Galois actions of finitely generated groups rarely have model companions
(joint work with Özlem Beyarslan)

Piotr Kowalski

Instytut Matematyczny
Uniwersytetu Wrocławskiego

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Existentially closed models

Let us fix a language L and let T be an L-theory.

Definition

Let $M \models T$. We say that M is an existentially closed (abbreviated e.c.) model of T, if for any quantifier free L_M-formula $\chi(x)$ and any L-extension $M \subseteq N$ of models of T, we have that:

$$N \models \exists x \chi(x) \quad \text{implies} \quad M \models \exists x \chi(x).$$

Intuitively, all solvable in an extension of M "systems of (in)equations" (parameters from M) can be already solved in M.

Example

- E.c. fields are algebraically closed fields.
- E.c. linear orders are dense linear orders without endpoints.

Inductive theories and model companion

Definition

A theory T is inductive, if for each chain of models of T, its union is also a model of T.

Classical results

1. A theory is inductive if and only if it can be axiomatized by $\forall\exists$-sentences.
2. Assume that T is inductive and $M \models T$. Then, there is an L-extension $M \subseteq N$ such that N is an e.c. model of T.

Definition

For an inductive L-theory T, we call an L-theory T^* a model companion of T if the class of models of T^* coincides with the class of e.c. models of T.

Model companions and non-companionable theories

1. The (empty) theory of sets has a model companion, which is the theory of infinite sets.
2. The theory of linear orders has a model companion, which is the theory of dense linear orders without endpoints.
3. The theory of fields has a model companion, which is the theory of algebraically closed fields.
4. The theory of fields with an automorphism has a model companion, which is called ACFA.
5. The theory of fields with a derivation has a model companion, which is called DCF.
6. The theory of commutative groups has a model companion, which is the theory of commutative divisible groups having infinitely many elements of order p for every prime p.
7. The theory of groups has no model companion.
8. The theory of commutative rings has no model companion.

Theory of G-fields

- Let us fix a group G.
- We use the following terminology: we call a pair consisting of a ring together with a G-action on this ring by a G-ring. Similarly, we consider G-fields, G-ring/G-field extensions, etc.
- We define the following language of G-rings:

$$L_G := L_{\mathrm{ring}} \cup \{\lambda_g \mid g \in G\},$$

where each λ_g is a unary function symbol.
- The theory of G-fields, abbreviated G-TF, is the following:

$$\text{Theory of fields} \cup \{\lambda_g \circ \lambda_h = \lambda_{gh} \mid g, h \in G\} \cup \{\lambda_e = \mathrm{id}_G\}$$

$$\cup \{\lambda_g \text{ is a field automorphism} \mid g \in G\}.$$

Existence of G-TCF

- The main question is:

  Does a model companion of the theory G-TF exist?

- If "Yes", then we call this model companion G-TCF and we say that "G-TCF exists".
- If $G = \mathbb{Z}$, then $\mathbb{Z}$-TF corresponds to the theory of difference fields and $\mathbb{Z}$-TCF exists (the theory ACFA).
- If $G = \mathbb{Z} \times \mathbb{Z}$, then $(\mathbb{Z} \times \mathbb{Z})$-TF corresponds to the theory of fields with two commuting automorphisms. Quite surprisingly, $(\mathbb{Z} \times \mathbb{Z})$-TCF does *not* exist (Hrushovski).
- If we drop the commutativity assumption (that is, we consider actions of the free group F_2), then a model companion exists. Similarly, for any free group F.
- If G is finite, then G-TCF exists (Sjögren and independently Hoffmann, K.).

Our mistake

- The following statement is true.

  If the group G is finite or free, then G-TCF exists.

- Therefore, Özlem and I asked: what about virtually free G? (That is: G has a free subgroup of finite index.)
- We also "answered".

Thm 3.26 *Model theory of fields w/ v.f. group actions* PLMS 2019

If G is finitely generated and virtually free, then G-TCF exists.

Unfortunately

- The proof of "Thm 3.26" above is wrong.
- The statement of "Thm 3.26" above is (very) false.

Why the proof is wrong?

We often used the following false claim:

A (fibered) product of K-irreducible K-varieties is K-irreducible.

It is true only when K is algebraically closed and when the product is not fibered.

Example

- Since
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C},$$
  it is not a domain!
  $\mathbb{C} \ \rightsquigarrow \ \mathrm{Spec}(\mathbb{R}[X]/(X^2+1))$: $\mathbb{R}$-irreducible $\mathbb{R}$-variety.
- The fibered product $\mathbb{A}^1 \times_{\mathbb{A}^1} \mathbb{A}^1$ with maps $x \mapsto x^2$ coincides with $\{(x, y) \in \mathbb{A}^2 \mid x^2 = y^2\}$ which is the union of two lines.

