THE ALGEBRAIC STRUCTURE OF QUASI-DEGREES

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Definition (Tennenbaum)

A set A is quasi-reducible to a set B ($A \leq_Q B$), if there is a computable function g such that for all $x \in \omega$,

$$x \in A \Leftrightarrow W_{g(x)} \subseteq A.$$

Example

- If $A \leq_m B$ via computable function $f(x)$ then $A \leq_Q B$ via computable function $g(x)$ such that $W_{g(x)} = \{f(x)\}$
- If $A \leq_Q B$ via computable function $g(x)$ then $\omega - A \leq_e \omega - B$ via c.e. set $W = \{< x, 2^y > | x \in \omega, y \in W_{g(x)}\}$, i.e. $(\forall x)(x \in \omega - A \Leftrightarrow \exists u(< x, u >\in W \& D_u \subseteq \omega - B))$
- If a c.e. set $W \leq_Q A$ then $W \leq_T A$

Quasi-reducibility and algebra

Theorem (Dobritsa, unpublished)

For every set of natural number X there is a word problem with the same Quasi-degree as that of X.

Theorem (Belegradek, 1974)

For any computably presented groups G and H, if G is a subgroup of every algebraically closed group of which H is a subgroup, then G's word problem must be quasi-reducible to that of H.

Quasi-reducibility and Post's problem

Question (Post, 1944)

Does there exist a computably enumerable set A with $\emptyset <_T A <_T \emptyset'$?

Theorem (Degtev, 1973)

There exists a noncomputable semirecursive η*-maximal set.*

Theorem (Marchenkov, 1976)

- *No* $\eta-$*hyperhypersimple set is Q-complete.*
- *Let A be c.e. and semirecursive,* $B \leq_T A$. *Then* $B \leq_Q A$.

Corollary (Positive solution of Post's Problem)

There exists a computably enumerable set A with $\emptyset <_T A <_T \emptyset'$.

Quasi-reducibility and Algorithmic complexity

Theorem (Kummer, 1996)

Every Q-complete set A has hard instances.

Corollary (Kummer, 1996)

Every strongly effective simple set has hard instances.

Theorem (Batyrshin, 2006)

The set $\mathcal{K} = \{(x, n) |\ x \in 2^{<\omega}, n \in \omega, K(x) \leq n\}$ *is Q-complete.*

Corollary (Batyrshin, 2006)

The halting probability $\Omega_U = \sum_{x \in dom(U)} 2^{-|x|}$ *is Q-complete.*

The algebraic structure of Quasi-degrees

Theorem (Downey, LaForte, Nies, 1998)

There exists a noncomputable c.e. set A a c.e. B with $A \equiv_T B$ such that A and B form a minimal pair in the c.e Q-degrees.

Theorem (Downey, LaForte, Nies, 1998)

For every c.e. $C \not\equiv 0$ there exists an c.e. set A, which is non-branching in the c.e. Q-degrees, such that $C \not\leq_Q A$.

Theorem (Downey, LaForte, Nies, 1998)

For every pair of c.e. sets $B <_Q A$ there exists an c.e. set C with $B <_Q B \oplus C <_Q A$.

The algebraic structure of Quasi-degrees

Theorem (Arslanov, Omanadze, ta in 2007, IJM)

There exists an n-c.e set M of properly n-c.e. Q-degree.

Theorem (Arslanov, Omanadze, ta in 2007, IJM)

For any $n \geq 2$ there is a $(2n)$-c.e. set M of properly $(2n)$-c.e. Q-degree such that for any c.e. W, if $W \leq_Q M$ then W is computable.

Theorem (Arslanov, Omanadze, ta in 2007, IJM)

Let V be a c.e. set such that $V <_Q K$. Then there exist c.e. sets A and B such that $V <_Q A - B <_Q K$ and the Q-degree of $A - B$ does not contain c.e. sets.

The algebraic structure of Quasi-degrees

Theorem (Arslanov, Batyrshin, Omanadze, ta)

Let A and B be c.e. sets such that $A - B \not\equiv 0$. Then A is a disjoint union of c.e. sets A_0 and A_1 such that $A_i - B \leq_Q A - B$ and $A_i - B \not\leq_Q A_{1-i} - B, i = 0, 1$.

Corollary

Given a d.c.e set $A - B \not\equiv 0$ there exist two Q-incomparable d-c.e below it.

The algebraic structure of Quasi-degrees

Theorem (Arslanov, Batyrshin, Omanadze, ta)

There is a c.e. set $A <_Q K$ such that for all noncomputable c.e. sets W_e there is a noncomputable c.e. set X_e such that $X \leq_Q A$ and $X \leq_Q W_e$.

Theorem (Arslanov, Batyrshin, Omanadze, ta)

Let A be a c.e. set such that $K \not\leq_Q A$. Then there exist noncomputable c.e. sets A_0 and A_1 such that $A \oplus A_i \not\leq_Q A \oplus A_{1-i}, i = 0, 1,$ and A_0 and A_1 for a minimal pair in the c.e. Q-degrees.

The algebraic structure of Quasi-degrees

Theorem

For every pair of c.e. degrees $\mathbf{a} <_{\mathbf{Q}} \mathbf{b}$ *there exists a properly d.c.e. degree* $\mathbf{d}$, $\mathbf{a} <_{\mathbf{Q}} \mathbf{d} <_{\mathbf{Q}} \mathbf{b}$ *such that intervals* $(\mathbf{a}, \mathbf{d}]$ *and* $[\mathbf{d}, \mathbf{b})$ *don't contain c.e. degrees.*

Corollary

Given a c.e. degree $\mathbf{a}$ *with* $\mathbf{0} <_{\mathbf{Q}} \mathbf{a} <_{\mathbf{Q}} \mathbf{0}'$ *there exists a d.c.e degree* $\mathbf{d}$ *such that* $\mathbf{a} \not\leq_{\mathbf{Q}} \mathbf{d}$ *and* $\mathbf{d} \not\leq_{\mathbf{Q}} \mathbf{a}$.

The algebraic structure of Quasi-degrees

Theorem

For every pair of d.c.e. degrees $\mathbf{a} <_{\mathbf{Q}} \mathbf{b}$ *either the interval* $(\mathbf{a}, \mathbf{b})$ *don't contain c.e. degrees or there exists a d.c.e. degree* $\mathbf{d}$*,* $\mathbf{a} <_{\mathbf{Q}} \mathbf{d} <_{\mathbf{Q}} \mathbf{b}$ *such that intervals* $(\mathbf{a}, \mathbf{d}]$ *and* $[\mathbf{d}, \mathbf{b})$ *don't contain c.e. degrees.*

Corollary

For every d.c.e degree $\mathbf{a} > \mathbf{0}$ *there exist a d.c.e degree* $\mathbf{b} <_{\mathbf{Q}} \mathbf{a}$ *such that the interval* $[\mathbf{b}, \mathbf{a})$ *don't contain c.e. degrees.*