Higher-Order Reverse Topology

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Outline

Review of second-order reverse math

Traditional reverse math studies subsystems of *second-order arithmetic*.

- **Language:** Number (type-0) and set (type-1) variables; $\{0, +, \cdot, \text{etc.}\}$; "$=_0$" for numbers (but not sets); "$\in$" relates numbers and sets.
- Base theory, $\mathbf{RCA_0}$: Axioms for number-theoretic $\mathbb{N}$; induction schema for Σ^0_1 formulas; comprehension schema for Δ^0_1 formulas.
  - The second-order part of the minimal ω-model of RCA_0 consists of all computable (recursive) sets.
  - The first-order part of the theory RCA_0 is Σ^0_1-PA [4].
- A stronger theory, $\mathbf{ACA_0}$: Axioms for RCA_0; comprehension schema for arithmetical (or "Π^0_∞") formulas.
  - The second-order part of the minimal ω-model of ACA_0 consists of all arithmetical sets.
  - The first-order part of the theory ACA_0 is PA [4].

Finite types

Definition

The **finite types** *are defined inductively:*

- 0 *is a type.*
- *If* σ *and* τ *are types then* $(\sigma \to \tau)$ *is a type.*

0 is the type of natural numbers; $(\sigma \to \tau)$ is the type of a functional mapping type-σ elements to type-τ elements.

Definition

The **standard types** *are defined inductively:*

- 0 *is a standard type.*
- *If* n *is a standard type, then* $n + 1 := (n \to 0)$ *is a standard type.*

Example: reals are of type 1.

Higher-order reverse math

The language of second-order arithmetic may be too restrictive. In a recent paper [3], Kohlenbach described a base theory in a more-flexible, higher-order language.

- **Language:** Variables of all finite types; $\{0, +, \cdot, \text{etc.}\}$ as before; "$=_0$" only for (type-0) numbers, as before; plus—
  - **Combinators** $\Pi_{\rho,\tau}$, $\Sigma_{\sigma,\rho,\tau}$ (for λ-**abstraction**);
  - Some symbol for **application**, not shown here; and
  - Symbol R_0, for **primitive recursion**.
- Base theory, $\mathbf{RCA}_0^\omega$ ($=$ E-PRA$^\omega$ $+$ QF-AC^{1,0}): Axioms for number-theoretic $\mathbb{N}$, as before; induction schema for quantifier-free formulas; axioms defining R_0, the $\Pi_{\rho,\tau}$'s, and the $\Sigma_{\sigma,\rho,\tau}$'s; extensionality axioms; and QF-AC^{1,0}:

$$\forall x^1 \exists n^0 (\Phi(x, n)) \rightarrow \exists F^{(1\rightarrow 0)} \forall x^1 (\Phi(x, F(x)),$$

where Φ is a quantifier-free formula.

The axioms (E_1) and (E_2)

Definition

The axiom (E_1) is the statement:

$$\exists E_1{}^2 \left[\forall x^1 (E_1(x) =_0 1) \leftrightarrow \exists n^0 (x(n) \neq_0 0)\right].$$

Definition

The axiom (E_2) is the statement:

$$\exists E_2{}^3 \left[\forall X^2 (E_2(X) =_0 1) \leftrightarrow \exists x^1 (X(x) \neq_0 0)\right].$$

- Higher-order equality is defined inductively. E.g., $x^1 =_1 y^1 \iff \forall n^0 (x(n) =_0 y(n))$.
- Think of E_1 as a functional determining type-1 equality:

$$x^1 =_1 y^1 \iff E_1(\lambda n^0.(x(n) - y(n))) =_0 0.$$

Conservation results

Proposition (Kohlenbach [3])

RCA_0^ω is conservative over and implies RCA_0.

Proposition (H.)

