There may be infinitely many near coherence classes under $\mathfrak{u} < \mathfrak{d}$

Heike Mildenberger

Kurt Gödel Research Center for Mathematical Logic, University of Vienna

Logic Colloquium 2007

Wrocław

July 13 – 20, 2007

Outline

Mappings between filters

Definition

A filter is a non-principal proper filter on ω.

Definition

Let $f : \omega \to \omega$ be finite-to-one. We set $f(\mathscr{F}) = \{X : f^{-1}X \in \mathscr{F}\}$.

Mappings between filters

Definition

A filter is a non-principal proper filter on ω.

Definition

Let $f : \omega \to \omega$ be finite-to-one. We set $f(\mathscr{F}) = \{X \, : \, f^{-1}X \in \mathscr{F}\}$.

$f(\mathscr{F})$ contains less information than $\mathscr{F}$:

Mappings between filters

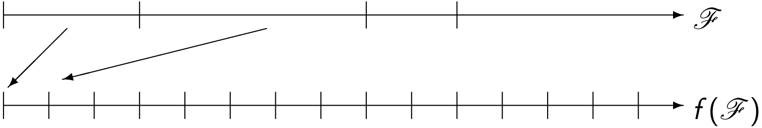
Definition

A filter is a non-principal proper filter on ω.

Definition

Let $f : \omega \to \omega$ be finite-to-one. We set $f(\mathscr{F}) = \{X \ : \ f^{-1}X \in \mathscr{F}\}$.

$f(\mathscr{F})$ contains less information than $\mathscr{F}$:



Near coherence of filters

Definition

Two filters $\mathscr{F}$ and $\mathscr{G}$ on ω are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a proper filter.

If $\mathscr{U}$ is an ultrafilter, then also $f(\mathscr{U})$ is an ultrafilter.

Near coherence of filters

Definition

Two filters $\mathscr{F}$ and $\mathscr{G}$ on ω are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a proper filter.

If $\mathscr{U}$ is an ultrafilter, then also $f(\mathscr{U})$ is an ultrafilter.

Two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{U}) = f(\mathscr{V})$.

Near coherence of filters

Definition

Two filters $\mathscr{F}$ and $\mathscr{G}$ on ω are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a proper filter.

If $\mathscr{U}$ is an ultrafilter, then also $f(\mathscr{U})$ is an ultrafilter.

Two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{U}) = f(\mathscr{V})$.

Near coherence of filters

Definition

Two filters $\mathscr{F}$ and $\mathscr{G}$ on ω are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a proper filter.

If $\mathscr{U}$ is an ultrafilter, then also $f(\mathscr{U})$ is an ultrafilter.

Two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathscr{U}) = f(\mathscr{V})$.

Near coherence of ultrafilters

If $f(\mathscr{U}) = f(\mathscr{V})$ and $g(\mathscr{V}) = g(\mathscr{W})$, then there is a slower growing finite-to-one function h such that $h(\mathscr{U}) = h(\mathscr{W})$.

Fact

The near-coherence relation is an equivalence relation on the ultrafilters on ω.

Near coherence of ultrafilters

If $f(\mathscr{U}) = f(\mathscr{V})$ and $g(\mathscr{V}) = g(\mathscr{W})$, then there is a slower growing finite-to-one function h such that $h(\mathscr{U}) = h(\mathscr{W})$.

Fact

The near-coherence relation is an equivalence relation on the ultrafilters on ω.

Its classes are called near-coherence classes of ultrafilters.

Near coherence of ultrafilters

If $f(\mathscr{U}) = f(\mathscr{V})$ and $g(\mathscr{V}) = g(\mathscr{W})$, then there is a slower growing finite-to-one function h such that $h(\mathscr{U}) = h(\mathscr{W})$.

Fact

The near-coherence relation is an equivalence relation on the ultrafilters on ω.

Its classes are called near-coherence classes of ultrafilters.

Two filters $\mathscr{F}$ and $\mathscr{G}$ are nearly coherent iff there are nearly coherent ultrafilters $\mathscr{U}$ and $\mathscr{V}$ such that $\mathscr{U} \supseteq \mathscr{F}$ and $\mathscr{V} \supseteq \mathscr{G}$. So "NCU" implies NCF.

Near coherence of ultrafilters

If $f(\mathscr{U}) = f(\mathscr{V})$ and $g(\mathscr{V}) = g(\mathscr{W})$, then there is a slower growing finite-to-one function h such that $h(\mathscr{U}) = h(\mathscr{W})$.

Fact

The near-coherence relation is an equivalence relation on the ultrafilters on ω.

Its classes are called near-coherence classes of ultrafilters.

Two filters $\mathscr{F}$ and $\mathscr{G}$ are nearly coherent iff there are nearly coherent ultrafilters $\mathscr{U}$ and $\mathscr{V}$ such that $\mathscr{U} \supseteq \mathscr{F}$ and $\mathscr{V} \supseteq \mathscr{G}$. So "NCU" implies NCF.

Possible numbers of near-coherence classes

Theorem. Booth, Galvin, Mary-Ellen Rudin, Blass

Under CH, there are 2^{2^ω} near-coherence classes of ultrafilters.

Theorem. Blass, Shelah, 1987

It is consistent relative to ZFC that there is just one near-coherence class of ultrafilters.

Possible numbers of near-coherence classes

Theorem. Booth, Galvin, Mary-Ellen Rudin, Blass

Under CH, there are 2^{2^ω} near-coherence classes of ultrafilters.

Theorem. Blass, Shelah, 1987

It is consistent relative to ZFC that there is just one near-coherence class of ultrafilters.

The fact, that there is just one near-coherence class is called the principle of near coherence of (ultra)filters, NCF.

Possible numbers of near-coherence classes

Theorem. Booth, Galvin, Mary-Ellen Rudin, Blass

Under CH, there are 2^{2^ω} near-coherence classes of ultrafilters.

Theorem. Blass, Shelah, 1987

It is consistent relative to ZFC that there is just one near-coherence class of ultrafilters.

The fact, that there is just one near-coherence class is called the principle of near coherence of (ultra)filters, NCF.

Conjecture: There is a model with exactly two near-coherence classes.

Possible numbers of near-coherence classes

Theorem. Booth, Galvin, Mary-Ellen Rudin, Blass

Under CH, there are 2^{2^ω} near-coherence classes of ultrafilters.

Theorem. Blass, Shelah, 1987

It is consistent relative to ZFC that there is just one near-coherence class of ultrafilters.

The fact, that there is just one near-coherence class is called the principle of near coherence of (ultra)filters, NCF.

Conjecture: There is a model with exactly two near-coherence classes. Big open question: Other finite numbers.

Possible numbers of near-coherence classes

Theorem. Booth, Galvin, Mary-Ellen Rudin, Blass

Under CH, there are 2^{2^ω} near-coherence classes of ultrafilters.

Theorem. Blass, Shelah, 1987

It is consistent relative to ZFC that there is just one near-coherence class of ultrafilters.

The fact, that there is just one near-coherence class is called the principle of near coherence of (ultra)filters, NCF.
Conjecture: There is a model with exactly two near-coherence classes. Big open question: Other finite numbers.

Excluded numbers

Theorem. Banakh, Blass, 2005

If there are infinitely many near-coherence classes of ultrafilters then there are 2^{2^ω} classes.

