# On a new functional interpretation

FERNANDO FERREIRA
Universidade de Lisboa

LOGIC COLLOQUIUM 2007

Wrocław, July 17

at

Universitas Wratislaviensis

"This blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. It lies in a widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward non-finitary reasoning."

Kurt Gödel

"On a hitherto unutilized extension of the finitary standpoint"

K. Gödel (1958)

operation, always performable (and constructively recognized as such) on given computable functionals

as immediately intelligible

## Type (variables)

$0$ - type (of natural numbers)

$\tau \to \rho$ - if $\tau$ and $\rho$ are types

$x^\tau$ - variable of type $\tau$

$S^{0\to 0}$ - constant for the sucessor function

$$P^{\tau\to\tau},\ h^\tau,\ x^0 \xrightarrow[It_\tau]{} P^x(h)$$

$$It_\tau(P, h, Sx) =_\tau P(It_\tau(P, h, x))$$

Primitive recursive functionals in the sense of Gödel the mathematicians will probably raise objections against that, because contemporary mathematics is thoroughly extensional.

K. Gödel

$$Q^{\tau\to 0}\left( It_\tau \left(P, h\, Sx\right)\right) =_0 Q\left(P\left(It_\tau\left(P, h, x\right)\right)\right)$$

Gödel's <u>Dialectica</u> interpretation:

$$PA \rightsquigarrow HA \rightsquigarrow T$$

trade-off between quantifier-complexity and raising of types

$$PA^\omega \rightsquigarrow HA^\omega \rightsquigarrow T$$

$$A \rightsquigarrow A^D := \exists \underline{x} \forall \underline{y} \underbrace{A_D(\bar{x}, \bar{y})}_{\text{quantifier-free}}$$

$$(A \wedge B)^D := \exists x, z \forall y, w \left( \underline{A_D(x,y) \wedge B_D(z,w)} \right)$$

$$(A \vee B)^D := \exists n^0, x, z \forall y, w \left( \underline{(n=0 \to A_D(x,y)) \wedge} \right.$$
$$\left. \underline{\wedge (n \neq 0 \to B_D(z,w))} \right)$$

$$(\forall z A(z))^D := \exists f \forall z, y \ \underline{A_D(fz, y, z)}$$

$$(\exists z A(z))^D := \exists z, x \forall y \ \underline{A_D(x, y, z)}$$

$$B^D := \exists z \forall w \ B_D(z, w)$$

$$(A \to B)^D :=$$

$$\exists f, g\, \forall x, w \left( \underline{A_D(x, gxw) \to B_D(fx, w)} \right)$$

Motivation:

$$\exists x \forall y\, A_D(x, y) \longrightarrow \exists z \forall w\, B_D(z, w)$$

$$\forall x \left( \forall y\, A_D(x, y) \to \exists z \forall w\, B_D(z, w) \right)$$

$$\forall x \exists z \left( \forall y\, A_D(x, y) \to \forall w\, B_D(z, w) \right)$$

$$\forall x \exists z \forall w \left( \forall y\, A_D(x, y) \to B_D(z, w) \right)$$

$$\forall x \exists z \forall w \exists y \left( A_D(x, y) \to B_D(z, w) \right)$$

$$\exists f\, \forall x \forall w \exists y \left( A_D(x, y) \to B_D(fx, w) \right)$$

$$\exists f, g\, \forall x, w \left( A_D(x, gxw) \to B_D(fx, w) \right)$$

## Soundnes Theorem (Gödel '58)

If $HA^\omega \vdash A$ then there are closed

$HA \vdash A$

terms $t$ such that

$$HA^\omega \vdash \forall y \, A_D(t, y)$$

$$T \vdash A_D(t, y)$$

## Monotone functional interpretation (Kohlenbach '96)

$$HA^\omega \vdash \exists x \leq^* q \, \forall y \, A_D(x, y)$$

$q$ a closed monotone term.

weakening of conclusion

If $\quad HA^\omega + AC^\omega + IP^\omega_\forall + MP^\omega + \Delta \vdash A$

then $\quad \dots \quad HA^\omega + \Delta \vdash \forall y \; A_D(t, y)$

$AC^\omega$ : $\quad \forall x \exists y \; A(x,y) \to \exists f \forall x \; A(x, fx)$

$IP^\omega_\forall$ : $\quad (C_\forall \to \exists z \; B(z)) \to \exists z \, (C_\forall \to B(z))$

$MP^\omega$ : $\quad \neg \forall z \; A_{qf}(z) \to \exists z \neg A_{qf}(z)$

If $\quad HA^\omega + AC^\omega + IP^\omega_\forall + MP^\omega + \Delta \vdash \forall x \exists y \; A_{qf}(x,y)$

then $\quad HA^\omega + \Delta \vdash \forall x \exists y \; A_{qf}(x,y)$

Howard - Bezem strong majorizability:

$$x \leq_0^* y \quad \leftrightarrow \quad x \leq y$$

$$x \leq_{\tau\to\rho}^* y \quad \leftrightarrow \quad \forall v^\tau \, \forall u \leq_\tau^* v \, ( xu \leq_\rho^* yv \wedge yu \leq_\rho^* yv )$$

<u>Theorem</u> (Howard '73)

For each <u>closed</u> term $t$ there is a closed term $q$ such that $HA^\omega \vdash t \leq^* q$.

