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Multiplicative quantifiers in fuzzy and substructural logics

Libor Běhounek

Joint work with **Petr Cintula** and **Rostislav Horčík**

Institute of Computer Science
Academy of Sciences of the Czech Republic

Substructural logics (of Ono 2003) = logics of residuated lattices

This talk focuses on the following subclass:

Deductive fuzzy logics = Ono's substructural logics with
(i) exchange (commutative conjunction)
(ii) prelinearity . . . $\models (\varphi \to \psi) \vee (\psi \to \varphi)$

They include the usual systems of t-norm fuzzy logics:
Łukasiewicz logic, Gödel–Dummett logic, Hájek's BL, . . .

Some definitions and results can be extended to broader classes of substructural logics

For simplicity, in this talk we assume
weakening and full propositional language (&,→,∧,∨,0,1)

Recall:

Substructural logics have *two* naturally defined conjunctions and disjunctions:

$\wedge$ . . . weak / lattice / "additive" conjunction

$$\varphi \otimes \psi \to \chi \;\equiv\; \varphi \to (\psi \to \chi)$$

$\otimes$ . . . strong / group / "multiplicative" conjunction

$$\varphi \wedge \psi \to \chi \;\equiv\; (\varphi \to \chi) \vee (\psi \to \chi)$$

$\varphi \otimes \psi$ = both φ and ψ

$\varphi \wedge \psi$ = any of φ and ψ

Denote $\underbrace{\varphi \otimes \ldots \otimes \varphi}_{n}$ by φ^n

First-order substructural logics:

Easy to define $\forall, \exists$ as the lattice infima and suprema $\bigwedge, \bigvee$

Rasiowa: *An Algebraic Approach to Non-Classical Logics*, 1974

$(\forall x)\varphi(x) \to \varphi(t)$ if t free for x in $\varphi(x)$

$\varphi(t) \to (\exists x)\varphi(x)$ "

$(\forall x)(\chi \to \varphi(x)) \to (\chi \to (\forall x)\varphi(x))$ if x not free in $\varphi(x)$

$(\forall x)(\varphi(x) \to \chi) \to ((\exists x)\varphi(x) \to \chi)$ "

$\varphi \ / \ (\forall x)\varphi$

Subtlety:

In incomplete lattices, the required $\bigwedge, \bigvee$ need not be defined

Logics of complete lattices need not be axiomatizable (BL, Ł)

$\Rightarrow$ use *Rasiowa's interpretations* = Hájek's *safe structures*

= those in which all necessary $\bigwedge, \bigvee$ exist

$\bigwedge, \bigvee$ are the *weak* quantifiers:

$$\vdash (\forall x)\varphi(x) \to \varphi(a) \wedge \varphi(b) \wedge \ldots$$

$$\not\vdash (\forall x)\varphi(x) \to \varphi(a) \otimes \varphi(b) \otimes \ldots$$

$\forall$ = ANY (rather than ALL):

$(\forall x)\varphi(x)$ implies *any* single instance of $\varphi(x)$,

but not *all* of them *at once* (ie, with $\otimes$)

Question: How should strong quantifiers be defined?

- Long-standing problem in substructural logics
- Without strong quantifiers,
  substructural quantification theory is incomplete
- First-order substructural logics with only weak quantifiers
  are viewed as a cheat by many

Requirements of *strong quantifiers*

(to be well-defined, well-behaved, and well-motivated)

- To be *universal,* a quantifier Π should satisfy:

$$\text{If } \models \varphi(x), \text{ then } \models (\Pi x)\varphi(x)$$

- To be *multiplicative,* Π should satisfy:

$$\models (\Pi x)\varphi(x) \to \bigotimes_{t \in M} \varphi(t) \text{ for any multiset } M \text{ of terms}$$

- To be semantically well-defined, the truth value of $(\Pi x)\varphi(x)$ in a model M should be determined by the truth values of $\varphi(a)$ for all individuals $a \in M$ (*truth-functionality*):

$$\|(\Pi x)\varphi(x)\|_{M,v} = F_{\Pi}(\{\langle a, \|\varphi(a)\|_{M,v}\rangle \mid a \in M\})$$

- It is natural to assume *monotony:*
  If $\|\varphi(a)\|_{M,v} \le \|(\psi(a)\|_{M,v}$ for all $a \in M$
  then $\|(\Pi x)\varphi(x)\|_{M,v} \le \|(\Pi x)\psi(x)\|_{M,v}$

On single-element universes, truth-functional quantifiers reduce to unary propositional connectives

$\Rightarrow$ Strong quantifiers generate unary connectives * such that

$\models \varphi^* \to \varphi^n$ for all n

if $\|\varphi\| \leq \|\psi\|$ then $\|\varphi^*\| \leq \|\psi^*\|$

if $\models \varphi$ then $\models \varphi^*$

We call them *exponentials* here

(cf. Girard's exponentials; better terminology?)

