Complexity of the isomorphism relation for countable models of ω-stable theories

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Motivation: Intuitive notion of complexity of a theory

Example: Vector spaces over a fixed field versus arbitary graphs

We compare two notions of complexity for a theory:

- Shelah's Classification Theory (stability hierarchy, NDOP, depth)
- Borel reducibility (H. Friedman, L. Stanley): a notion coming from descriptive set theory

Borel reducibility

- A *Borel space* is a set X equipped with a σ-algebra $\mathcal{B}$
- $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ is a *Borel map* if for all $A \in \mathcal{B}'$, $f^{-1}[A] \in \mathcal{B}$.
- To a topological space $(X, \mathcal{T})$ we associate a Borel space $(X, \mathcal{B})$: $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{T}$.
- $(X, \mathcal{T})$ is *polish* iff it is completely metrisable and separable. The associated Borel spaces are called standard Borel spaces.
- Examples: finite or countable spaces with discrete topology, $\mathbb{R}$, $\mathcal{C} = 2^\omega$, $\mathcal{N} = \omega^\omega$

Proposition 1. *Let $(X, \mathcal{B})$, $(X', \mathcal{B}')$ be standard. Then*

(1) X is finite, countable or of cardinality $2^{\aleph_0}$

(2) if $|X| = |Y|$ then X and Y are Borel isomorphic

(3) The category of standard Borel spaces is closed under countable products

For X, Y standard and $E \subset X \times X$ and $F \subset Y \times Y$ equivalence relations, we define $E \leq_B F$ iff there is $f : X \to Y$ Borel s.t. for all $a, b \in X$, $a \; E \; b$ iff $f(a) \; F \; f(b)$.

Connection to model theory

Let L be countable. X_L, the set of L-structures living on ω is standard Borel. Example: If $L = \{c, R, f\}$ then

$$X_L = \omega \times 2^{\omega\times\cdots\times\omega} \times \omega^{\omega\times\cdots\times\omega}$$

If $\sigma \in L_{\omega_1\omega}$ then $\mathrm{Mod}(\sigma) \subset X_L$ is *invariant Borel* and thus standard (the converse is also true).

Notation: $\cong_\sigma \quad = \quad \cong_L \restriction \mathrm{Mod}(\sigma)^2, \qquad =_n \ (n \leq \omega), \qquad =_{\mathbb{R}}$

- E is *Borel* if it is a Borel subset of $X \times X$
- E is *smooth* if $E \leq_B =_{\mathbb{R}}$
- E is *countable* if all E-equivalence classes are
- E is *essentially countable* if $E \leq_B F$ for some countable F

ω-stability

A complete first-order theory T is ω-stable if $|S_1(A)| = |A|$ for all infinite A.

We then have

- Morley rank of definable sets
- non-forking extensions of types (Morley rank does not decrease)
- a notion of independence: $A \underset{C}{\downarrow} B$ (i.e. $t(A/BC) \supset t(A/C)$ is non-forking)
- strongly regular types (for which dimension is well-defined)
- prime models over any set

And we can define the property NDOP and depth for T

We have to adapt these notions to the case of countable models:

- ENI types (those which can have *finite* dimension)
- ENI-NDOP
- eni-depth

First results: the extremes

Let T a complete, ω-stable first order theory with infinite models.

Theorem 2. *(Laskowski-Shelah)*

(1) If T has ENI-DOP, then $\cong_T \quad \approx_B \quad \cong_{\text{graphs}}$

(2) If T has ENI-NDOP and is eni-deep, then $\cong_T \quad \approx_B \quad \cong_{\text{graphs}}$

Proposition 3. *If T has $\kappa \leq \aleph_0$ countable models, then* $\cong_T \quad \approx_B \quad \cong_n$

Theorem 4. *If T has ENI-NDOP and eni-depth 1, then* $\cong_T$ *is smooth.*

A cofinal sequence of increasing complexity

Theorem 5. *There exists a sequence* $(T_\alpha)_{1\leq\alpha<\omega_1}$ *of* ω*-stable theories having ENI-NDOP with the following properties : for all* $\alpha < \omega_1$,

- T_α *has depth and eni-depth* α
- $\cong_{T_\alpha}$ *est Borel*
- *for all* β *with* $\alpha < \beta < \omega_1$, $\cong_{T_\alpha} \; <_B \; \cong_{T_\beta}$ *(i.e.* $\cong_{T_\alpha} \; \leq_B \; \cong_{T_\beta}$ *and* $\cong_{T_\beta} \; \not\leq_B \; \cong_{T_\alpha}$*)*

Moreover, $(T_\alpha)_{1\leq\alpha<\omega_1}$ *is Borel-cofinal in the sense that for each countable* L *and* $\sigma \in L_{\omega_1\omega}$, *if* $\cong_\sigma$ *is Borel, then* $\cong_\sigma \; \leq_B \; \cong_{T_\alpha}$ *for some* $\alpha < \omega_1$*.*

A non-Borel theory of depth 2

Theorem 6. *There is a complete first order ω-stable theory having ENI-NDOP, of (eni-) depth 2 whose isomorphism relation is not Borel.*

Let $L = \{\pi_i^j, S_i\}_{i<\omega, j\leq i+1}$ be the language with sorts U, V_i, C_i $(i < \omega)$ and

$\pi_i^j : V_i \to C_j$ for $j \leq i$

$\pi_i^{i+1} : V_i \to U$

$S_i : V_i \to V_i$

and let T be the L-theory that states

(1) $|U| = \infty$ and $|C_i| = 2$ for all $i < \omega$

(2) $\pi_i : V_i \to C_0 \times C_1 \times \cdots \times C_i \times U$ is onto, where $\pi_i(x) = (\pi_i^0(x), \pi_i^1(x), \ldots, \pi_i^{i+1}(x))$

(3) for all $i < \omega$, S_i is a successor function on V_i

(4) $\pi_i \circ S_i = \pi_i$

Description of the isomorphism relation

We fix for all $i < \omega$ $C_i = \{a_i^0, a_i^1\}$.

