Complexity of models of fuzzy predicate logics with witnessed semantics

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The basic fuzzy propositional calculus.

The real unit interval $[0,1]$ is taken to be the *standard set of truth values*; *comparative notion of truth.*

Continuous t-norms are taken as possible truth functions of *conjunction.*

Binary operation $*$ on $[0,1]$ is a t-norm if it is commutative ($x*y = y*x$), associative ($x*(y*z) = (x*y)*z$), non-decreasing in each argument (if $x \leq x'$ then $x*y \leq x'*y$ and dually) and 1 is a unit element ($1*x = x$).

$$
\begin{array}{ll}
x * y = \max(0, x+y-1) & (\text{Łukasiewicz } t\text{-norm}), \\
x * y = \min(x, y) & (\text{Gödel } t\text{-norm}), \\
x * y = x \cdot y & (\text{product } t\text{-norm}).
\end{array}
$$

The truth function of *implication* is the *residuum* of the corresponding t-norm.

$$x \Rightarrow y = \max\{z | x * z \leq y\}.$$

$$x \Rightarrow y = 1 \text{ iff } x \leq y; \text{ for } x > y$$

$$\begin{array}{ll} x \Rightarrow y = 1 - x + y & \text{(Łukasiewicz)}, \\ x \Rightarrow y = y & \text{(Gödel)}, \\ x \rightarrow y = y/x & \text{(product)}. \end{array}$$

negation $(-)x = x \Rightarrow 0$ $(-)x = 1 - x$ for Łukasiewicz, Gödel and product: $(-)0 = 1$, $(-)x = 0$ for $x > 0$

Basic propositional fuzzy logic BL:

propositional variables $p, q, \ldots$
connectives $\&, \rightarrow$, truth constant $\bar{0}$
Given a continuous t-norm $*$ (and its residuum $\Rightarrow$), each evaluation of variables extends to an evaluation of all formulas.
$*$-tautology: a formula φ such that $e_*(\varphi) = 1$ for each evaluation e.
t-tautology: $*$-tautology for each continuous t-norm $*$.

Axioms for connectives:

(A1)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2)	$(\varphi \& \psi) \rightarrow \varphi$
(A3)	$(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
(A4)	$(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi))$
(A5a)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
(A5b)	$((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6)	$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7)	$\bar{0} \rightarrow \varphi$

Deduction rule: modus ponens.

Łukasiewicz logic BL + $\neg\neg\varphi \to \varphi$
Gödel logic G: BL + $\varphi \to (\varphi \& \varphi)$
product logic Π: BL + $(\varphi \to \neg\varphi) \to \neg\varphi$ +
$\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$

We write
$\neg\varphi$ for $\varphi \to \bar{0}$,
$\varphi \wedge \psi$ for $\varphi \& (\varphi \to \psi)$,
$\varphi \vee \psi$ for $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$

Truth function of $\neg$: $\neg x = 1 - x$ for Łukasiewicz, $\neg 0 = 1$, $\neg x = 0$ for x positive – Gödel, product (Gödel negation)
Truth function of $\wedge$, $\vee$ is minimum, maximum for each $*$.

Standard Completeness: BL proves exactly all t-tautologies.
Ł proves exactly all $[0,1]_{Ł}$-tautologies.
G proves exactly all $[01,]_{G}$-tautologies.
Π proves exactly all $[0,1]_{\Pi}$-tautologies.
(Cignoli-Esteva-Godo-Torrens)

General semantics.

A BL-*algebra* is a residuated lattice

$$\mathbf{L} = (L, \leq, *, \Rightarrow, 0_L, 1_L)$$

satisfying two additional conditions:

$$x \cap y = x * (x \Rightarrow y),$$

$$(x \Rightarrow y) \cup (y \Rightarrow x) = 1_L$$

$[0,1]_{Ł}$, $[0,1]_G$, $[0,1]_{\Pi}$ – Łukasiewicz, Gödel and product t-algebra respectively.

Theorem *strong completeness* (for provability in theories over BL): For each theory T over BL, T proves φ iff for each [linearly ordered] BL-algebra $\mathbf{L}$, φ is true in all $\mathbf{L}$-models of T. (Here e is an $\mathbf{L}$ model of T if $e_{\mathbf{L}}(\alpha) = 1_{\mathbf{L}}$ for each axiom α of T.)

Basic fuzzy predicate calculus BL$\forall$:

Predicates, variables, connectives, quantifiers $\forall, \exists$.