Theorem. Blass, 1987

$\mathfrak{d} \leq \mathfrak{u}$ implies that there are infinitely many near-coherence classes of ultrafilters.

Excluded numbers

Theorem. Banakh, Blass, 2005

If there are infinitely many near-coherence classes of ultrafilters then there are 2^{2^ω} classes.

Theorem. Blass, 1987

$\mathfrak{d} \leq \mathfrak{u}$ implies that there are infinitely many near-coherence classes of ultrafilters.

Question. Banakh, Blass, 2005

Does $\mathfrak{u} < \mathfrak{d}$ imply that there are only finitely many near-coherence classes of ultrafilters?

Excluded numbers

Theorem. Banakh, Blass, 2005

If there are infinitely many near-coherence classes of ultrafilters then there are 2^{2^ω} classes.

Theorem. Blass, 1987

$\mathfrak{d} \leq \mathfrak{u}$ implies that there are infinitely many near-coherence classes of ultrafilters.

Question. Banakh, Blass, 2005

Does $\mathfrak{u} < \mathfrak{d}$ imply that there are only finitely many near-coherence classes of ultrafilters?

Bases and characters, $\mathfrak{u}$

Definition

A set $\mathscr{B} \subseteq \mathscr{F}$ is called a base for $\mathscr{F}$ if $(\forall F \in \mathscr{F})(\exists B \in \mathscr{B})(B \subseteq F)$.

A set $\mathscr{B} \subseteq [\omega]^\omega$ is called a pseudobase for $\mathscr{F}$ if $(\forall F \in \mathscr{F})(\exists B \in \mathscr{B})(B \subseteq F)$.

Bases and characters, $\mathfrak{u}$

Definition

A set $\mathscr{B} \subseteq \mathscr{F}$ is called a base for $\mathscr{F}$ if
$(\forall F \in \mathscr{F})(\exists B \in \mathscr{B})(B \subseteq F)$.
A set $\mathscr{B} \subseteq [\omega]^\omega$ is called a pseudobase for $\mathscr{F}$ if
$(\forall F \in \mathscr{F})(\exists B \in \mathscr{B})(B \subseteq F)$.
The smallest size of a base of $\mathscr{F}$ is called $\chi(\mathscr{F})$, the character of $\mathscr{F}$.

Bases and characters, $\mathfrak{u}$

Definition

A set $\mathscr{B} \subseteq \mathscr{F}$ is called a base for $\mathscr{F}$ if $(\forall F \in \mathscr{F})(\exists B \in \mathscr{B})(B \subseteq F)$.
A set $\mathscr{B} \subseteq [\omega]^\omega$ is called a pseudobase for $\mathscr{F}$ if $(\forall F \in \mathscr{F})(\exists B \in \mathscr{B})(B \subseteq F)$.
The smallest size of a base of $\mathscr{F}$ is called $\chi(\mathscr{F})$, the character of $\mathscr{F}$.

Taking the minimum over all ultrafilters

Definition

The ultrafilter characteristic $\mathfrak{u}$ is the minimal $\chi(\mathscr{U})$ for a non-principal ultrafilter $\mathscr{U}$.

The reaping number $\mathfrak{r}$ is the minimal cardinality of a pseudobase for a non-principal ultrafilter $\mathscr{U}$.

Taking the minimum over all ultrafilters

Definition

The ultrafilter characteristic $\mathfrak{u}$ is the minimal $\chi(\mathscr{U})$ for a non-principal ultrafilter $\mathscr{U}$.
The reaping number $\mathfrak{r}$ is the minimal cardinality of a pseudobase for a non-principal ultrafilter $\mathscr{U}$.

Theorem, Goldstern, Shelah, 1990

$\mathfrak{r} < \mathfrak{u}$ is consistent relative to ZFC.

But then $\mathfrak{d} \leq \mathfrak{u}$ by a theorem of Aubrey.

Taking the minimum over all ultrafilters

Definition

The ultrafilter characteristic $\mathfrak{u}$ is the minimal $\chi(\mathscr{U})$ for a non-principal ultrafilter $\mathscr{U}$.
The reaping number $\mathfrak{r}$ is the minimal cardinality of a pseudobase for a non-principal ultrafilter $\mathscr{U}$.

Theorem, Goldstern, Shelah, 1990

$\mathfrak{r} < \mathfrak{u}$ is consistent relative to ZFC.

But then $\mathfrak{d} \leq \mathfrak{u}$ by a theorem of Aubrey.

Dominating numbers, $\mathfrak{d}$

Definition

We consider the order of eventual domination: $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$.

For a filter $\mathscr{F}$, we define the reduced order $f \leq_{\mathscr{F}} g$ iff $\{n : f(n) \leq g(n)\} \in \mathscr{F}$.

Dominating numbers, $\mathfrak{d}$

Definition

We consider the order of eventual domination: $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$.
For a filter $\mathscr{F}$, we define the reduced order $f \leq_{\mathscr{F}} g$ iff $\{n : f(n) \leq g(n)\} \in \mathscr{F}$.

Definition

A family D is dominating [$\mathscr{F}$-dominating] iff for every $f \in {}^\omega\omega$ there is some $g \in D$ such that $f \leq^* g$ [$f \leq_{\mathscr{F}} g$].

Dominating numbers, $\mathfrak{d}$

Definition

We consider the order of eventual domination: $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$.
For a filter $\mathscr{F}$, we define the reduced order $f \leq_{\mathscr{F}} g$ iff $\{n : f(n) \leq g(n)\} \in \mathscr{F}$.

Definition

A family D is dominating [$\mathscr{F}$-dominating] iff for every $f \in {}^\omega\omega$ there is some $g \in D$ such that $f \leq^* g$ [$f \leq_{\mathscr{F}} g$].
The dominating number $\mathfrak{d}$ [the dominating number of $\mathscr{F}$, $\mathfrak{d}(\mathscr{F})$,] is the smallest cardinal of a dominating [$\mathscr{F}$-dominating] family $D \subseteq {}^\omega\omega$.

Dominating numbers, $\mathfrak{d}$

Definition

We consider the order of eventual domination: $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$.
For a filter $\mathscr{F}$, we define the reduced order $f \leq_{\mathscr{F}} g$ iff $\{n : f(n) \leq g(n)\} \in \mathscr{F}$.

Definition

A family D is dominating [$\mathscr{F}$-dominating] iff for every $f \in {}^\omega\omega$ there is some $g \in D$ such that $f \leq^* g$ [$f \leq_{\mathscr{F}} g$].
The dominating number $\mathfrak{d}$ [the dominating number of $\mathscr{F}$, $\mathfrak{d}(\mathscr{F})$,] is the smallest cardinal of a dominating [$\mathscr{F}$-dominating] family $D \subseteq {}^\omega\omega$.

The role of $\mathfrak{u}$ and $\mathfrak{d}$

$\mathfrak{u}$ comes in as the minimal number of steps in constructing one representative of one class.

Proposition. Blass, 1987

There is a set D, a so-called test set, of size $\mathfrak{d}$ such that any two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent, if there is some $f \in D$ with $f(\mathscr{U}) = f(\mathscr{V})$.

The role of $\mathfrak{u}$ and $\mathfrak{d}$

$\mathfrak{u}$ comes in as the minimal number of steps in constructing one representative of one class.

