Brownian Motion and Kolmogorov Complexity

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- “Algorithm”: an informal, intuitive concept.
- “Turing machine”: a precise mathematical concept.

Random real numbers

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- Must restrict attention to a countable collection of measure zero sets.
- The "computable" measure zero sets. Various definitions.
- Definition of random real numbers motivated by the Church-Turing thesis.

Mathematical Brownian Motion

- The basic process in modeling of the stock market in Mathematical Finance, and important in physics and biology.

Brownian Motion

Figure: Botanist Robert Brown (1773-1858)

Brownian Motion

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Pollen grains suspended in water perform a continued swarming motion.

Brownian Motion?

Figure: The fluctuations of the CAC40 index

Mathematical Brownian Motion

A path of Brownian motion is a function $f \in C[0,1]$ or $f \in C(\mathbb{R})$ that is typical with respect to Wiener measure.

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- $f(t)$ is a normally distributed random variable with variance t and mean 0.
- *Stationarity.* $f(1)$ and $f(2006) - f(2005)$ have the same probability distribution.

Brownian Motion and Random Real Numbers

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- Definition of Martin-Löf random continuous functions with respect to Wiener measure: Asarin (1986).

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- Work by Asarin, Pokrovskii, Fouché.

Khintchine's Law of the Iterated Logarithm

The Law of the Iterated Logarithm
holds for $f \in C[0,1]$ at $t \in [0,1]$ if

$$\limsup_{h \to 0} \frac{|f(t+h) - f(t)|}{\sqrt{2|h| \log\log(1/|h|)}} = 1.$$

Theorem (Khintchine)

Fix t. Then almost surely, the LIL holds at t.

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For each Schnorr random Brownian motion, the LIL holds almost everywhere.

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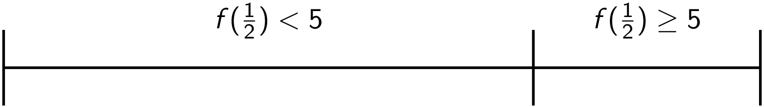
- Method: use *Wiener-Carathéodory measure algebra isomorphism theorem* to translate the problem from $C[0,1]$ into more familiar terrain: $[0,1]$.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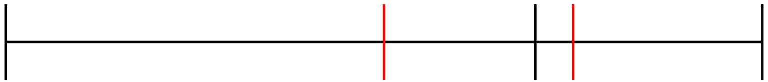