Essentially, the 1-types are the following:

- $r(x) = \{U(x)\}$
- for $s \in 2^{<\omega} \setminus \{\emptyset\}$ and $b \in U$:
  $p_s^b(x) = \{\pi_{|s|}(x) = (a_{s(0)}, a_{s(1)}, \ldots, a_{|s|-1}, b)\}$

Let $\mathcal{A} = \{f | f : 2^{<\omega} \setminus \{\emptyset\} \to \omega + 1\}$ and define an action of $\mathcal{C}$ on $\mathcal{A}$ by

$$\text{for } \sigma \in \mathcal{C},\ \delta \in \mathcal{A}, \quad \sigma\delta(s) = \delta(s + \sigma \restriction |s|)$$

For $M \models T$ countable and $b \in U(M)$ we define $\delta_b^M \in \mathcal{A}$ by $\delta_b^M(s) = \dim_M(p_s^b) - 1$. Then we can prove

Proposition 7. *Countable $M, N \models T$ are isomorphic if and only if there exists $\sigma \in \mathcal{C}$ and bijective $f : U(M) \to U(N)$ such that for all $b \in U(M)$, $\delta_b^M = \sigma\delta_{f(b)}^N$.*

So, roughly speaking, isomorphism types are countable sets of countably coloured complete binary trees up to "simultaneous flips" of levels.

An idea of the proof of non-Borelness (part I)

(1) We show that SH=∞ ("Scott Height"), which is equivalent to non-Borelness. Goal : define for all $\alpha < \omega_1$ models M, N such that $M \not\cong N$ and $M \equiv_\alpha N$. Recall that

- $(M, \bar{a}) \equiv_0 (N, \bar{b})$ iff $\bar{a}$ and $\bar{b}$ have same quantifier-free type
- $(M, \bar{a}) \equiv_\lambda (N, \bar{b})$ iff $\forall \beta < \lambda$ $(M, \bar{a}) \equiv_\beta (N, \bar{b})$
- $(M, \bar{a}) \equiv_{\alpha+1} (N, \bar{b})$ iff $\forall x \in M \exists y \in N$ $(M, \bar{a}^\frown x) \equiv_\alpha (N, \bar{b}^\frown y)$ and vice versa

(2) What pairs of models $((M, \bar{u}), (N, \bar{v}))$ we consider :

- Fix $(X_i)_{i<\omega}$ independent
- Define trees δ_i with stabiliser $\mathcal{C}^{X_i} = \{\sigma \in \mathcal{C} | \forall n \notin X_i \ \sigma(n) = 0\}$
- M, N realise only orbits $o_i = \mathcal{C}\delta_i$
- For all $i < \omega$, $M^{o_i} \cong N^{o_i}$
- $\bar{u}$ and $\bar{v}$ have same type
- $\{\sigma \in \mathcal{C} | t(\bar{u}/\mathrm{acl}(\emptyset)), t(\bar{v}/\sigma\mathrm{acl}(\emptyset))$ are conjugate$\} \neq \emptyset$

(3) Define configurations : $c = (D, X, d)$ with

- $D : \omega \to \mathcal{P}(\mathcal{C})$
- $D : \omega \to \mathcal{P}(\omega)$
- $d \in \mathcal{P}(\omega)$

An idea of the proof of non-Borelness (part II)

(4) Assign $((M,\bar{u}),(N,\bar{v})) \mapsto c^{((M,\bar{u}),(N,\bar{v}))} = (D, X, d)$:

- $D(i) = \{\sigma | \sigma \text{ allows } M^{o_i} \cong N^{o_i}\}$
- $X(i) = X_i$
- $d = \{\sigma \in \mathcal{C} | t(\bar{u}/\mathrm{acl}(\emptyset)), t(\bar{v}/\sigma\mathrm{acl}(\emptyset)) \text{ are conjugate}\}$

(5) Define *thin* configurations (for $\not\cong$): $\bigcap_{i \in I} D(i) \cap d = \emptyset$ for all $I \subset \omega$ infinite

Define α-*rich* configurations (for $\equiv_{\alpha+\omega}$)
e.g. 0-rich : $\forall i < \omega \; D(i) \cap d \neq \emptyset$

(6) Construction of α-rich configurations for all $\alpha < \omega_1$ and show they are thin.

Some open questions

- Is the non-Borel depth 2 theory as complicated as graphs?
- Are there non-smooth first-order theories which are essentially countable?
- Are there "simple" eni-depth α theories, e.g. smooth ones for $\alpha > 2$?
- What can be said about the superstable case?