## Benefits of MFI (part I)

$\Delta$ : $\quad \forall b \, \exists u \leq r b \, \forall v \; B_{qf}(v, u, b)$

r closed term

## Benefits of MFI (part II)

Uniform bounds

$$\forall x^1 \, \forall z \leq_1 s x \, \exists y^0 \; A(x, y, z)$$

get

$$\forall x^1 \, \exists z \leq_1 s x \, \exists y \leq t x \; A(x, y, z)$$

# Proof Mining

# Benefits of MFI (part III)

applies to full second-order arithmetic

via Spector's 1962 far-fetched generalization of the Dialectica interpretation, using bar-recursion.

Uniform bounds still exist

# Bounded functional interpretation (FF, Oliva, 2005)

Intensional (rule-governed) majorizability

$$x \trianglelefteq_0 y \quad \leftrightarrow \quad x \leq y$$

$$x \trianglelefteq_{\tau \to \rho} y \quad \to \quad \forall v^{\tau} \forall u \trianglelefteq_{\tau} v \ (xu \trianglelefteq_{\rho} yv \wedge yu \trianglelefteq_{\rho} yv)$$

$$\frac{A_{bd} \wedge u \trianglelefteq v \longrightarrow su \trianglelefteq tv \wedge tu \trianglelefteq tv}{A_{bd} \longrightarrow s \trianglelefteq t}$$

$$\underbrace{\begin{aligned} &\forall x \trianglelefteq t\ (\ldots) \quad \leftrightarrow \quad \forall x\,(x \trianglelefteq t \to \ldots) \\ &\exists x \trianglelefteq t\ (\ldots) \quad \leftrightarrow \quad \exists x\,(x \trianglelefteq t \wedge \ldots) \end{aligned}}_{\text{new syntax}}$$

$$A \rightsquigarrow A^B := \tilde{\exists} \underline{x}\, \tilde{\forall} \underline{y}\ \underbrace{A_B(\underline{x}, \underline{y})}_{\text{bounded}}$$

$$(A \vee B)^B := \tilde{\exists} x, z\, \tilde{\forall} y, w\, \big( \underline{\tilde{\forall} y' \trianglelefteq y\ A_B(x, y') \vee}$$
$$\underline{\vee\ \forall w' \trianglelefteq w\ B_B(x, w')} \big)$$

$$\left(\forall z \trianglelefteq t\ A(z)\right)^B := \tilde{\exists} x\, \tilde{\forall} y\, \underline{\forall z \trianglelefteq t\ A_B(x, y, z)}$$

$$\left(\exists z\ A(z)\right)^B := \tilde{\exists} z, x\, \tilde{\forall} y\, \underline{\exists z' \trianglelefteq z\, \tilde{\forall} y' \trianglelefteq y\ A_B(x, y', z')}$$

$$\vdots$$

$\tilde{\exists} x\, (\cdots)$ is $\exists x\, (x \trianglelefteq x \wedge \cdots)$

$\tilde{\forall} x\, (\cdots)$ is $\forall x\, (x \trianglelefteq x \rightarrow \cdots)$

## Soundness Theorem (Oliva, FF, 2005)

If $HA^\omega_\trianglelefteq + \text{PRINCIPLES} + \Delta \vdash A$ then there are closed monotone terms $t$ such that

$$HA^\omega_\trianglelefteq + \Delta_{weak} \vdash \tilde{\forall} y\, A_B(t, y)$$

Expecting: $\exists w \trianglelefteq t\, \tilde{\forall} y\, A_B(w, y)$

## Conservation result

$HA^\omega_\trianglelefteq + \text{PRINCIPLES}$ is $\Pi^0_2$-conservative over $HA^\omega_\trianglelefteq$. Actually, over $HA^\omega$ (even $HA$).