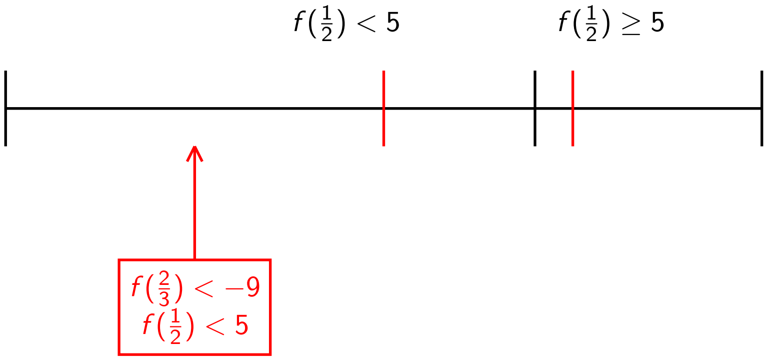
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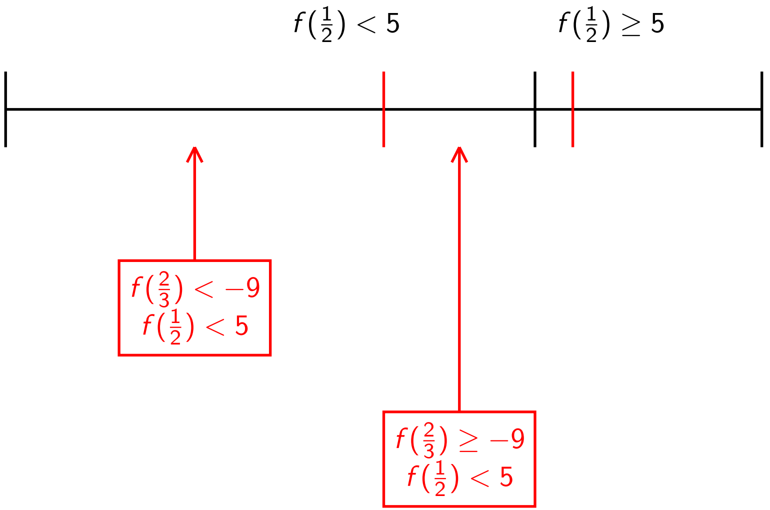
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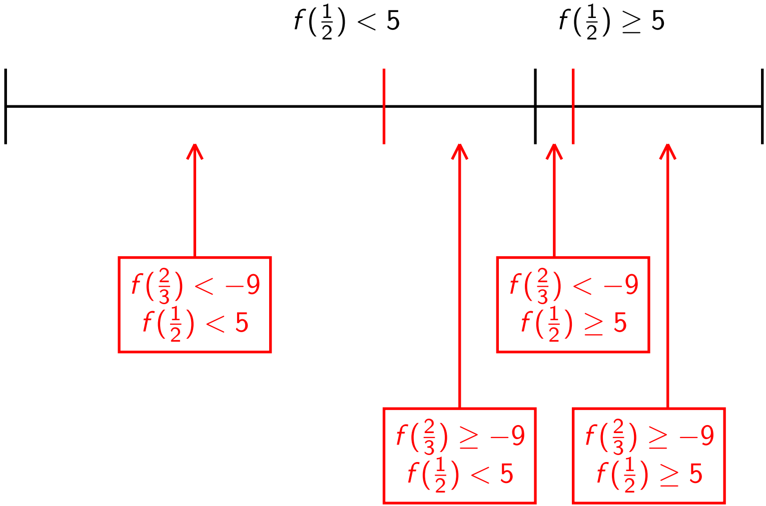
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- The complexity $K(\sigma)$ of a binary string σ is the length of the shortest description of σ by a fixed universal Turing machine having prefix-free domain.
- For a real number $x = 0.x_1x_2\cdots$ we can look at the complexity of the prefixes $x_0\cdots x_n$.

Definition

Let $f \in C[0,1]$, $t \in [0,1]$, and $c \in \mathbb{R}$.
t is a *c-fast time* of f if

$$\limsup_{h \to 0} \frac{|f(t+h) - f(t)|}{\sqrt{2|h| \log 1/|h|}} \geq c.$$

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- Both slow and fast times almost surely exist (and form dense sets) [Orey and Taylor 1974, Davis, Greenwood and Perkins 1983].

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- But actually, all slow points have high Kolmogorov complexity.
- Can prove this using either computability theory or probability theory.

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A set $A \subseteq \mathbb{N}$ is infinitely often c.e. traceable if there is a computable function $p(n)$ such that for all $f : \mathbb{N} \to \mathbb{N}$, if f is computable in A then there is a uniformly c.e. sequence of finite sets E_n of size $\leq p(n)$ such that

$$\exists^\infty n \;\; f(n) \in E_n.$$

Definition

An infinite binary sequence x is autocomplex if there is a function $f : \mathbb{N} \to \mathbb{N}$ with $\lim_n f(n) = \infty$, f computable from x, and

$$K(x \upharpoonright n) \geq f(n).$$

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A sequence x is Kurtz random if $x \notin C$ for any Π^0_1 class C of measure 0.

Theorem (K, Merkle, Stephan)

x is infinitely often c.e. traceable iff x is not autocomplex.

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- If $t \in [0, 1]$ is not of high Kolmogorov complexity then each sufficiently random $f \in C[0, 1]$ is such that t is not a slow point of f.

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On the other hand, these methods can be applied to computability-theoretic problems.

Two notions of *random closed set*

Two probability distributions on closed subsets of Cantor space.

1. "Random closed set" (Barmpalias, Brodhead, Cenzer, Dashti, and Weber (2007)). $1/3$ probability each of: keeping only left branch, keeping only right branch, keeping both branches.
2. Percolation limit set (Hawkes, R. Lyons (1990)). $2/3$ probability of keeping the left branch, and independently $2/3$ probability of keeping the right branch.