Proposition. Blass, 1987

There is a set D, a so-called test set, of size $\mathfrak{d}$ such that any two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent, if there is some $f \in D$ with $f(\mathscr{U}) = f(\mathscr{V})$.

The construction of two non-nearly-coherent ultrafilters can be seen as a diagonalization with $\mathfrak{u}$ steps and $\mathfrak{d}$ tasks.

The role of $\mathfrak{u}$ and $\mathfrak{d}$

$\mathfrak{u}$ comes in as the minimal number of steps in constructing one representative of one class.

Proposition. Blass, 1987

There is a set D, a so-called test set, of size $\mathfrak{d}$ such that any two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent, if there is some $f \in D$ with $f(\mathscr{U}) = f(\mathscr{V})$.

The construction of two non-nearly-coherent ultrafilters can be seen as a diagonalization with $\mathfrak{u}$ steps and $\mathfrak{d}$ tasks.

Candidates

Models of $\mathfrak{u} < \mathfrak{d}$: The known models with *countable support* iterations fulfil the stronger inequality $\mathfrak{u} < \mathfrak{g}$, which implies NCF.

There is one type of model (from [BsSh:257], 1989) of $\mathfrak{u} < \mathfrak{d}$ gotten with a *finite support iteration* of c.c.c. partial orders.

Candidates

Models of $\mathfrak{u} < \mathfrak{d}$: The known models with *countable support* iterations fulfil the stronger inequality $\mathfrak{u} < \mathfrak{g}$, which implies NCF. There is one type of model (from [BsSh:257], 1989) of $\mathfrak{u} < \mathfrak{d}$ gotten with a *finite support iteration* of c.c.c. partial orders.

Theorem. M.

It is consistent relative to ZFC that there are infinitely many near-coherence classes of ultrafilters and $\mathfrak{u} < \mathfrak{d}$.

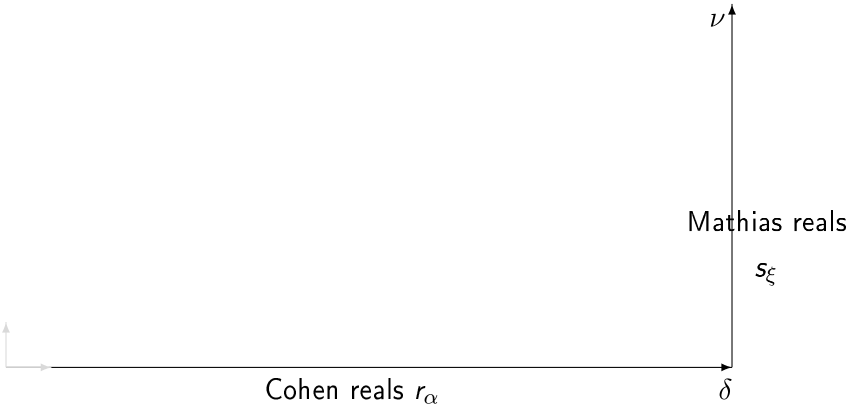
Candidates

Models of $\mathfrak{u} < \mathfrak{d}$: The known models with *countable support* iterations fulfil the stronger inequality $\mathfrak{u} < \mathfrak{g}$, which implies NCF. There is one type of model (from [BsSh:257], 1989) of $\mathfrak{u} < \mathfrak{d}$ gotten with a *finite support iteration* of c.c.c. partial orders.

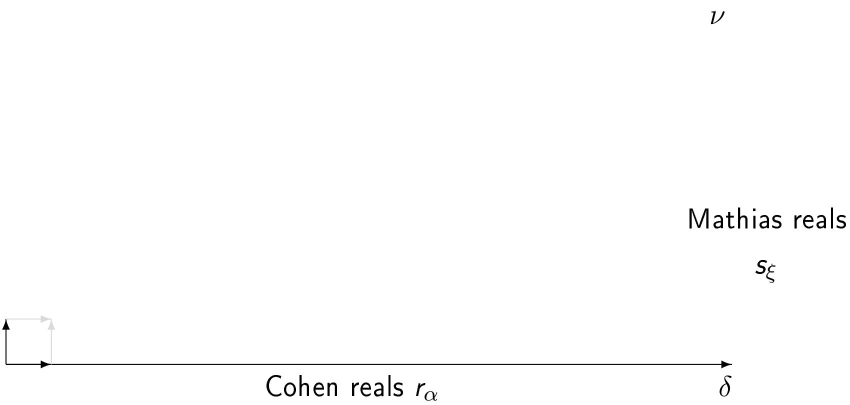
Theorem. M.

It is consistent relative to ZFC that there are infinitely many near-coherence classes of ultrafilters and $\mathfrak{u} < \mathfrak{d}$.

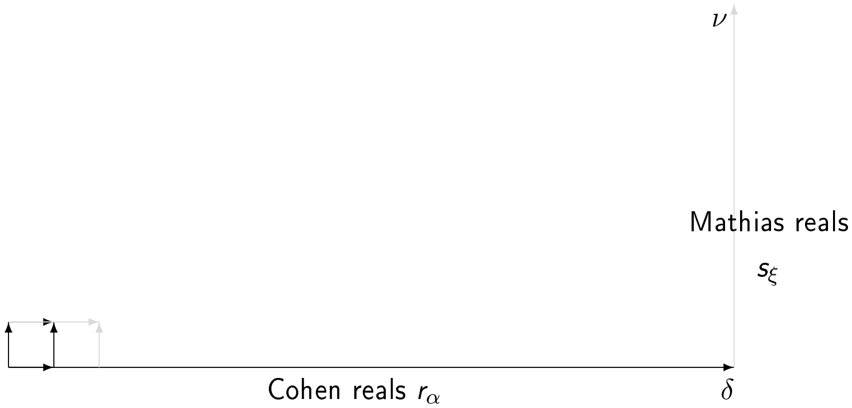
An iteration with a rectangular structure



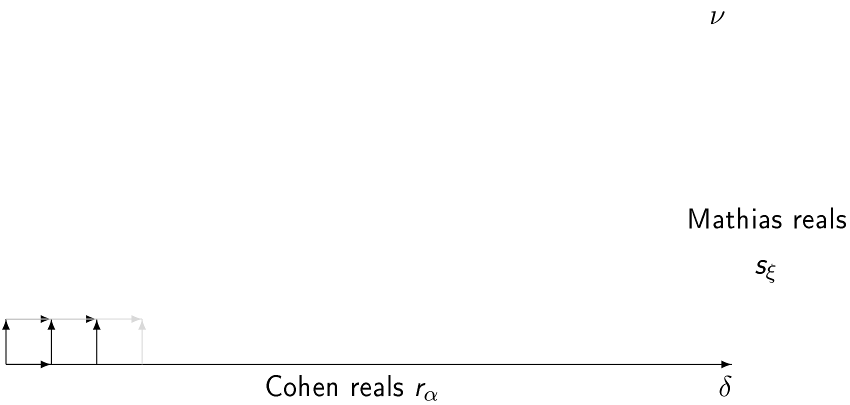
An iteration with a rectangular structure