What *is* true?

Theorem (Beyarslan, K; BLMS; published online 7 Dec. 2023)

Assume that G is finitely generated and virtually free. Then: G-TCF exists if and only if G is finite or G is free.

Therefore, in all “new” cases G-TCF does not exist!

Strong negation

- If α is a sentence of the form $\forall x \varphi(x)$, then let us call the sentence $\forall x \neg\varphi(x)$ a strong negation of α.
- Names like a “common folk negation” or a “politician’s negation” were suggested as well.
- The BLMS statement is a strong negation of the PLMS statement if the universal quantifier means: “For all ‘new’ cases”.

Proof 1: profinite groups

- For a perfect field K, we denote by $\mathrm{Gal}(K)$ the absolute Galois group $\mathrm{Aut}(K^{\mathrm{alg}}/K)$ of K. It is a profinite group.
- We are interested in $\mathrm{Gal}(K)$ for e.c. G-fields (some fixed G).
- To describe them, we need the notion of a Frattini cover $f : \mathcal{G} \twoheadrightarrow \mathcal{H}$ (no proper closed $\mathcal{G}_0 < \mathcal{G}$ such that $f(\mathcal{G}_0) = \mathcal{H}$).
- There is a universal Frattini cover $\widetilde{\mathcal{G}} \to \mathcal{G}$, e.g. $\mathbb{Z}_p = \widetilde{\mathbb{Z}/p\mathbb{Z}}$.

Theorem (Sjögren)

If G is finitely generated and K is an e.c. G-field, then there is:

$$\mathrm{Gal}(K) \twoheadrightarrow \mathcal{K}_G := \ker\left(\widetilde{\widehat{G}} \to \widehat{G}\right),$$

where $\widehat{G}$ is the profinite completion of G.

Proof 2: two lemmas

Lemma 1

If G is finitely generated and $\mathcal{K}_G$ is *not* small (there is $n > 0$ s. t. there are infinitely many closed subgroups of $\mathcal{K}_G$ if index n), then $\mathrm{Gal}(K)$ is not small.

Lemma 2

If G is countable and K is an e.c. G-field, then $\mathrm{Gal}(K)$ is a separable topological space ($\mathrm{Gal}(K)$ has a countable dense subset).

Morally:

- Lemma 1 says that "$\mathrm{Gal}(K)$ is large";
- Lemma 2 says that "$\mathrm{Gal}(K)$ is not large".

Proof 3: Main result and strong negation

Theorem (Beyarslan, K.)

If G is finitely generated and $\mathcal{K}_G$ is not small, then G-TCF does not exist.

Proof

Assume that G-TCF exists. Lemma 1 gives an e.c. G-field K s.t.

$$|\operatorname{Gal}(K)| > \beth_2 := 2^{2^{\aleph_0}}.$$

Lemma 2 says that $|\operatorname{Gal}(K)| \leqslant \beth_2$, since $\beth_2$ is the maximal cardinality of a separable Hausdorff topological space.

The above theorem provide a general criterion for non-companionability which is not common (and nice!).

Proof 4: Strong negation

Corollary

If G is infinite, finitely generated, virtually free and not free, then G-TCF does not exist.

It follows from the last theorem and the following.

Lemma 3 (PLMS paper!)

If G is infinite, finitely generated, virtually free and not free, then $\mathcal{K}_G$ is not small.

- We "used" Lemma 3 in the PLMS paper to "show" how the (non-existing) theory G-TCF fits to Shelah's dividing lines.
- Now, the dividing lines are much stronger: existence vs non-existence.

Nilpotent case and Hrushovski's result

Theorem (Beyarslan, K.)

If N is finitely generated, infinite, nilpotent and not cyclic, then $\mathcal{K}_N$ is not small.

Corollary

For N as above, N-TCF does not exist. In particular, we get another proof of Hrushovski's result about the non-existence of $(\mathbb{Z} \times \mathbb{Z})$-TCF (or: the theory of fields with two commuting automorphisms has no model companion).

Questions and the commutative torsion case

- Assume that G is infinite and finitely generated. Is it true that:

  G-TCF exists if and only if G is free?

  It is true for G virtually free or for G nilpotent.
  Problematic cases: "strange groups" like Tarski monsters.
- Suppose that $H < G$ and G-TCF exists. Is it true that H-TCF exists?

Regarding other types of groups, we proved the following.

Theorem (Beyarslan, K.; *J. Inst. Math. Jussieu* 2023)

If A is a commutative torsion group, then:
A-TCF exists if and only if for each prime p, the p-primary part of A is either finite or it is the Prüfer p-group.