From Philosophical Logic to Computer Science — and back again

G. Aldo Antonelli

aldo.antonelli@uci.edu

Dept. of Logic & Philosophy of Science

University of California, Irvine

Logic Colloquium

Wroclaw, July 14-19, 2007

Outline

Philosophical Logic

- Philosophical Logic is not distinguished from Mathematical Logic either in methods or in subject matter (it is, in each case, the study of formal languages by mathematical methods), but rather in *inspiration*.
- Traditionally, Philosophical Logic derived its problems from the analysis of philosophical issues. In this, Philosophical Logic was part and parcel with the *linguistic turn* in philosophy — the idea that many traditional philosophical problems can be explained (or explained *away*) by linguistic analysis. Logic — this time *simpliciter* — provided the formal tools for philosophical analysis.
- This enterprise was not initially regarded as conceptually distinct from the application of logical tools to the analysis of mathematical reasoning. This unity was reflected in 1936 when the ASL was founded — it was all, perhaps redundantly, *symbolic logic*.

The logic of possibility and necessity

- Modal Logic has quintessentially philosophical origins in the study of the alethic modalities: *possibility* and *necessity*.
- Philosophers have dealt with modalities ever since Aristotle, but especially with Leibniz and Kant (both of whom recognized the duality of possibility and necessity).
- Modal *logic* began with Lewis & Langford's *Symbolic Logic* (!) (1932). L&L argued against Russell's use of the material conditional $A \supset B$ in *Principia* in favor of necessary implication $\Box(A \supset B)$.
- This has led to the development of the philosopher's favorite style ("plain vanilla") of (mono-) modal logic and its different system, K, T, B, $S4$, $S5$, $\ldots$

Modal Logics

- Just like there is more to quantifiers than $\exists$ and $\forall$, so there is more to modal logic than $\Box$ and $\Diamond$.
- One useful characterization of modal logic is that is perhaps the simplest way to describe *relational structures*, i.e., structures of the form (A, R), where $R \subseteq A^2$.
- Of course there are many ways to talk about relational structures, beginning with first- and second-order logic. Modal logic differs from all these by taking an *internal* viewpoint, i.e., by asking what the structure looks like *from within*.
- The difference is that not all of the structure (A, R) may be accessible from any given point $a \in A$. This expressive limitation has proved immensely fruitful.
- A further characterization is due to Tarski, who analyzed modal logic in terms of Boolean algebras with operators.

Transition systems

- Perhaps the deepest and broadest characterization regards modal logic as the theory of various kinds of *transitions* (represented by $\Box$) between *states* of a given system.
- $\Box A$ is true at state s iff A is true at every state $s' \leftarrow s$. (Dually for $\Diamond A$).
  - As an immediate consequence, the schema K is valid: $\Box(A \supset B) \supset (\Box A \supset \Box B)$.
- *Correspondence theory* is the characterization of given properties of $\rightarrow$ by means of linguistic schemata, e.g.:
  - Transitivity is characterized by the schema 4: $\Box A \supset \Box\Box A$
  - Euclideanness: $\forall s, t, u$: if $s \rightarrow t$ and $s \rightarrow u$ then $t \rightarrow u$ by the schema 5: $\Diamond A \supset \Box\Diamond A$
  - Converse well-foundedness (not a first-order condition!) by the Löb schema $\Box(\Box A \rightarrow A) \rightarrow \Box A$ (in the context of 4).