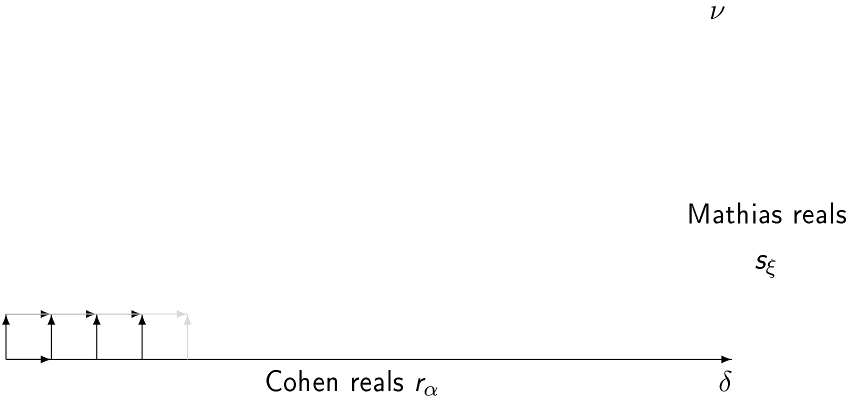
An iteration with a rectangular structure



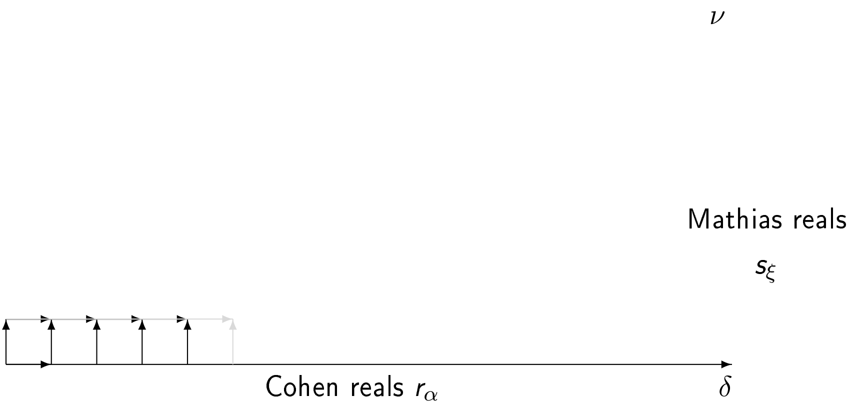
An iteration with a rectangular structure



An iteration with a rectangular structure

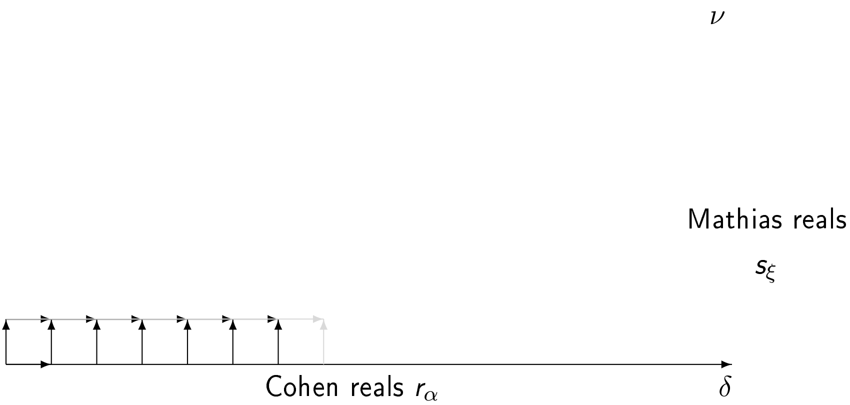


An iteration with a rectangular structure

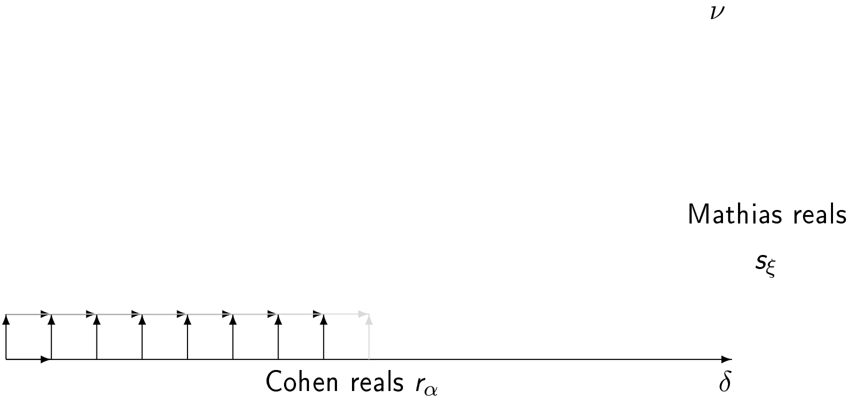


An iteration with a rectangular structure

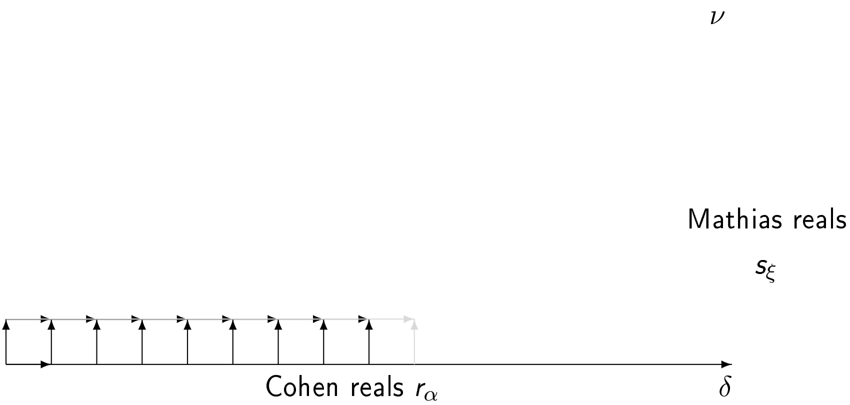
An iteration with a rectangular structure



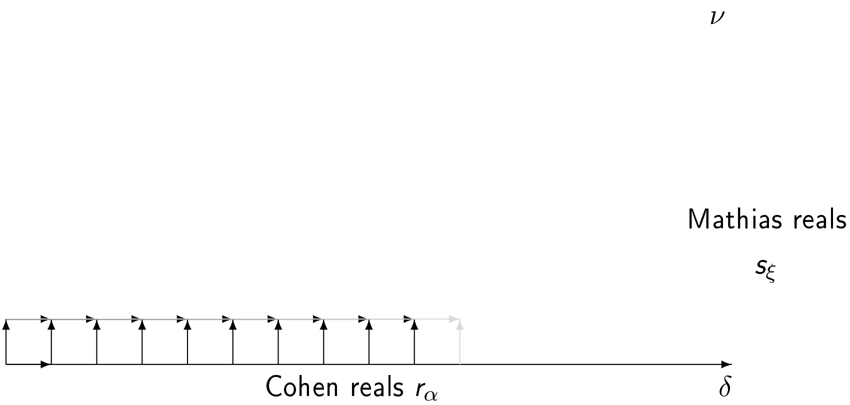
An iteration with a rectangular structure