1. *$RCA_0^\omega + (E_1)$ is conservative over and implies ACA_0.*
2. *$RCA_0^\omega + QF\text{-}AC^{0,1}$ is conservative over and implies $\Sigma_1^1\text{-}AC_0$.*
3. *$RCA_0^\omega + (E_2)$ is conservative over and implies $\Pi_\infty^1\text{-}CA_0$.*
4. *Etc.*

The proof of the second proposition uses term models and is analogous to the proof of the first.

Sets, families

Definition

1. *A* **real** *is a (type-1) function.*
2. *A* **set** *is a (type-2) functional X such that* $\forall x^1(X(x) =_0 0 \vee X(x) =_0 1)$.
3. *A* **family** *is a (type-3) functional $\mathcal{F}$ such that* $\forall X^2(\mathcal{F}(x) =_0 0 \vee \mathcal{F}(X) =_0 1)$.

(We write "$x \in X$" as shorthand for "$X(x) =_0 1$.")

We consider only topologies on the reals.

Topologies

Definition

A family $\mathcal{F}$ is a topology iff:

1. $\emptyset := (\lambda x^1.0) \in \mathcal{F}$;
2. ${}^{\mathbb{N}}\mathbb{N} := (\lambda x^1.1) \in \mathcal{F}$;
3. *if* $X \in \mathcal{F}$ *and* $Y \in \mathcal{F}$ *then* $X \cap Y := (\lambda x. \min(X(x), Y(x))) \in \mathcal{F}$; *and*
4. *if* $\mathcal{G} \subseteq_2 \mathcal{F}$ *and* $\bigcup \mathcal{G} := \{x : \exists X^2 \in \mathcal{G}\,(x \in X)\}$ *exists, then* $\bigcup \mathcal{G} \in \mathcal{F}$.

Examples: The indiscrete topology is $\{\emptyset, {}^{\mathbb{N}}\mathbb{N}\}$, and the discrete topology is $(\lambda X^2.1)$.

Simple equivalences

Proposition (H. and folklore)

Over RCA_0^ω, we have the following equivalences:

1. $(E_2) \iff$ *there is a topology for a connected space (i.e., only $\emptyset$ and ${}^{\mathbb{N}}\mathbb{N}$ are clopen).*
2. $(E_2) \iff$ *there is a topology with a dense, nowhere-dense set.*
3. $(E_2) \iff$ *there is a topology generated by a countable enumeration for a basis.*

A consequence of (3) is that any topological statement examined in second-order reverse math follows, in higher-order reverse math, from the existence of such a **formal** topology. So second-order reverse topology does not carry over nicely to higher-order theories.

More simple equivalences

Proposition (H. and folklore)

Over $RCA_0^\omega + (E_1)$, *we have the following equivalences:*

1. $(E_2) \iff$ *there is a topology with a countable dense set.*
2. $(E_2) \iff$ *there is a topology for a space that is the countable union of nowhere-dense sets (i.e., is of first category).*

Topology in $\mathsf{RCA}_0^\omega + (E_1)$

Proposition (H.)

If $\mathcal{T}$ is a topology existing in a minimal term model of $\mathsf{RCA}_0^\omega + (E_1)$ then $\mathcal{T}$ is equivalent to $T \times \mathcal{P}({}^{\mathbb{N}}\mathbb{N} \setminus X)$, where $X = \{x_0, x_1, \dots\}$ is a countable set and T is a topology on X.

(In other words $\mathcal{T}$ is essentially just a topology on $\mathbb{N}$.)

Open questions

Over $\mathrm{RCA}_0^\omega + (E_2)$:

- $\mathrm{QF\text{-}AC}^{1,2} \implies$ "every T_2 space has a witnessing functional" $\implies \mathrm{QF\text{-}AC}^{1,1}$.
- "Every T_2 space has a witnessing functional" $\implies$:
  - every compact T_2 space is T_4.
  - every compact T_2 space has a basis of size $\leq 2^{\aleph_0}$.
- The principle $(E_3) \implies$ that every compact, T_2 space is T_4.

Open question: what about reversals?

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