The PRINCIPLES include:

- $mAC^{\omega}_{\trianglelefteq}$ : $\tilde{\forall} x \tilde{\exists} y\ A(x,y) \rightarrow \tilde{\exists} f\ \tilde{\forall} x\ \tilde{\exists} y \trianglelefteq fx\ \ A(x,y)$

- $bC^{\omega}_{\trianglelefteq}$: $\forall z \trianglelefteq x\ \exists y\ A(x,y) \longrightarrow \tilde{\exists} w\ \forall z \trianglelefteq x\ \exists y \trianglelefteq w\ \ A(x,y)$

FAN:

$$\forall z \leq_1 1\ \exists m^0\ A(z,m) \longrightarrow \exists m^0\ \forall z \leq_1 1\ \exists m \leq m\ \ A(z,m)$$

(for extensional $A$)

- $bBCC^{\omega}_{\trianglelefteq}$: $\tilde{\forall} w\ \exists z \trianglelefteq x\ \forall y \trianglelefteq w\ A_{bd}(x,y) \longrightarrow \exists z \trianglelefteq x\ \forall y\ A_{bd}(x,y)$

ideal element
uniform for each $y$

WKL follows

- MAJ : $\forall x\ \exists y\ \ x \trianglelefteq y$

- LLPO

$$\forall m^0, k^0 \left( A_{qf}(m) \vee B_{qf}(k) \right) \to \forall m\, A_{qf}(m) \vee \forall k\, B_{qf}(k)$$

- <u>Refutes</u> LPO

$$\forall x^1 \left( \exists m^0\, x(m) = 0 \quad \vee \quad \forall m\, x(m) \neq 0 \right)$$

- <u>Refutes</u> extensionality

$$\forall \Phi^2\, \forall x^1, y^1 \left( \forall k^0 \left( x(k) = y(k) \right) \to \Phi(x) =_0 \Phi(y) \right)$$

- Every extensional function from the Cantor space to $\mathbb{N}$ is uniformly continuous.

<u>Lemma</u> (Flattening) If $HA^{\omega}_{\trianglelefteq} \vdash A$ then $HA^{\omega} \vdash A^{*}$.

- The flattening of $HA^{\omega}_{\trianglelefteq}$ + PRINCIPLES is <u>inconsistent</u>.

# Second-order arithmetic

The intuitionistic theory

$$WKL_0 + LLPO + IP_\forall + MP + AC^{\mathbb{N}} + FAN$$

is $\Pi^0_2$-conservative over $PRA$.

$AC^{\mathbb{N}}$ : $\forall n \exists k\, A(n,k) \rightarrow \exists X\, (Func(X) \wedge \forall n\, A(n, X(n)))$

FAN : $\forall X \exists n\, A(X,n) \rightarrow \exists k \forall X \exists n \le k\, A(X,k)$

$n$ type 0

$X$ type 1 $x \leq_1 1$

$m \in X$ $x(m) = 0$

$$\mathrm{WKL}_0 + \mathrm{LLPO} + \mathrm{IP}_\forall + \mathrm{MP} + \mathrm{AC}^{\mathbb{N}} + \mathrm{FAN}$$

$$\rightsquigarrow$$

$$\mathrm{PRA}^\omega_\trianglelefteq + \text{PRINCIPLES}$$

$$\rightsquigarrow \quad \Pi^0_2$$

$$\mathrm{PRA}^\omega_\trianglelefteq$$

$$\rightsquigarrow$$

$$\mathrm{PRA}^\omega$$

$$\rightsquigarrow$$

$$\mathrm{PRA}$$

H. Friedman: $\mathrm{WKL}_0$ is $\Pi^0_2$-conservative over $\mathrm{RCA}_0$.

# Injecting uniformities into Peano arithmetic

$$\tilde{\forall} b\, \tilde{\exists} c\; A_{bd}(b,c) \;\rightarrow\; \tilde{\exists} f\, \tilde{\forall} b\, \tilde{\exists} c \le f(b)\; A_{bd}(b,c)$$

$$\forall z \le c\, \exists y\; A_{bd}(y,z) \;\rightarrow\; \tilde{\exists} b\, \forall z \le c\, \exists y \le b\; A_{bd}(y,z)$$

$$\forall x\, \exists y\, (x \le y)$$

into mathematics

- F. Ferreira & P. Oliva
  Bounded Functional Interpretation
  APAL 135, pp. 73-112 (2005)

- F. Ferreira
  Proof Interpretations
  Textos de Matemática (série B), nº 38
  Universidade de Coimbra (2006), 42 pp
  http://www.ciul.ul.pt/~ferferr/Proofinterpretations.pdf

- U. Kohlenbach & P. Oliva
  Proof Mining: a Systematic Way of Analysing Proofs
  Proceedings of the Steklov Institute of Mathematics 242, pp. 136-164 (2003)

- U. Kohlenbach
  Applied Proof Theory: Proof Interpretations and their Use in Mathematics

  to appear in Springer Monographs in Mathematics