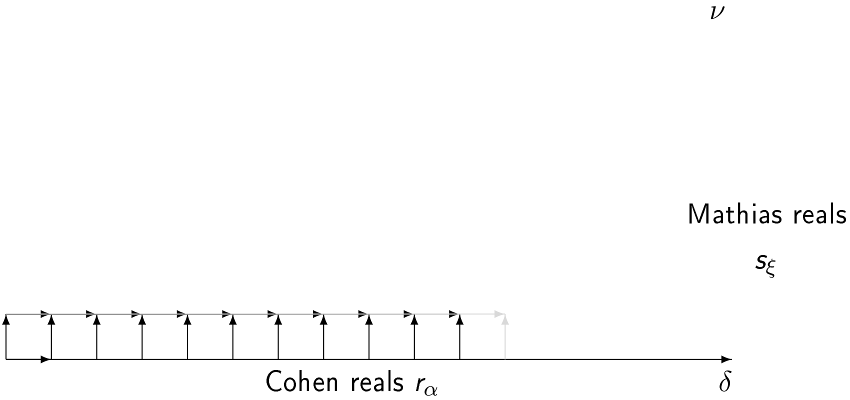
An iteration with a rectangular structure



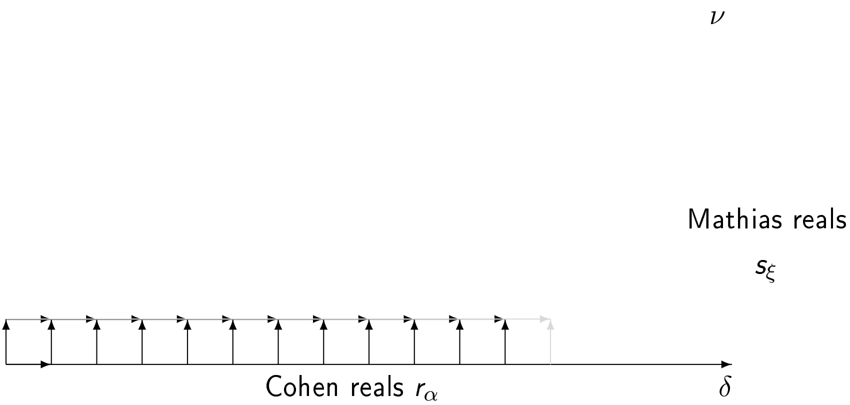
An iteration with a rectangular structure



