Sets and the Concept of Set

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Outline of talk

(1) The concept of set.

(2) Axioms for set theory and the negative thesis.

(3) The positive thesis and its consequences.

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Basic Question. Does the concept of set determine what object the set with given members has to be?

The iterative set hierarchy

Stage 0	$V_0 = \mathbb{N}$
Stage 1	$V_1 = \mathbb{N} \cup \mathcal{P}(\mathbb{N}) = V_0 \cup \mathcal{P}(V_0)$
Stage 2	$V_2 = V_1 \cup \mathcal{P}(V_1)$
Etc.	Through the transfinite

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Informal Separation Axiom. Let x be a set. For any property P, there is a set whose members are those members of x that have property P.

Power Set Axiom. Let x be a set. There is a set whose members are the subsets of x.

Union Axiom. Let x be a set. There is a set whose members are the members of members of x.

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0th Thesis. *The iterative concept of set is not—or, at least, is not merely—a concept of what it is for an object to be a set. It is a concept of what it is to be a universe of sets.*

1st Thesis. *The Informal Separation Axiom is sharp enough to guarantee that any two candidates for* V_1 *(for the natural numbers and the sets of natural numbers) are isomorphic, that any two candidates for* V_2 *are isomorophic, and so one for* V_3, V_4, *etc. Indeed, the iterative concept of set is sharp enough that any two instantiations of it are isomorphic.*

2nd Thesis. *In our present state of knowledge, it is an open question whether the Informal Separation Axiom (or even the iterative concept of set) is sharp enough to determine a truth-value for the Continuum Hypothesis.*

What about the the Natural Numbers?

Basic Question for Natural Numbers	No
0th Thesis for Natural Numbers	Yes
1st Thesis for Natural Numbers	Yes
2nd Thesis for Natural Numbers	No

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ZFC axiom systems

- Informal ZFC axioms
- First-order ZFC axioms
- Second order ZFC axioms

ZFC axioms based on plural logic

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Large cardinal axioms

- Large cardinal axioms imply answers to most important questions about V_1.
- Large cardinal axioms imply nothing about the Continuum Hypothesis, which is a statement about V_2.

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Isomorphism. *any two structures instantiating the concept of set have definably isomorphic V_2's.*

$$\mathfrak{M}_1 \qquad \mathfrak{M}_2$$

$$V_0(\mathfrak{M}_1) \overset{\pi_0}{\cong} V_0(\mathfrak{M}_2)$$

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1st Thesis for V_2 + 2nd Thesis $\Rightarrow$

We do not at present know that there is a structure instantiating the concept of set.