Axioms for quantifiers:

$$
\begin{array}{ll}
(\forall 1) & (\forall x)\varphi(x) \to \varphi(y) \\
(\exists 1) & \varphi(y) \to (\forall x)\varphi(x) \\
(\forall 2) & (\forall x)(\chi \to \psi) \to (\chi \to (\forall x)\psi) \\
(\exists 2) & (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi) \\
(\forall 3) & (\forall x)(\varphi \vee \chi) \to ((\forall x)\varphi \vee \chi)
\end{array}
$$

Ł$\forall$, G$\forall$, $\Pi\forall$, BL$\forall$

Given a BL-algebra **L**, an **L**-*interpretation* is a structure $\mathbf{M} = (M, (r_P)_{P\ predicate})$ where $M \neq \emptyset$ and for each predicate P of arity n, r_P is an n-ary **L**-fuzzy relation on M, i.e. $r_P : M^n \to \mathbf{L}$.
$\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ – Tarski style conditions,
$\|P(x,y)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(v(x), v(y))$,
$\|\varphi \& \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} * \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$,
$\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$,
$\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'} | v' \equiv_x v\}$
$\|(\exists x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} | v' \equiv_x v\}$
This is always defined if **L** is a t-algebra (all infima and suprema exist). For a general BL-algebra **L** we call **M** **L**-*safe* if all truth values $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ are well defined. For closed φ write $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$.

A closed formula φ of predicate logic is an **L**-*tautology* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$ for all **L**-safe **M**. φ is **L**-*satisfiable* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$ for some **L**-safe **M**.
φ is a *general* BL*-tautology* if φ is an **L**-tautology for each **linearly ordered** BL-algebra (for each BL-chain).
φ is a *standard* BL-tautology (or a t-tautology) if it is a tautology for each t-algebra $[0,1]_*$.
Generally BL-satisfiable, standardly BL-satisfiable – obvious.

Theorem (Completeness). Let T be a theory over BL∀, let φ be a formula, $T \vdash \varphi$ (over BL∀) iff φ is true in all **L**-models of T, **L** being an arbitrary BL-chain.

$(\mathbf{M}, \ominus)$ is **witnessed** if for each formula $\varphi(x, y, \ldots)$ and each $b, \ldots \in M$,
$\|(\forall x)\varphi(x, b, \ldots)\|^{\ominus}_{\mathbf{M}} = \min_a \|\varphi(a, b, \ldots)\|^{\ominus}_{\mathbf{M}}$,
$\|(\exists x)\varphi(x, b, \ldots)\|^{\ominus}_{\mathbf{M}} = \max_a \|\varphi(a, b, \ldots)\|^{\ominus}_{\mathbf{M}}$,
(I.e. there is an a with minimal (maximal) value of $\|\varphi(a, b, \ldots)\|$.)

Theorem 1. Over Ł∀ with standard semantic, each countable model **M** is an elementary submodel of a witnessed model $\mathbf{M}'$ (i.e. for each α,
$\|\alpha\|^{Ł}_{\mathbf{M}} = \|\alpha\|^{Ł}_{\mathbf{M}'}$).

But e.g. for standard Gödel – example:
$M = \{1, 2, \ldots\}, r_P(n) = \frac{1}{n+1}$.
Not witnessed: $\|(\forall x)P(x)\| = 0$, satisfies
$\neg(\forall x)\varphi(x) \& \neg(\exists x)\neg\varphi(x)$
(not elem. embed. into witnessed).

H.–Cintula: On theories and models in fuzzy logic, JSL:

Axiom schemas:

$(C\forall)$ $(\exists x)(\varphi(x) \to (\forall y)\varphi(y))$

$(C\exists)$ $(\exists x)((\exists y)\varphi(y) \to \varphi(x))$

For logic $\mathcal{L}\forall$, $\mathcal{L}\forall^w$ is $\mathcal{L}$ extended by $(C\forall), (C\exists)$.

Theorem 2. (1) $(\mathbf{M}, \ominus)$ is elementarily embeddable into a witnessed model iff $(C\forall), (C\exists)$ are true in $(\mathbf{M}, \ominus)$.

(2) For our logics $\mathcal{L}$, the logic $\mathcal{L}\forall^w$ is strongly complete w.r.t. witnessed models.

16 classes of formulas
for each predicate calculus:
$\{-, w\}$ arbitrary $\times$ witnessed models
$\{St, Gen\}$ standard $\times$ general semantics
$\{1, Pos\}$ designated: 1 x positive values
$\{Taut, Sat\}$ tautologies, satisfiable.