An iteration with a rectangular structure



An iteration with a rectangular structure

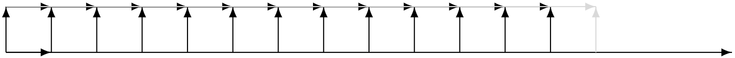


An iteration with a rectangular structure

ν

Mathias reals

s_ξ



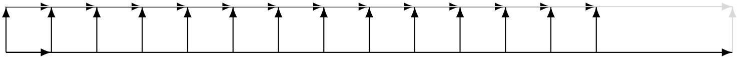
Cohen reals r_α δ

An iteration with a rectangular structure

ν

Mathias reals

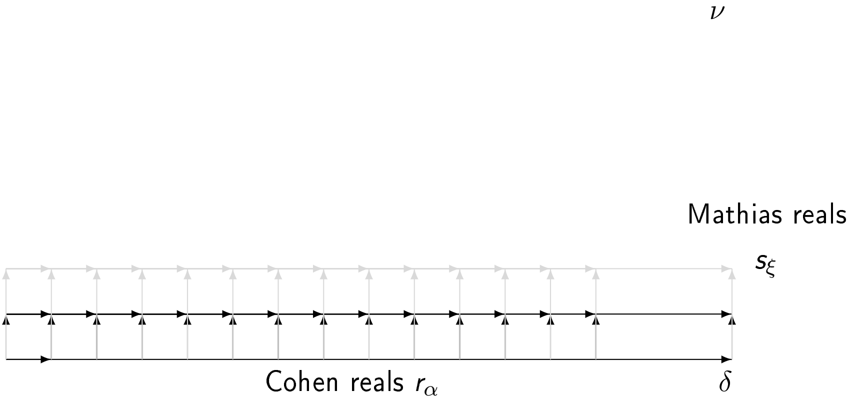
s_ξ



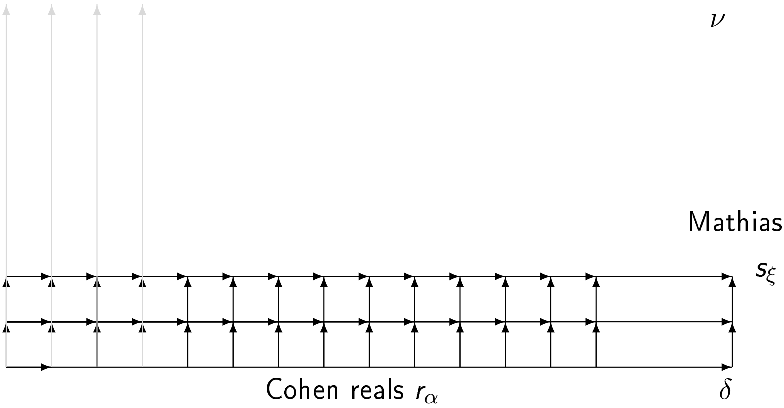
Cohen reals r_α

δ

An iteration with a rectangular structure



An iteration with a rectangular structure



An iteration with a rectangular structure

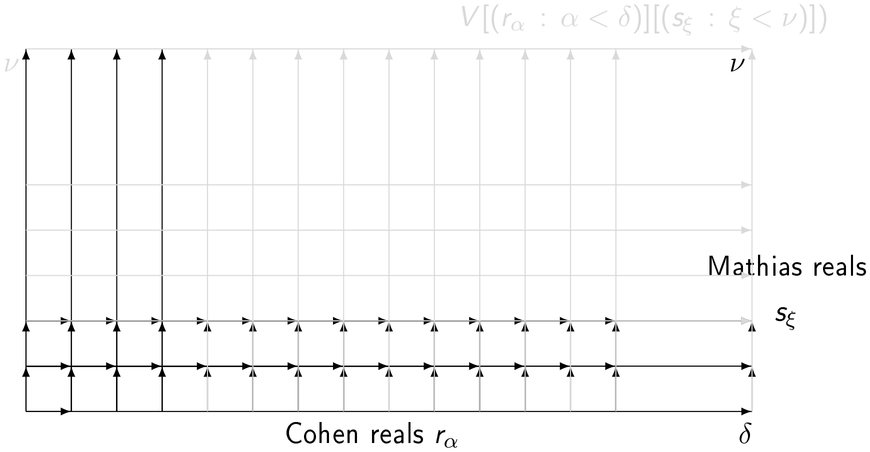


Figure 1: A sketch of $V[(r_\alpha : \alpha < \delta)][(s_\xi : \xi < \nu)]$

An iteration with a rectangular structure

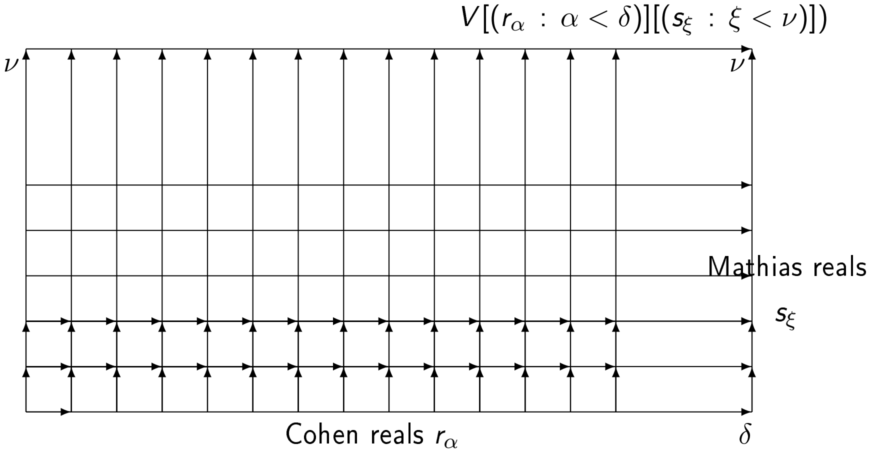


Figure 1: A sketch of $V[(r_\alpha : \alpha < \delta)][(s_\xi : \xi < \nu)]$

Splitting families and reaping

Definition

$\mathscr{S} \subseteq [\omega]^\omega$ is a splitting family iff $(\forall X \in [\omega]^\omega)(\exists S \in \mathscr{S})(X \cap S$ and $X \setminus S$ are both infinite). The splitting number $\mathfrak{s}$ is the smallest size of a splitting family.

Splitting families and reaping

Definition

$\mathscr{S} \subseteq [\omega]^\omega$ is a splitting family iff $(\forall X \in [\omega]^\omega)(\exists S \in \mathscr{S})(X \cap S$ and $X \setminus S$ are both infinite). The splitting number $\mathfrak{s}$ is the smallest size of a splitting family.

Theorem. Blass, M., 1999

If $\mathfrak{s} > \mathfrak{r}$ then there are at most two near-coherence classes.

Splitting families and reaping

Definition

$\mathscr{S} \subseteq [\omega]^\omega$ is a splitting family iff $(\forall X \in [\omega]^\omega)(\exists S \in \mathscr{S})(X \cap S$ and $X \setminus S$ are both infinite). The splitting number $\mathfrak{s}$ is the smallest size of a splitting family.

Theorem. Blass, M., 1999

If $\mathfrak{s} > \mathfrak{r}$ then there are at most two near-coherence classes.

Theorem. Aubrey, 2004

If $\mathfrak{r} < \mathfrak{d}$ then $\mathfrak{r} = \mathfrak{u}$.

Splitting families and reaping

Definition

$\mathscr{S} \subseteq [\omega]^\omega$ is a splitting family iff $(\forall X \in [\omega]^\omega)(\exists S \in \mathscr{S})(X \cap S$ and $X \setminus S$ are both infinite). The splitting number $\mathfrak{s}$ is the smallest size of a splitting family.

Theorem. Blass, M., 1999

If $\mathfrak{s} > \mathfrak{r}$ then there are at most two near-coherence classes.

Theorem. Aubrey, 2004

If $\mathfrak{r} < \mathfrak{d}$ then $\mathfrak{r} = \mathfrak{u}$.

Conclusion: In $V^{\mathbb{P}}$, we have $\nu = \mathfrak{u} = \mathfrak{r}$.

Splitting families and reaping

Definition

$\mathscr{S} \subseteq [\omega]^\omega$ is a splitting family iff $(\forall X \in [\omega]^\omega)(\exists S \in \mathscr{S})(X \cap S$ and $X \setminus S$ are both infinite). The splitting number $\mathfrak{s}$ is the smallest size of a splitting family.

Theorem. Blass, M., 1999

If $\mathfrak{s} > \mathfrak{r}$ then there are at most two near-coherence classes.

Theorem. Aubrey, 2004

If $\mathfrak{r} < \mathfrak{d}$ then $\mathfrak{r} = \mathfrak{u}$.

Conclusion: In $V^{\mathbb{P}}$, we have $\nu = \mathfrak{u} = \mathfrak{r}$.

A small splitting family

Proposition

In $V^{\mathbb{P}}$, $\mathfrak{s} \leq \nu$.

Sketch of proof: Remember, s_ξ, $\xi < \nu$, are the Mathias reals. We set

$$X_\xi = \{n \in \omega \ : \ |s_\xi \cap n| \text{ is even}\}.$$

A small splitting family

Proposition

In $V^{\mathbb{P}}$, $\mathfrak{s} \leq \nu$.

Sketch of proof: Remember, s_ξ, $\xi < \nu$, are the Mathias reals. We set

$$X_\xi = \{n \in \omega \ : \ |s_\xi \cap n| \text{ is even}\}.$$

Then $\{X_\xi \ : \ \xi < \nu\}$ is a splitting family witnessing $\mathfrak{s} \leq \mathfrak{r}$. □

A small splitting family

Proposition

In $V^{\mathbb{P}}$, $\mathfrak{s} \leq \nu$.

Sketch of proof: Remember, s_ξ, $\xi < \nu$, are the Mathias reals. We set

$$X_\xi = \{n \in \omega \; : \; |s_\xi \cap n| \text{ is even}\}.$$

Then $\{X_\xi \; : \; \xi < \nu\}$ is a splitting family witnessing $\mathfrak{s} \leq \mathfrak{r}$. □

(We have $\mathfrak{s} = \nu$ in these models.)

A small splitting family

Proposition

In $V^{\mathbb{P}}$, $\mathfrak{s} \leq \nu$.

Sketch of proof: Remember, s_ξ, $\xi < \nu$, are the Mathias reals. We set

$$X_\xi = \{n \in \omega \,:\, |s_\xi \cap n| \text{ is even}\}.$$

Then $\{X_\xi \,:\, \xi < \nu\}$ is a splitting family witnessing $\mathfrak{s} \leq \mathfrak{r}$. □
(We have $\mathfrak{s} = \nu$ in these models.)

Conclusion: In $V^{\mathbb{P}}$, we have $\mathfrak{s} \leq \nu = \mathfrak{u} = \mathfrak{r}$ and hence the $\mathfrak{r} < \mathfrak{s}$-Theorem does not apply.

A small splitting family

Proposition

In $V^{\mathbb{P}}$, $\mathfrak{s} \leq \nu$.

Sketch of proof: Remember, s_ξ, $\xi < \nu$, are the Mathias reals. We set

$$X_\xi = \{n \in \omega \,:\, |s_\xi \cap n| \text{ is even}\}.$$

Then $\{X_\xi \,:\, \xi < \nu\}$ is a splitting family witnessing $\mathfrak{s} \leq \mathfrak{r}$. □

(We have $\mathfrak{s} = \nu$ in these models.)

Conclusion: In $V^{\mathbb{P}}$, we have $\mathfrak{s} \leq \nu = \mathfrak{u} = \mathfrak{r}$ and hence the $\mathfrak{r} < \mathfrak{s}$-Theorem does not apply.

The answer

Claim

In the model $V^{\mathbb{P}}$ there are infinitely near-coherence classes of ultrafilters.

Sketch of proof: We have the ultrafilter $\mathscr{U}_P$ that is generated by the Mathias reals s_ξ, $\xi < \nu$.

The answer

Claim

In the model $V^{\mathbb{P}}$ there are infinitely near-coherence classes of ultrafilters.