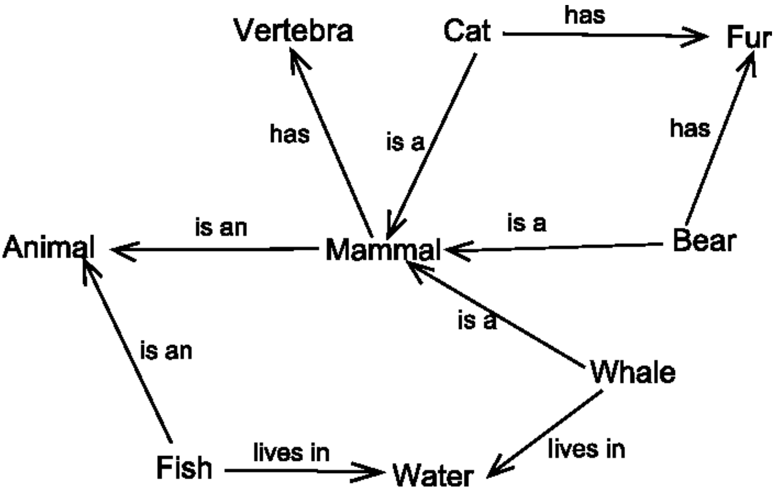
Beyond Basic Modal Logic

- The view of modal logic as the theory of state transitions subsumes other accounts:
  1. The alethic modality $\Box$ connects a state s to a state s' representing a state of affairs that is possible relative to s.
  2. The deontic modality $\bigcirc$ connects a state s to a state s' where all s-obligations are fulfilled.
  3. The epistemic modality **K** connects a state s to a state s' which is consistent with what the agent knows at s.
- The account can also be generalized along several different directions:
  1. Allowing more than one kind of transition (poly-modal logic or labeled transition systems);
  2. Constraining the number of out-going arrows from s (graded modalities);
  3. Using binary modalities such as $\mathit{until}(p, q)$.

Semantic Networks

- In the 1970's a number of *direct* (i.e., not logic-based) approaches were developed for the representation of specialized knowledge bases.
- Among these are *semantic networks*, where nodes refer to classes of individuals, edges represent IS-A (subsumption) links, as well as, possibly value restrictions.
- Such networks support assertions obtained by chaining through IS-A links, and they provide a simple yet powerful mechanism for knowledge representation.
- The problem is that such networks lack a well-defined *semantics*.

A semantic network



Description Logics

- *Description Logics*, initially known also as *terminological systems*, provide a mathematically precise representation for this kind of networks.
  - Description logics are used to provide "ontologies" for many different fields, from medicine, to software enginnering, to library science.
- The language of DL is built up from *concepts* $C, D, \ldots$ (1-place preds) and *roles* (2-place preds) $R, S, \ldots$ by means of several operations:

$$C, D \quad \rightarrow \quad \mathsf{c} \mid \top \mid C \sqcap D \mid \neg C \mid \forall R \,.\, C$$

- Notice that in this version of DL only atomic roles are allowed, but we have full negation.

Semantics for DL

- Given a non-empty, possibly infinite domain U, we define an interpretation $\mathscr{E}$ assigning subsets of U to atomic concepts and subsets of U^2 to (atomic) roles. The interpretation can then be lifted as follows:
  - $\mathscr{E}[\top] = U$
  - $\mathscr{E}[C \sqcap D] = \mathscr{E}[C] \cap \mathscr{E}[D]$
  - $\mathscr{E}[\neg C] = U \setminus \mathscr{E}[C]$
  - $\mathscr{E}[\forall R\,.\,C] = \{d \in U : \forall e \in U(\langle d, e\rangle \in \mathscr{E}[R] \rightarrow e \in \mathscr{E}[C])\}$
- We can then take $\exists$, $\sqcup$, and $\bot$ as defined ...
- ... or extend the language by *number restrictions*:

$$\mathscr{E}[\leq nR] = \{d : \mathsf{card}\{e : \mathscr{E}[R](d, e)\} \leq n\}$$

- and non-atomic, i.e., compound, roles (more about this later).

Examples in DL

- The set of all women: $\mathsf{Person} \sqcap \mathsf{Female}$
- The set of all parents: $\mathsf{Person} \sqcap \exists \mathsf{HasChild} \,.\, \top$
- The set of parents of only daughters: $\mathsf{Person} \sqcap \forall \mathsf{HasChild} \,.\, \mathsf{Female}$
- the set of all childless people: $\mathsf{Person} \sqcap \forall \mathsf{HasChild} \,.\, \bot$
- the set of parents of only children: $\mathsf{Person} \sqcap \exists \mathsf{HasChild} \,.\, \top \sqcap \leq 1 \mathsf{HasChild}$

Notice that all these statements are *variable-free*.

From Description Logic to Modal Logic

- It was first noticed by K. Schild (1991) that it is natural to interpret the domain U as a set of possible worlds and concepts C as *propositions*, i.e., sets of possible worlds at which the proposition holds.
- On this interpretation, the $\forall\,.$ operator of DL (with only atomic roles) becomes a modal operator and each atomic role r becomes an accessibility relation.
- This way we obtain a translation into $\mathbf{K}_m$, multi-modal K.

Axiomatizing DL

- Say that D *subsumes* C, written $\models C \sqsubseteq D$, iff $\mathscr{E}[C] \subseteq \mathscr{E}[D]$ for every interpretation $\mathscr{E}$.
  - Similarly say that C and D are *equivalent*, written $\models C = D$, iff $\mathscr{E}[C] = \mathscr{E}[D]$ for every interpretation $\mathscr{E}$.
  - Subsumption is reducible to equivalence for $C \sqsubseteq D$ iff $C \sqcap D = C$
- We are interested in axiomatizing the equational theory of DL (with atomic roles only).
- The translation into $\mathbf{K}_m$ immediately gives the following axioms:
  - Axioms forcing $(\top, \sqcap, \neg)$ to be a Boolean Algebra;
  - $\forall R\,.\,\top = \top$
  - $\forall R\,.(C \sqcap D) = (\forall R\,.\,C) \sqcap (\forall R\,.\,D)$.
  - From $\mathbf{K}_m$ we also obtain that subsumption is (not only decidable, but) PSPACE-complete.