E.g. $Gen1Taut$(Ł)
$wStPosSat$(Π)
etc.

Also:
BoolTaut, BoolSat

Plan:
– some general theorems
– Tables showing, for given $*$,
  equality of some classes,
  arithmetical complexity,
– conclusion, problems.

Some theorems

Theorem 3. Each logic $\mathcal{L}\forall^w$ has prenex normal form theorem: each formula is logically equivalent to a prenex formula.

Theorem 4. For each $*$, $Gen1Taut(*)$ and $wGen1Taut(*)$ are Σ_1 (complete), $Gen1Sat(*)$ and $wGen1Sat(*)$ are Π_1 (complete).

Theorem 5. $PC(*)\forall$ proves $C\exists, C\forall$ iff $*$ is Łukasiewicz.

Tables

Given $\mathcal{L}$ – 16 sets of formulas. Are some of them equal? What is their arithmetical complexity?
Ł, G, Π, Ł⊕, Gödel negation.
Notation:

	stand		gen	
	1	Pos	1	Pos
$Taut$	A	C	E	G
Sat	B	D	F	H
$wTaut$	I	K	P	R
$wSat$	J	L	Q	S

Furthermore, X is the set of all classical (Boolean) tautologies and Y the set of all classically satisfiable formulas. Note: $(\exists x)P_1(x) \in$ all $Sat, \notin$ any $Taut$.
In all cases, E and P are in Σ_1; moreover, F and Q are inΠ_1. Moreover, G and R are in Σ_1 and H and S are in Π_1.

Łukasiewicz

	$St1$	$StPos$	$G1$	$GPos$
$Taut$	A	C	E	C
Sat	B	D	B	H
$wTaut$	A	C	E	C
$wSat$	B	D	B	H
$Taut$	$\Pi_2 c$	$\Sigma_1 c$	$\Sigma_1 c$	$\Sigma_1 c$
Sat	$\Pi_1 c$	$\Sigma_2 c$	$\Pi_1 c$	$\Pi_1 c$
$wTaut$		the	same	
$wSat$		as	above	

$A \neq E, D \neq H$ from arithm.

$A \neq C, C \neq E - (\forall x)(P_x \vee \neg P_x)$

$B \neq D, B \neq H - (\exists x)(P_x \wedge \neg P_x)$

Gödel

	$St1$	$StPos$	$G1$	$GPos$
$Taut$	A	C		
Sat	B	B		the
$wTaut$	I	X		same
$wSat$	Y	Y		
$Taut$	$\Sigma_1 c$	$\Sigma_1 c$		
Sat	$\Pi_1 c$	$\Pi_1 c$		the
$wTaut$	$\Sigma_1 c$	$\Sigma_1 c$		same
$wSat$	$\Pi_1 c$	$\Pi_1 c$		

$A \neq I - (C\exists, C\forall)$

Product

	$St1$	$StPos$	$G1$	$GPos$
$Taut$	A	C	E	G
Sat	B	D	F	H
$wTaut$	I	X	P	X
$wSat$	Y	Y	Y	Y
$Taut$	NA	NA	$\Sigma_1 c$	$\Sigma_1 c$
Sat	NA	NA	$\Pi_1 c$	$\Pi_1 c$
$wTaut$	Π_2-hard	$\Sigma_1 c$	$\Sigma_1 c$	$\Sigma_1 c$
$wSat$	$\Pi_1 c$	$\Pi_1 c$	$\Pi_1 c$	$\Pi_1 c$

Ł$\oplus$

	$St1$	$StPos$	$G1$	$GPos$
$Taut$	A	C	E	C
Sat	B	D	B	H
$wTaut$	I	C	P	C
$wSat$	B	D	B	H
$Taut$	Π_2-hard	$\Sigma_1 c$	$\Sigma_1 c$	$\Sigma_1 c$
Sat	$\Pi_1 c$	$\Sigma_2 c$	$\Pi_1 c$	$\Pi_1 c$
$wTaut$	Π_2-hard	$\Sigma_1 c$	$\Sigma_1 c$	$\Sigma_1 c$
$wSat$	$\Pi_1 c$	$\Sigma_2 c$	$\Pi_1 c$	$\Pi_1 c$

(Composed t-norms with Gödel negation)