Sketch of proof: We have the ultrafilter $\mathscr{U}_P$ that is generated by the Mathias reals s_ξ, $\xi < \nu$.

By [BM] all ultrafilters $\mathscr{U}$ with $< \mathfrak{d}$ generators have $\mathrm{cf}(\omega^\omega/\mathscr{U}) = \mathfrak{d} > \mathfrak{r}$ and hence are nearly coherent to $\mathscr{U}_P$.

The answer

Claim

In the model $V^{\mathbb{P}}$ there are infinitely near-coherence classes of ultrafilters.

Sketch of proof: We have the ultrafilter $\mathscr{U}_P$ that is generated by the Mathias reals s_ξ, $\xi < \nu$.
By [BM] all ultrafilters $\mathscr{U}$ with $< \mathfrak{d}$ generators have $\mathrm{cf}(\omega^\omega/\mathscr{U}) = \mathfrak{d} > \mathfrak{r}$ and hence are nearly coherent to $\mathscr{U}_P$.
We shall show that there is a filter $\mathscr{H}_0$ that is non-nearly-coherent to $\mathscr{U}_P$ such that $\mathscr{H}_0$ extended by fewer than $\mathfrak{d}(\mathscr{H}_0)$ sets is not almost ultra.

The answer

Claim

In the model $V^{\mathbb{P}}$ there are infinitely near-coherence classes of ultrafilters.

Sketch of proof: We have the ultrafilter $\mathscr{U}_P$ that is generated by the Mathias reals s_ξ, $\xi < \nu$.
By [BM] all ultrafilters $\mathscr{U}$ with $< \mathfrak{d}$ generators have $\mathrm{cf}(\omega^\omega/\mathscr{U}) = \mathfrak{d} > \mathfrak{r}$ and hence are nearly coherent to $\mathscr{U}_P$.
We shall show that there is a filter $\mathscr{H}_0$ that is non-nearly-coherent to $\mathscr{U}_P$ such that $\mathscr{H}_0$ extended by fewer than $\mathfrak{d}(\mathscr{H}_0)$ sets is not almost ultra.

We shall get $\mathscr{H}_0$ from the Cohen reals.

The answer

Claim

In the model $V^{\mathbb{P}}$ there are infinitely near-coherence classes of ultrafilters.

Sketch of proof: We have the ultrafilter $\mathscr{U}_P$ that is generated by the Mathias reals s_ξ, $\xi < \nu$.
By [BM] all ultrafilters $\mathscr{U}$ with $< \mathfrak{d}$ generators have $\mathrm{cf}(\omega^\omega/\mathscr{U}) = \mathfrak{d} > \mathfrak{r}$ and hence are nearly coherent to $\mathscr{U}_P$.
We shall show that there is a filter $\mathscr{H}_0$ that is non-nearly-coherent to $\mathscr{U}_P$ such that $\mathscr{H}_0$ extended by fewer than $\mathfrak{d}(\mathscr{H}_0)$ sets is not almost ultra.
We shall get $\mathscr{H}_0$ from the Cohen reals.

Filters not nearly coherent to $\mathscr{U}_P$

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$X_{\alpha,\xi} = \{r_\alpha(n) : |s_\xi \cap n| \text{ even}\},$$

Filters not nearly coherent to $\mathscr{U}_P$

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$X_{\alpha,\xi} = \{r_\alpha(n) \,:\, |s_\xi \cap n| \text{ even}\},$$

$$\mathscr{H}_\xi = \{X_{\alpha,\xi} \,:\, \alpha < \delta\},$$

Filters not nearly coherent to $\mathscr{U}_P$

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$\begin{aligned} X_{\alpha,\xi} &= \{r_\alpha(n) : |s_\xi \cap n| \text{ even}\}, \\ \mathscr{H}_\xi &= \{X_{\alpha,\xi} : \alpha < \delta\}, \\ \mathscr{H} &= \{X_{\alpha,\xi} : \alpha < \delta, \xi < \nu, \xi \text{ is an even ordinal}\}, \end{aligned}$$

Filters not nearly coherent to $\mathscr{U}_P$

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$\begin{aligned}
X_{\alpha,\xi} &= \{r_\alpha(n) : |s_\xi \cap n| \text{ even}\},\\
\mathscr{H}_\xi &= \{X_{\alpha,\xi} : \alpha < \delta\},\\
\mathscr{H} &= \{X_{\alpha,\xi} : \alpha < \delta, \xi < \nu, \xi \text{ is an even ordinal}\},\\
\mathscr{H}_{\text{full}} &= \{X_{\alpha,\xi} : \alpha < \delta, \xi < \nu\}.
\end{aligned}$$

Filters not nearly coherent to $\mathscr{U}_P$

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$\begin{aligned} X_{\alpha,\xi} &= \{r_\alpha(n) \,:\, |s_\xi \cap n| \text{ even}\}, \\ \mathscr{H}_\xi &= \{X_{\alpha,\xi} \,:\, \alpha < \delta\}, \\ \mathscr{H} &= \{X_{\alpha,\xi} \,:\, \alpha < \delta, \xi < \nu, \xi \text{ is an even ordinal}\}, \\ \mathscr{H}_{\rm full} &= \{X_{\alpha,\xi} \,:\, \alpha < \delta, \xi < \nu\}. \end{aligned}$$

Lemma

$\mathscr{H}_{\rm full}$ *has the finite intersection property.*

Filters not nearly coherent to $\mathscr{U}_P$

We think of the Cohen reals as subsets of ω and let the Cohen reals r_α, $\alpha < \delta$, be their strictly increasing enumerations. We set

$$\begin{aligned} X_{\alpha,\xi} &= \{r_\alpha(n) \,:\, |s_\xi \cap n| \text{ even}\}, \\ \mathscr{H}_\xi &= \{X_{\alpha,\xi} \,:\, \alpha < \delta\}, \\ \mathscr{H} &= \{X_{\alpha,\xi} \,:\, \alpha < \delta, \xi < \nu, \xi \text{ is an even ordinal}\}, \\ \mathscr{H}_{\mathrm{full}} &= \{X_{\alpha,\xi} \,:\, \alpha < \delta, \xi < \nu\}. \end{aligned}$$

Lemma

$\mathscr{H}_{\mathrm{full}}$ *has the finite intersection property.*

$f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$

Lemma

For every $\xi < \nu$, for every $Y \in V(\delta, \xi)$ for every $\alpha_i < \delta$, $i < k$, we have: If $Y \cap \bigcap_{i<k} \mathrm{range}(r_{\alpha_i})$ is infinite, then the set

$$Y \cap \bigcap_{0 \leq i < k} X_{\alpha_i, \xi}$$

is infinite.

Lemma

For every finite-to-one f, $f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$.

$f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$

Lemma

For every $\xi < \nu$, for every $Y \in V(\delta, \xi)$ for every $\alpha_i < \delta$, $i < k$, we have: If $Y \cap \bigcap_{i<k} \mathrm{range}(r_{\alpha_i})$ is infinite, then the set

$$Y \cap \bigcap_{0 \leq i < k} X_{\alpha_i, \xi}$$

is infinite.

Lemma

For every finite-to-one f, $f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$.

So $\mathscr{H}_0$ and $\mathscr{U}_P$ are not nearly coherent, and thus we have at least two near coherence classes of ultrafilters.