Propositional Dynamic Logic

- PDL is based on Vaughan Pratt's idea to associate with each *non-deterministic* program α a distinct modality $[\alpha]$.
- PDL takes the idea of multiple modalities very seriously and applies it to the analysis and representation of computation.
- The language of PDL is built up from *formulas* and *programs*, recursively defined:

$$A, B \rightarrow \mathsf{a} \mid 0 \mid \neg A \mid A \vee B \mid [\alpha]A$$

$$\alpha, \beta \rightarrow \mathsf{p} \mid (\alpha\,;\beta) \mid (\alpha \cup \beta) \mid \alpha^* \mid A?$$

- The abbreviations 1, $\wedge$, $\rightarrow$, and $\langle\alpha\rangle$ are as usual.
- Standard programming constructs can be represented, e.g.:
  - **if** A **then** α **else** β as $((A?\,;\alpha) \cup ((\neg A?\,;\beta)$
  - **while** A **do** α as $((A?\,;\alpha)^*\,;\neg A?)$

Kripke semantics for PDL

- A model structure $\mathfrak{M}$ for PDL is a triple (W, V, R), where W is set of worlds, V maps atomic propositions into subsets of W and R maps atomic programs into subsets of W^2.
- V and R can be lifted to complex formulas and programs by simultaneous recursion. Here are the clauses for R:

$$\begin{aligned}
R(\alpha\,;\beta)(u,v) &\iff \exists w[R(\alpha)(u,w)\ \&\ R(\beta)(w,v)) \\
R(\alpha\cup\beta)(u,v) &\iff R(\alpha)(u,v) \text{ or } R(\beta)(u,v) \\
R(\alpha^*)(u,v) &\iff \langle u,v\rangle \text{ is in the trasitive} \\
&\qquad \text{closure of } R(\alpha) \\
R(A?)(u,v) &\iff u=v\ \&\ u\in V(A)
\end{aligned}$$

Axiomatizing PDL

- The following set of axioms is sound and complete for PDL:
  1. $[\alpha\,;\beta]A \leftrightarrow [\alpha][\beta]A$
  2. $[\alpha \cup \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$
  3. $[\alpha^*]A \leftrightarrow A \wedge [\alpha][\alpha^*]A$
  4. $[A?]B \leftrightarrow (A \rightarrow B)$
- with the following rule:

$$\frac{A \rightarrow [\alpha]A}{A \rightarrow [\alpha^*]A}$$

- The last axiom does *not* allow the eliminations of tests from PDL!
- Moreover, PDL has the finite model property and is therefore decidable. Satisfiability for PDL is in fact NEXPTIME-complete.

Variants of PDL

- There are many variants and extensions for PDL. We mention two.
- PDL with deterministic programs: then $R(\alpha)$ is a partial *function* on W.
  - DPDL is axiomatized by adding to PDL the single axiom schema:

$$\langle\alpha\rangle A \rightarrow [\alpha]A$$

- PDL with converse: we introduce a converse operator on accessibility relations, α^{-1}, where $R(\alpha^{-1})(u, v)$ holds iff $R(\alpha)(v, u)$.
  - An axiomatization is obtained by adding the axioms:

$$A \rightarrow [\alpha]\langle\alpha^{-1}\rangle A$$
$$A \rightarrow [\alpha^{-1}]\langle\alpha\rangle A$$

- PDL with converse is not significantly different in complexity from PDL.

Description Logics with Compound Roles

- We extend DL to allow for compound roles:

$$R, S \quad \rightarrow \quad \mathsf{r} \mid R \circ S \mid R \sqcup S \mid R^{*} \mid R^{-1} \mid \mathsf{id}(C)$$

  where $\mathsf{id}(C)$ is the identity on C, and extend the semantic interpretation function $\mathscr{E}$ accordingly.
  - The converse operator $^{-1}$ commutes with $\sqcup$, $\circ$, and * and so it only need be applied to atomic roles.
  - Also define $R^{+} := R \circ R^{*}$ and $\mathsf{self} := \mathsf{id}(\top)$.
- This version of DL is just a notational variant of PDL with converse — roles are re-interpreted as *non-deterministic programs*; e.g:
  1. $\forall R \,.\, C$: C holds whenever R terminates;
  2. $R_1 \circ R_2$: run R_1 and then R_2;
  3. $R_1 \sqcup R_2$: non-det'ly run one of R_1, R_2;
  4. R^{*}: non-det'ly pick $n \geq 0$ and run $\underbrace{R \circ \cdots \circ R}_{n \text{ times}}$.