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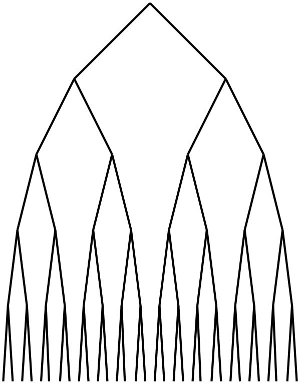

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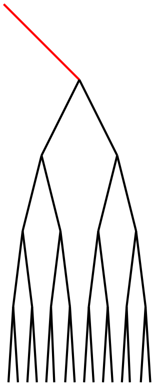

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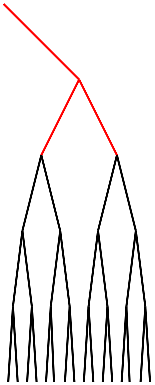

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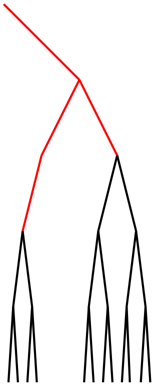

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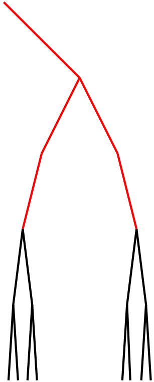

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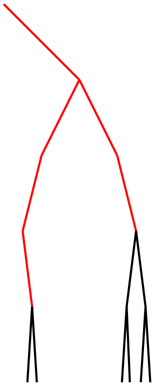

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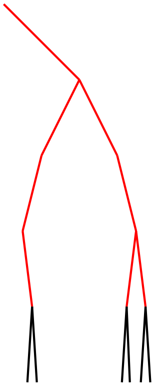

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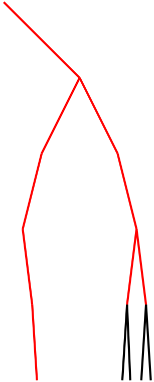

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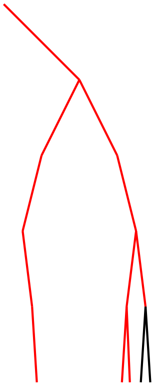

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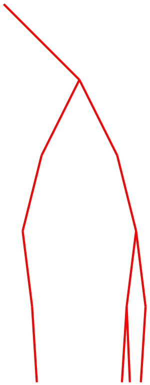

Let $\gamma = \log_2(3/2)$ and $\alpha = 1 - \gamma = \log_2(4/3)$.
Barmpalias, Brodhead, Cenzer, Dashti, and Weber define *(Martin-Löf-)random closed sets* and show that they all have dimension α.
We denote Hausdorff dimension by dim and effective Hausdorff dimension by $\dim^{\emptyset}$. Then

$$\dim^{\emptyset}(x) = \liminf_n \frac{K(x \upharpoonright n)}{n}$$

$$= \sup\{s : x \text{ is } s\text{-Martin-Löf-random}\}.$$

We define a strengthening of Reimann and Stephan's strong γ-randomness, *vehement γ-randomness*. Both notions coincide with Martin-Löf γ-randomness for $\gamma = 1$.

Definition

Let $\rho : 2^{<\omega} \to \mathbb{R}$, $\rho(\sigma) = 2^{-|\sigma|\gamma}$ for some fixed $\gamma \in [0, 1]$. For a set of strings V,

$$\rho(V) := \sum_{\sigma \in V} \rho(\sigma)$$

and

$$[V] := \bigcup \{[\sigma] : \sigma \in V\}$$

.

Definition

A ML-γ-test is a uniformly c.e. sequence $(U_n)_{n<\omega}$ of sets of strings such that for all n,

$$\rho(U_n) \leq 2^{-n}.$$

A strong ML-γ-test is a uniformly c.e. sequence $(U_n)_{n<\omega}$ of sets of strings such that

$$(\forall n)(\forall V \subseteq U_n)[V \text{ prefix-free } \Rightarrow \rho(V) \leq 2^{-n}].$$

A vehement ML-γ-test is a uniformly c.e. sequence $(U_n)_{n<\omega}$ such that for each n there is a set of strings V_n with $[V_n] = [U_n]$ and $\rho(V) \leq 2^{-n}$.

Lemma

Vehemently γ*-random* $\Rightarrow$ *strongly* γ*-random* $\Rightarrow$ γ*-random.*

Theorem

Let $\gamma = \log_2(3/2)$ and let x be a real. We have (1)⇔(2)⇒(3)⇒(4)⇒(5).

