Strong Jump-Traceability

The Computably Enumerable Case

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Preprint available: Cholak, Downey, and Greenberg, *Strong Jump-Traceability I: the Computably Enumerable Case*.

Reals with little *value* as oracles

Are there any? How low do they go? Are they all the same?

Try to understand the relation between reals with *low initial segment complexity* as measured by Kolmogorov complexity and reals with *low computational power* (as measured by the halting set relative to the real).

Example: Loveland showed the a real α is computable iff the sequence $C(\alpha \upharpoonright n) - C(n)$ is bounded, where C is plain Kolmogorov complexity.

K-Trivial Reals

Reals with very low initial segment complexity

Definition

If the sequence $K(A \restriction n) - K(n)$ is bounded then A is K-*trivial*, where K is prefix-free Kolmogorov complexity.

Theorem (Chatin, Downey, Hirschfeldt, *Nies*, Solovay, Stephan)

The K-trivial reals form a robust nontrivial ideal of low Δ_2^0 degrees.

Cost Functions

How to build an K-trivial real. Or how do you prove your results.

Definition

The *cost* (or weight) of x at stage s is

$$c(x,s) = \sum_{x<n<s} 2^{-K_s(n)}.$$

Example: Define a computably enumerable set $A = \bigcup_s A_s$ by putting $x \in A_{s+1} - A_s$ if $W_{e,s} \cap A_s = \emptyset$, $x > 2e$, $x \in W_{e,s}$ and $c(x,s) < 2^{-(e+1)}$. Then A is simple and K-trivial.

C.e. Traceability

Computationally Feeble

Definition

- A (c.e.) *trace* is an uniformly c.e. sequence $\langle T_x \rangle$ of finite sets. (Equivalently there is a computable function g such that for all x, $T_x = W_{g(x)}$.)
- A trace *traces* a function f if for all x, $f(x) \in T_x$.
- A function $h \colon \omega \to \omega \setminus \{0\}$ is an *order* if h is computable, nondecreasing and $\lim_s h(s) = \infty$.
- The tracing *obeys* an order h if for all x, $|T_x| \leq h(x)$.
- A degree **a** is *c.e. traceable* if there is an order h such that every $f \leq_T$ **a** can be traced by some trace obeying h.

Theorem (Zambella)

If A is K-trivial then $\deg(A)$ *is c.e. traceable.*

Jump Traceable

More Computationally Feeble

Definition

A is *jump-traceable* if there is some order h and a c.e. trace $\langle T_x \rangle$ obeying h and tracing $\{e\}^X(e)$ (if $\{e\}^X(e)\downarrow$) then $\{e\}^X(e) \in T_e$).

Theorem (Nies)

Jump-traceability and superlowness are the same on the c.e. sets. There are non K-trivial jump traceable sets.

Theorem (Nies, Figueira, and Stephan)

If A is K-trivial, then A is jump traceable with respect to an order roughly $h(n) = n \log n$.

Strongly Jump Traceable

Even More Computationally Feeble

Definition

A is *strongly jump-traceable* iff $\{e\}^X(e)$ can be traced obeying any order.

Theorem (Nies, Figueira, and Stephan)

There are non-computable, strongly jump-traceable, computably enumerable reals. Strong jump-traceability is weaker than jump-traceability on the c.e. reals.

Question (Nies and Miller)

Is the class of K-trivials exactly the class of strongly jump traceable reals? Is strongly jump traceability a combinatorial characterization of K-triviality?

NO!

The c.e. strongly jump-traceable degrees form a proper subideal of the K-trivials.

Theorem

Every c.e. strongly jump-traceable set is K-trivial.

Theorem

There is a K-trivial c.e. set that is not *strongly jump-traceable. Indeed it is not jump traceable with a bound of size roughly $\log\log n$.*

Theorem

The c.e. strongly jump-traceable degrees form an ideal.

Corollary (to the proof of the first theorem above)

If a set A is jump-traceable with respect to about $\sqrt{\log n}$ then it is K-trivial.

An hierarchy of jump-traceability?

Or a possible combinatorial characterization of the K-trivials.

$\sqrt{\log n} < n \log n$.

Question

Is A K-trivial iff for all orders h with $\sum_{n\in\mathbb{N}} 2^{-h(n)} < \infty$, A is jump traceable with order h?