Axiomatizing DL with Compound Roles

- The correspondence with converse PDL also gives an axiomatization:
  1. $\forall(R \sqcup S) . C = (\forall R . C) \sqcap (\forall S . C)$
  2. $\forall(R \circ S) . C = \forall R . \forall S . C$
  3. $\forall \mathsf{id}(C) . D = \neg C \sqcup D$
  4. $\forall R^* . C = C \sqcap \forall(R^+) . C$
  5. $C \sqcap \forall R^* .(\neg C \sqcup \forall R . C) \sqsubseteq \forall R^* . C$
  6. $C \sqsubseteq \forall \mathsf{r} \exists \mathsf{r}^{-1} . C$
  7. $C \sqsubseteq \forall \mathsf{r}^{-1} \exists \mathsf{r} . C$
- Just like converse PDL, DL with compound roles has the finite model property. Subsumption is therefore decidable, in fact decidable in NEXPTIME.
- Note that the finite model property is lost with *intersection* of roles:

$$\forall \mathsf{r}^* .((\exists \mathsf{r} . \top) \sqcap \forall(\mathsf{r}^+ \sqcap \mathsf{self}) . \bot$$

is an *axiom of infinity* giving an acyclic r-chain.

The modal μ-calculus

- The modal μ-calculus significantly extends PDL by introducing a *least fixed-point operator* μ. The result is strictly more expressive than PDL while still EXPTIME (-complete).
- Formulas are built up from propositional *variables* and propositional as well as program constants:

$$\phi, \psi \quad \rightarrow \quad x \mid 0 \mid \mathsf{p} \mid \phi \vee \psi \mid \neg\phi \mid \langle \mathsf{a} \rangle \phi \mid \mu x \,.\, \phi(x),$$

  the last clause with the proviso that X must occur positively in ϕ. Notice that only atomic programs are needed.
- We can define 1, $\wedge$, $\rightarrow$ and $[\mathsf{a}]$ as usual, as well *greatest fixed-points*:

$$\nu x \,.\, \phi(x) := \neg \mu x \,.\, \neg\phi(\neg x)$$

Semantics of the modal μ-calculus

- As in PDL, a model $\mathfrak{M}$ is a triple (W, V, R), with $V(0) = \emptyset$. A (partial) valuation assigns subsets of W to (some of) the variables $x, y, z, \ldots$
- If $\overline{A} = A_1, \ldots, A_n$ is a partial valuation (with each $A_i \subseteq W$) and $\phi(\bar{x})$ is a formula (with $\bar{x} = x_1, \ldots, x_n$), the extension $\phi^{\mathfrak{M}}(\overline{A})$ of ϕ is defined inductively by putting $x_i^{\mathfrak{M}}(\overline{A}) = A_i$ and $\mathsf{p}^{\mathfrak{M}}(\overline{A}) = V(\mathsf{p})$, and

$$\begin{aligned} \neg\phi^{\mathfrak{M}}(\overline{A}) &= W \setminus \phi^{\mathfrak{M}}(\overline{A}), \\ (\phi \vee \psi)^{\mathfrak{M}}(\overline{A}) &= \phi^{\mathfrak{M}}(\overline{A}) \cup \psi^{\mathfrak{M}}(\overline{A}), \\ \langle \mathsf{a} \rangle \phi^{\mathfrak{M}}(\overline{A}) &= \{w \in W : \exists v \in \phi^{\mathfrak{M}}(\overline{A}) \,.\, R(\mathsf{a})(v, w)\}, \\ \mu x \,.\, \phi(x)^{\mathfrak{M}}(\overline{A}) &= \bigcap \{B \subseteq W : \phi^{\mathfrak{M}}(B, \overline{A}) \subseteq B\} \end{aligned}$$

- The last line is justified because if x is positive in ϕ then $\phi^{\mathfrak{M}}$ defines a monotone operator over $\mathscr{P}(W)$, and so by the Knaster-Tarski theorem it has a least as well as a greatest fixed point.