For a strong quantifier Π, define:

$$\varphi^{*\Pi} \equiv_{\text{df}} (\Pi x)\varphi \quad \text{if } x \text{ is not free in } \varphi$$

Vice versa, if * is an exponential, then

$(\Pi_* x)\varphi(x) \equiv_{\text{df}} [(\forall x)\varphi(x)]^*$ is a strong quantifier

not $(\forall x)\varphi^*(x)$

Examples:

- Girard's exponentials (! in linear logic):
  Introduced proof-theoretically
  Essentially, just $!\varphi \to \varphi$ and $!\varphi \to !\varphi \otimes !\varphi$ required
  Truth value: any $\otimes$-idempotent below φ
  not necessarily the weakest one
- Globalization
  $\Box x = 1$ iff $x = 1$, otherwise $\Box x = 0$
  Adding $\Box$ to a fuzzy logic need not yield a fuzzy logic
- Baaz Δ operator
  The strongest exponential preserving fuzziness
  Coincides with globalization in linear algebras
  Too strong unless Crisp(φ^*) is required
  (notice: conditions of Girard's ! satisfied by $\Box, \Delta$)

- Montagna's storage operator

(Journal of Logic and Computation, 2004)

$\varphi^\star$ = the largest $\otimes$-idempotent below φ

(in algebras where it exists)

However, exponentials need not be idempotent

$\Rightarrow$ still unnecessarily strong,

unless repeatable usage is required of $\varphi^\star$, too

$\varphi^\star \otimes \varphi^\star = \varphi^\star, \quad (\varphi^*)^* = \varphi^*$

Question:

optimal (ie, the weakest) exponential (or strong quantifier)...?

The condition of optimality of $*$ is expressed by
the infinitary rule $\{\psi \to \varphi^n \mid n \in \omega\} \vdash \psi \to \varphi^*$

This defines the optimal (weakest) exponential φ^ω
(as far as we know, not studied in fuzzy logic as yet)

The corresponding multiplicative quantifier:

$$(\Omega x)\varphi(x) \;\equiv_{\mathrm{df}}\; ((\forall x)\varphi(x))^\omega$$

In semantics: $\varphi^\omega \;=_{\mathrm{df}}\; \inf_{n\in\omega} \varphi^n$ (in "ω-safe" algebras)

Not every algebra can be extended with $^\omega$
(cf Chang's MV-algebra: co-infinitesimals have no inf),
but if it can, then $^\omega$ is its weakest exponential

Example:

$\varphi^\omega = \varphi^n$ in n-contractive logics (ie, such that $\models \varphi^n \to \varphi^{n+1}$)

In general, Montagna's * differs from $^\omega$

Counter-example by Montagna (2004)

If they exist,

$\varphi^\star$ is the nearest $\otimes$-idempotent below φ

φ^ω is the supremum of the first Archimedean class below φ

Recall: $^\omega$ is introduced by an infinitary rule

Question: Can it be axiomatized (or approximated) finitarily?

Consider an operator $\overline{\omega}$ with the following axioms and rules:

$$\vdash \varphi^{\overline{\omega}} \to \varphi$$

$$\vdash ((\varphi \to \varphi^{\overline{\omega}}) \to \varphi^{\overline{\omega}}) \vee (\varphi^{\overline{\omega}} \to (\varphi^{\overline{\omega}})^2)$$

$$\psi \to \varphi, ((\varphi \to \psi) \to \psi) \vee (\psi \to \psi^2) \vdash \psi \to \varphi^{\overline{\omega}}$$

Then $^{\omega}$ satisfies the rules for $^{\overline{\omega}}$

In semantics, $^{\overline{\omega}}$ coincides with $^{\omega}$ if the latter is defined

However, $^{\omega}$ need not be defined even if $^{\overline{\omega}}$ is

(in Chang's MV-algebra: $\varphi^{\overline{\omega}} = \Delta\varphi$, while φ^{ω} is undefined)

Recall: In semantics, quantifiers are fuzzy sets of fuzzy sets

Why:

- quantifiers are operators on predicates
- semantic values of predicates are fuzzy sets

$\Rightarrow$ quantifiers take fuzzy sets to truth values

$\Rightarrow$ quantifiers are fuzzy sets of fuzzy sets

Recall: Sets of sets is the domain of higher-order logic

Notice: A system of Henkin-style higher-order fuzzy logic
(based on the weak quantifiers $\forall, \exists$ only!)
has recently been developed

Behounek, Cintula: Fuzzy class theory. *Fuzzy Sets and Systems* 2004

$\Rightarrow$ Multiplicative quantifiers can conveniently be studied in higher-order fuzzy logic

Propositional fuzzy logic:

any well-behaved expansion of $\mathsf{MTL}_\triangle$

First-order fuzzy logic (with weak quantifiers only)

add Rasiowa's axioms for $\forall, \exists$, crisp identity $=$

Henkin-style second-order fuzzy logic

= theory in 1st-order fuzzy logic:

- Sorts of objects $(x, y, \ldots)$, fuzzy sets $(X, Y, \ldots)$, tuples
- Axioms for tuples (crisp)
- Primitive membership predicate $\in$
- Comprehension axioms $(\exists Z)(\forall x)\, \triangle(x \in Z \leftrightarrow \varphi)$ for all φ
- Extensionality axiom $(\forall x)\, \triangle(x \in A \leftrightarrow x \in B) \to A = B$

Henkin-style higher-order fuzzy logic: iterate for all orders

Intended models = fuzzy subsets of all orders in a domain V

Fact: The definition of the weakest exponential $^{\omega}$ can be internalized in higher-order fuzzy logic. The weakest multiplicative quantifier is thus definable in higher-order fuzzy logic.

Subtlety: Henkin-style $\Rightarrow$ non-standard models
$\Rightarrow$ possibly non-standard semantics of the defined notions

Moral:
The lattice quantifiers $\forall, \exists$ suffice for developing higher-order fuzzy logic, in which multiplicative quantifiers become definable

$\Rightarrow$ Multiplicative quantifiers need not be present as primitives in first-order fuzzy logic: they can be bypassed by using lattice quantifiers, developing higher-order fuzzy logic by means of the latter, and defining the former within its framework

A similar approach should work for other substructural logics