	$St1$	$StPos$	$G1$	$GPos$
$Taut$	A	C	E	G
Sat	B	D	F	H
$wTaut$	I	X	P	X
$wSat$	Y	Y	Y	Y
$Taut$			$\Sigma_1 c$	$\Sigma_1 c$
Sat			$\Pi_1 c$	$\Pi_1 c$
$wTaut$		$\Sigma_1 c$	$\Sigma_1 c$	$\Sigma_1 c$
$wSat$	$\Pi_1 c$	$\Pi_1 c$	$\Pi_1 c$	$\Pi_1 c$

For $\Pi\oplus$: A, B, C, D are non-arithmetical.
For $G\oplus$: A is Π_2-hard, B is Π_1(-complete), C is Σ_1(-compl.), $D = B$ is Π_1(-compl.)
(Montagna's results)

Fuzzy modal logic(s) S5.
The logic $S5(\mathcal{L})$ ($\mathcal{L}$ a fuzzy propositional logic extending BL). The language: that of propositional calculus extended by modalities $\Box, \Diamond$. Kripke models: $K = (W, e, A)$ where W is a set of possible worlds, A is a BL-chain and $e(p, w) \in A$ for each prop. variable p and possible world w. This extends to $e(\varphi, w)$ for each formula φ using the algebra A of truth functions of connectives and $\Box, \Diamond$ work as universal and existential quantifier over possible worlds: $e(\Box\varphi, w) = \inf_{v \in W} e(\varphi, v)$, and similarly for $\Diamond$, sup. The model is safe if e is total. We also write $\|\varphi\|_{K,w}$ for $e(\varphi, w)$.

Formulas of $S5(\mathcal{L})$ are in the obvious one-one isomorphic correspondence with formulas of the monadic predicate calculus $m\mathcal{L}\forall$ with unary predicates and just one object variable x, the atomic formula $P_i(x)$ corresponding to propositional variable p_i and modalities corresponding to quantifiers.

Axioms for $S5(\mathcal{L})$ (from my book) (ν is a propositional combination of formulas beginning by $\Box$ or $\Diamond$):

($\Box$1) $\Box\varphi \to \varphi$

($\Diamond$1) $\varphi \to \Diamond\varphi$

($\Box$2) $\Box(\nu \to \varphi) \to (\nu \to \Box\varphi)$

($\Diamond$2) $\Box(\varphi \to \nu) \to (\Diamond\varphi \to \nu)$

($\Box$3) $\Box(\nu \vee \varphi) \to (\nu \vee \Box\varphi)$

The problem whether the above axioms for $S5(\mathcal{L})$ are complete remained open in the book.

Theorem. The modal logic $S5(\mathcal{L})$ is strongly complete with respect to its general semantics.

Definition. (1) A Kripke model $K = (W, e, A)$ is witnessed if for each modal formula φ the truth value $\|\Box\varphi\|_K$ is the minimum of the truth values $\|\varphi\|_{K,w}$ $(w \in W)$ and similarly truth value $\|\Diamond\varphi\|_K$ is the maximum of the truth values $\|\varphi\|_{K,w}$ $(w \in W)$. A $w \in W$ such that $\|\Box\varphi\|_K = \|\varphi\|_{K,w}$ is called a witness for $\Box\varphi$ (in K); similarly for $\Diamond\varphi$.
(2) In $S5(\mathcal{L})$ introduce the following axiom schemata:
$(C\Box)$ $\Diamond(\varphi \to \Box\varphi)$,
$(C\Diamond)$ $\Diamond(\Diamond\varphi \to \varphi)$.
$S5(\mathcal{L})^w$ is the extension of the logic $S5(\mathcal{L})$ by these two axiom schemata.

Theorem The logic $S5(\mathcal{L})^w$ is strongly complete with respect to witnessed Kripke models as well as to finite Kripke models. For each logic $\mathcal{L}$ in question the set $TAUT(S5(\mathcal{L})^w)$ of all tautologies of $S5(\mathcal{L})^w$ is decidable and so is the the set $SAT(S5(\mathcal{L})^w)$ of its satisfiable formulas.

Summary - moral?

t-norm based fuzzy predicate logic ($BL\forall$ and variants) is a rich and well behaved many-valued logic.
Double semantics: standard and general.
Arithmetical complexity - varying.

Now quadruple semantics:
only witnessed models?
Straccia: fuzzy descriptive logic??
Here: fuzzy modal S5 with finite/witnessed models.
(Arithmetical complexity.)
Other uses? Let's see.