Expressive power of the modal μ-calculus

- PDL is properly subsumed by the modal μ-calculus.
- *Some* fixed points can be represented in PDL, *viz.*, those of the form $\langle \mathsf{a}^* \rangle \phi$.
  - In fact, $\langle \mathsf{a}^* \rangle \phi$ is the least fixed point of the operator $\phi \vee \langle \mathsf{a} \rangle x$, and is therefore represented by

$$\mu x \,.\, \phi \vee \langle \mathsf{a} \rangle x$$

  - The operator $\phi \vee \langle \mathsf{a} \rangle x$ is continuous in x (commutes with $\bigcup$) and it therefore closes at the ordinal ω.
- Consider instead

$$\mu x \,.[\mathsf{a}]x = \{w : \text{there are no infinite a-paths out of } w\}$$

- The operator $[\mathsf{a}]x$ is not continuous: it closes at $\omega + 1$ on the tree of all sequences $\langle n, s \rangle$, where $n > 0$ and s is a finite word over a 1-letter alphabet of length $\leq n$, with $\langle\rangle$ as a top node.
- In fact $\mu x \,.[\mathsf{a}]x$ is not equivalent to any PDL-formula.

Second-order propositional bi-modal logic

- The language of 2S5 is obtained from propositional variables, connectives, two distinct S5-modalities, $\boxed{1}$ and $\boxed{2}$, and propositional quantifiers $\forall p$.
- A model $\mathfrak{M}$ is, as before, a tuple (W, R_1, R_2, V), where each R_i is an equivalence over W, and V assigns subsets of W to the propositional variables.
- Propositional quantifiers are given the standard interpretation, in that they range over $\mathscr{P}(W)$.
  - $\mathfrak{M}, w \models \forall p \phi$ iff $\mathfrak{M}[X/p] \models \phi$ for every $X \subseteq W$, where $\mathfrak{M}[X/p]$ assigns X to p and is otherwise like $\mathfrak{M}$.
- In 2S5 we can write, for instance:

$$\forall p \boxed{1} (p \rightarrow \exists q \langle 2 \rangle q)$$

The complexity of 2S5

- **The mono-modal case:** mono-modal S5 with propositional quantifiers can be interpreted into *monadic second-order logic* and is therefore *decidable* (as proved independently by D. Kaplan and K. Fine in 1970).
- **Asymmetric modal logics:** each of the modal logics K, B, T, K4, S4, when augmented with propositional quantifiers (with the standard interpretation) becomes equivalent to full second-order logic. The asymmetric nature of the accessibility relation allows to define a notion of order that can be used to represent second-order logic.
- **The bi-modal case**: The set of validities 2S5 is effectively equivalent to full second-order logic (under the standard interpretation), and hence not axiomatizable (Antonelli & Thomason, 2001). In this respect, the decidability of second-order S5 is an anomaly, and the poly-modal case is the more natural one.

The epistemic interpretation

- S5 is a natural first candidate for epistemic modalities.
  - S5 conveys that knowledge implies truth, as well as both positive (4) and negative (5) introspection:

$$\Box\phi \longrightarrow \phi$$
$$\Box\phi \longrightarrow \Box\Box\phi$$
$$\neg\Box\phi \longrightarrow \Box\neg\Box\phi$$

- In the case of multiple agents, each reasoning about the others' as well as his or her own knowledge, one needs multiple modalities.
- In fact, certain reasoning tasks require *common knowledge* (puzzle of the muddy children).

Common Knowledge

- Agents i and j have common knowledge (CK) of a proposition p iff the following proposition q is true:

$$p\text{, and both } i \text{ and } j \text{ know that } q.$$

- The inherent *circularity* can be analyzed iteratively as the infinite conjunction:

$$p \wedge \Box_i p \wedge \Box_j p \wedge \Box_i \Box_i p \wedge \Box_j \Box_j p \wedge \Box_i \Box_j p \wedge \Box_j \Box_i p \wedge \dots$$

- Agents i and j have common knowledge that p (at a world w) iff p is true at all worlds v that can be reached from w by a finite sequence of i- and j-links.
- Thus common knowledge requires the transitive closure of the R_i relations. Such transitive closure is *explicitly definable* in 2S5, which provides a natural framework for the formalization of this kind of reasoning tasks.

Conclusion

- We have come full circle: we started out with some quintessentially philosophical issues (the nature of possibility and necessity), then moved on to the analysis of transition systems of many different kinds.
- This has led us to re-evaluate the significance of the poly-modal framework and how distinctly different it can be from the mono-modal case.
- In turn, this leads to further extending the propositional language at the second-order, in search of more and more expressive resources.
- And finally we come to the application of the resulting logical framework to the analysis of philosophical problems, such as the nature of reflective knowledge.
- Thus we should re-evaluate well-entrenched distinctions between philosophical, mathematical, computational logic:

It's all just (symbolic) logic.