rational difference \mathcal{G} -modules is Abelian with enough injectives so we can define in a usual way:

$$H_{\sigma \text{ rat}}^n(\mathcal{G}, \mathcal{M}) := \text{Ext}^n((\mathbf{k}, \sigma^{-1}), \mathcal{M})$$

where $\mathcal{M} = (M, \sigma_M)$. We get a result as in the discrete case.

Theorem (Chałupnik, K.)

We have the following spectral sequence

$$E_2^{n,m} = H_{\sigma \text{ rat}}^n(\mathbf{1}, H_{\text{rat}}^m(\mathbf{G}, M)) \Rightarrow H_{\sigma \text{ rat}}^{n+m}(\mathcal{G}, \mathcal{M})$$

and a short exact sequence

$$0 \rightarrow H_{\text{rat}}^{n-1}(\mathbf{G}, M)_{\sigma} \rightarrow H_{\sigma \text{ rat}}^n(\mathcal{G}, \mathcal{M}) \rightarrow H_{\text{rat}}^n(\mathbf{G}, M)^{\sigma} \rightarrow 0.$$

Telescope and rational cohomology

To any rational \mathbf{G} -module V , we can functorially associate the **telescope** difference rational \mathcal{G} -module

$$V^\infty := \bigoplus_{i=0}^{\infty} (\sigma_{\mathbf{G}}^i)^*(V).$$

Since $(\sigma_{\mathbf{G}})^*(V^\infty) = \bigoplus_{i=1}^{\infty} (\sigma_{\mathbf{G}}^i)^*(V)$, the inclusion

$$\bigoplus_{i=1}^{\infty} (\sigma_{\mathbf{G}}^i)^*(V) \subset \bigoplus_{i=0}^{\infty} (\sigma_{\mathbf{G}}^i)^*(V)$$

defines on V^∞ the structure of a difference rational \mathcal{G} -module.

Theorem (Chałupnik, K.)

We have the following isomorphism

$$H_{\text{sta}}^{n-1}(\mathbf{G}, V) \cong H_{\text{rat}}^n(\mathcal{G}, V^\infty).$$

More functorial approach

This is really work in progress. For any left $\mathbf{k}[\sigma]$ -module $\mathcal{M} = (M, \sigma_M)$ we define the following functor

$$F_{\mathcal{M}} : \text{Alg}_{(\mathbf{k}, \sigma)} \rightarrow \text{Mod}_{\mathbf{k}[\sigma]}, \quad F_{\mathcal{M}}(A, \sigma_A) := A[\sigma_A] \otimes_{\mathbf{k}[\sigma]} \mathcal{M}.$$

In this “Attempt 2 case”, we have the following.

Definition

We call \mathcal{M} a **difference rational \mathcal{G} -module** if there is an action of the functor $F_{\mathcal{G}}$ on the functor $F_{\mathcal{M}}$ by $\mathbf{k}[\sigma]$ -linear maps.

Difference comodules in this case

In this case, it is not even easy to find the right notion of a comodule map capturing this definition of a difference rational \mathcal{G} -module. We have recently achieved it, the new comodule map Δ_M should fit into the following commutative diagram.

$$\begin{array}{ccc}
 M & \xrightarrow{\Delta_M} & \mathbf{k}[\mathbf{G}][\sigma] \otimes_{\sigma} M \\
 \Delta_M \downarrow & & \downarrow 1 \otimes \Delta_M \\
 \mathbf{k}[\mathbf{G}][\sigma] \otimes_{\sigma} M & & \mathbf{k}[\mathbf{G}][\sigma] \otimes_{\sigma} (\mathbf{k}[\mathbf{G}][\sigma] \otimes_{\sigma}^{\text{op}} M) \\
 & \searrow \Delta_G^{\sigma} \otimes 1 & \downarrow \text{br} \\
 & & \mathbf{k}[\mathbf{G}][\sigma] \otimes_{\sigma} \mathbf{k}[\mathbf{G}][\sigma] \otimes_{\sigma} M
 \end{array}$$

Yetter-Drinfeld modules

- The map br in this diagram has some properties of the **braiding map** appearing in the context of **Yetter-Drinfeld** modules, but we do not have satisfactory understanding yet.
- We have not shown yet that the resulting category is Abelian with enough injectives (a Yetter-Drinfeld module interpretation would help).
- This “Attempt 2 notion” is **different** than the “Attempt 1 notion”! The second notion generalizes the first one only in the case of $\sigma_G = \text{id}_G$.

Comparison to the other difference representation theory

- There is a theory of representations of difference algebraic groups: Ovchinnikov/Wibmer, Kamensky, ...
- Lemma 3.1.2. from Wibmer's Habilitation "Affine Difference Algebraic Groups" amounts to saying that the category of difference representations (in their sense) of $\mathcal{G} = (\mathbf{G}, \sigma_{\mathbf{G}})$ is equivalent to the category of rational representations of \mathbf{G} . So, the cohomology groups are the same as the rational ones.
- The difference rational representations (again, in their sense) of $(\mathbf{G}, \sigma_{\mathbf{G}})$ coincide with the difference rational representations of $(\mathbf{G}, \text{id}_{\mathbf{G}})$, so this notion of a difference representation fits both into "Attempt 1 case" and into "Attempt 2 case".