$f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$

Lemma

For every $\xi < \nu$, for every $Y \in V(\delta,\xi)$ for every $\alpha_i < \delta$, $i < k$, we have: If $Y \cap \bigcap_{i<k} \mathrm{range}(r_{\alpha_i})$ is infinite, then the set

$$Y \cap \bigcap_{0 \le i < k} X_{\alpha_i,\xi}$$

is infinite.

Lemma

For every finite-to-one f, $f(\mathscr{H}_0) \not\subseteq f(\mathscr{U}_P)$.

So $\mathscr{H}_0$ and $\mathscr{U}_P$ are not nearly coherent, and thus we have at least two near coherence classes of ultrafilters.

Small dominating families modulo filter orderings

Aim: Find a tree of pairwise non-nearly coherent ultrafilters among the supersets of $\mathscr{H}_0$.

Proposition. Banakh, Blass, 2005

If a filter $\mathscr{F}$ and a ultrafilter $\mathscr{U}$ are not nearly coherent, then $\mathfrak{d}(\mathscr{F}) \leq \chi(\mathscr{U})$.

Small dominating families modulo filter orderings

Aim: Find a tree of pairwise non-nearly coherent ultrafilters among the supersets of $\mathscr{H}_0$.

Proposition. Banakh, Blass, 2005

If a filter $\mathscr{F}$ and a ultrafilter $\mathscr{U}$ are not nearly coherent, then $\mathfrak{d}(\mathscr{F}) \leq \chi(\mathscr{U})$.

So in $V^{\mathbb{P}}$, $\mathfrak{d}(\mathscr{H}_0) \leq \nu$.

Small dominating families modulo filter orderings

Aim: Find a tree of pairwise non-nearly coherent ultrafilters among the supersets of $\mathscr{H}_0$.

Proposition. Banakh, Blass, 2005

If a filter $\mathscr{F}$ and a ultrafilter $\mathscr{U}$ are not nearly coherent, then $\mathfrak{d}(\mathscr{F}) \leq \chi(\mathscr{U})$.

So in $V^{\mathbb{P}}$, $\mathfrak{d}(\mathscr{H}_0) \leq \nu$.

Small test sets

Let $t(\mathscr{F})$ be the smallest size of a test set for near coherence in $[\mathscr{F}] = \{\mathscr{G} : \mathscr{G} \text{ filter}, \mathscr{G} \supseteq \mathscr{F}\}$.

Proposition. Banakh, Blass, 2005

$t(\mathscr{F}) \leq \mathfrak{d}(\mathscr{F})$.

Small test sets

Let $t(\mathscr{F})$ be the smallest size of a test set for near coherence in $[\mathscr{F}] = \{\mathscr{G} : \mathscr{G} \text{ filter}, \mathscr{G} \supseteq \mathscr{F}\}$.

Proposition. Banakh, Blass, 2005

$t(\mathscr{F}) \leq \mathfrak{d}(\mathscr{F})$.

So, in $V^{\mathbb{P}}$, $t(\mathscr{H}_0) \leq \nu$.

Small test sets

Let $t(\mathscr{F})$ be the smallest size of a test set for near coherence in $[\mathscr{F}] = \{\mathscr{G} : \mathscr{G} \text{ filter}, \mathscr{G} \supseteq \mathscr{F}\}$.

Proposition. Banakh, Blass, 2005

$t(\mathscr{F}) \leq \mathfrak{d}(\mathscr{F})$.

So, in $V^{\mathbb{P}}$, $t(\mathscr{H}_0) \leq \nu$.

A tree of near-coherence classes

Lemma. Slight generalization of Blass, 1987

If all extensions of $\mathscr{H}_0$ by fewer than $t(\mathscr{H}_0)$ sets are not almost ultra, then we can construct infinitely many pairwise non-nearly coherent ultrafilters by an induction of length $t(\mathscr{H}_0)$.

So our proof is finished with

Lemma

In $V^{\mathbb{P}}$, each extension of $\mathscr{H}_0$ by fewer than ν sets is not almost ultra.

and we do not need to construct the desired tree explicitly.

A tree of near-coherence classes

Lemma. Slight generalization of Blass, 1987

If all extensions of $\mathscr{H}_0$ by fewer than $t(\mathscr{H}_0)$ sets are not almost ultra, then we can construct infinitely many pairwise non-nearly coherent ultrafilters by an induction of length $t(\mathscr{H}_0)$.

So our proof is finished with

Lemma

In $V^{\mathbb{P}}$, each extension of $\mathscr{H}_0$ by fewer than ν sets is not almost ultra.

and we do not need to construct the desired tree explicitly.

Bibliography, part 1

Jason Aubrey. Combinatorics for the Dominating and the Unsplitting Numbers. J. Symbolic Logic, 69:482–498, 2004.

Taras Banakh and Andreas Blass. The Number of Near-Coherence Classes of Ultrafilters is Either Finite or $2^{\mathfrak{c}}$. Set Theory (Proceedings of the Special Year in Set Theory in Barcelona, 2003/2004). Pages 257 – 273. Eds. J. Bagaria and S. Todorcevic. Series: Trends in Mathematics. Birkhaeuser, 2006.

Andreas Blass. Near coherence of filters, II: Applications to operator ideals, the Stone-Čech remainder of a half-line, order ideals of sequences, and slenderness of groups. Trans. Amer. Math. Soc., 300:557–581, 1987.

Bibliography, part 2

Andreas Blass. Groupwise density and related cardinals. Arch. Math. Logic, 30:1–11, 1990.

Andreas Blass and Claude Laflamme. Consistency results about filters and the number of inequivalent growth types. J. Symbolic Logic, 54:50–56, 1989.

Andreas Blass and Heike Mildenberger. On the cofinality of ultrapowers. J. Symbolic Logic, 64:727–736, 1999.

Andreas Blass and Saharon Shelah. [BsSh:242]. There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed. Ann. Pure Appl. Logic, 33:213–243, [BsSh:242], 1987.

Bibliography, part 3

Andreas Blass and Saharon Shelah. [BsSh:287]. Near coherence of filters III: A simplified consistency proof. Notre Dame J. Formal Logic, 30:530–538, [BsSh:287], 1989.

Andreas Blass and Saharon Shelah. [BsSh:257]. Ultrafilters with small generating sets. Israel J. Math., 65:259–271, [BsSh:257], 1989.

Jörg Brendle. Distinguishing Groupwise Density Numbers. Preprint, 2006.

Martin Goldstern and Saharon Shelah. Ramsey ultrafilters and the reaping number— $\mathrm{Con}(\mathfrak{r} < \mathfrak{u})$. Ann. Pure Appl. Logic, 49, 1990.

Bibliography, part 4

Heike Mildenberger. Groupwise dense families. Arch. Math. Logic, 40:93–112, 2000.

Heike Mildenberger. There may be infinitely many near coherence classes under $\mathfrak{u} < \mathfrak{d}$, to appear in JSL

Heike Mildenberger. On the groupwise density number for filters. Acta Univ. Carolinae - Math. et Phys., 46:55–63, 2005.

Heike Mildenberger, Saharon Shelah, and Boaz Tsaban. Covering the Baire Space with Meager Sets, [MdShTs:847]. Ann. Pure Appl. Logic 140:60–71, 2006.