1. *x is 1-random;*
2. *x is vehemently 1-random;*
3. *x is vehemently $\gamma + \frac{1-\gamma}{2} \approx 0.8$-random;*
4. *x belongs to some random closed set;*
5. *x is vehemently $\gamma \approx 0.6$-random.*

Corollary (J. Miller and A. Montálban)

The implication from (1) to (4).

Theorem

Suppose x is a member of a random closed set. Then x is vehemently γ-random.

Proof: Random closed sets are denoted by Γ, whereas $\mathfrak{S}$ is the set of strings in the tree corresponding to Γ.
Let $i < 2$ and $\sigma \in 2^{<\omega}$. The probability that the concatenation $\sigma i \in \mathfrak{S}$ given that $\sigma \in \mathfrak{S}$ is, by definition of the BBCDW model,

$$\mathbb{P}\{\sigma i \in \mathfrak{S} | \sigma \in \mathfrak{S}\} = \frac{2}{3}.$$

Hence the absolute probability that σ survives is

$$\mathbb{P}\{\sigma \in \mathfrak{S}\} = \left(\frac{2}{3}\right)^{|\sigma|} = \left(2^{-\gamma}\right)^{|\sigma|} = \left(2^{-|\sigma|}\right)^{\gamma}$$

Suppose x is not vehemently γ-random. So there is some uniformly c.e. sequence $U_n = \{\sigma_{n,i} : i < \omega\}$, such that $x \in \cap_n [U_n]$, and for some $U'_n = \{\sigma'_{n,i} : i < \omega\}$ with $[U'_n] = [U_n]$,

$$\sum_{i=1}^{\infty} 2^{-|\sigma'_{n,i}|\gamma} \leq 2^{-n}.$$

Let

$$V_n := \{\Gamma : \exists i \, \sigma_{n,i} \in \mathfrak{S}\} = \{\Gamma : \exists i \, \sigma'_{n,i} \in \mathfrak{S}\}.$$

The first expression shows V_n is uniformly Σ^0_1. The equality is proved using the fact that $\mathfrak{S}$ is a tree without dead ends.

Now

$$\mathbb{P}V_n \leq \sum_{i\in\omega} \mathbb{P}\{\sigma'_{n,i} \in \mathfrak{S}\} = \sum_{i\in\omega} 2^{-|\sigma'_{n,i}|\gamma} \leq 2^{-n}.$$

That is, if $x \in \Gamma$ then x belongs to the effective null set $\cap_{n\in\omega} V_n$.
As Γ is ML-random, this is not the case.
End of proof.

Corollary

If x belongs to a random closed set, then

$$\dim^{\varnothing}(x) \geq \log_2(3/2).$$

Corollary (BBCDW)

No member of a random closed set is 1-generic.

Theorem

For each $\varepsilon > 0$, each random closed set contains a real x with

$$\dim^{\varnothing}(x) \leq \log_2(3/2) + \varepsilon.$$

Corollary (BBCDW)

Not every member of a random closed set is Martin-Löf random.

Open problems

We have seen that the members of random closed sets do not coincide with the reals of effective dimension $\geq \gamma$, although (1) they all have dimension $\geq \gamma$ and (2) they do not all have dimension $\geq \gamma + \varepsilon$ for any fixed $\epsilon > 0$.
There are (at least) two possible conjectures, and the answer may help determine whether *vehement* or ordinary γ-randomness is the most natural generalization of 1-randomness.

Conjecture (1)

The members of random closed sets are exactly the reals x such that for some $\varepsilon > 0$, x is $\gamma + \varepsilon$-random. (That is, x has effective dimension $> \gamma$.)

Conjecture (2)

The members of random closed sets are exactly the reals x such that for some $\varepsilon > 0$, x is vehemently $\gamma + \varepsilon$-random.

Conjecture 1 would imply that

$$\gamma + \varepsilon\text{-random} \Rightarrow \text{ vehemently } \gamma\text{-random.}$$

This seems unlikely, but J. Reimann has shown that

$$\gamma + \varepsilon\text{-random} \Rightarrow \text{ strongly } \gamma\text{-random.}$$

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Thank You