PROOF THEORY AND MEANING

Greg Restall · http://consequently.org

THE UNIVERSITY OF
MELBOURNE

Logic Colloquium 2007

Wrocław · JULY 16

Outline

Scene Setting

Propositional Logic

Quantification

Mathematics

Modality

SCENE SETTING

&–I	&–E	∨–I	∨–E
$\dfrac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$	$\dfrac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \quad \dfrac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$	$\dfrac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} \quad \dfrac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}$	$\dfrac{\mathfrak{A} \vee \mathfrak{B} \quad \begin{matrix}[\mathfrak{A}] \\ \mathfrak{C}\end{matrix} \quad \begin{matrix}[\mathfrak{B}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$
∀–I	**∀–E**	**∃–I**	**∃–E**
$\dfrac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$	$\dfrac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$	$\dfrac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$	$\dfrac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \begin{matrix}[\mathfrak{F}\mathfrak{a}] \\ \mathfrak{C}\end{matrix}}{\mathfrak{C}}$
⊃–I	**⊃–E**	**¬–I**	**¬–E**
$\dfrac{\begin{matrix}[\mathfrak{A}] \\ \mathfrak{B}\end{matrix}}{\mathfrak{A} \supset \mathfrak{B}}$	$\dfrac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$	$\dfrac{\begin{matrix}[\mathfrak{A}] \\ \curlywedge\end{matrix}}{\neg \mathfrak{A}}$	$\dfrac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \quad \dfrac{\curlywedge}{\mathfrak{D}}$.

Gerhard Gentzen: “Untersuchungen uber das logische Schliessen” *Math. Zeitschrift* 1934

Inference rules *define* connectives

$$\frac{A \quad B}{A \wedge B}\,[\wedge I] \qquad \frac{A \wedge B}{A}\,[\wedge E_1] \quad \frac{A \wedge B}{B}\,[\wedge E_2]$$

Inference rules *define* connectives

$$\frac{A \qquad B}{A \wedge B}[\wedge I] \qquad \frac{A \wedge B}{A}[\wedge E_1] \quad \frac{A \wedge B}{B}[\wedge E_2]$$

That's *all there is* to conjunction.

Inference rules *define* connectives

$$\frac{A \qquad B}{A \wedge B}\ [\wedge I] \qquad\qquad \frac{A \wedge B}{A}\ [\wedge E_1] \quad \frac{A \wedge B}{B}\ [\wedge E_2]$$

That's *all there is* to conjunction.

You don't need to give *truth conditions*,
satisfaction conditions
or any other sort of 'semantics.'

Inference rules *define* connectives

$$\frac{A \qquad B}{A \wedge B}\,[\wedge I] \qquad \frac{A \wedge B}{A}\,[\wedge E_1] \quad \frac{A \wedge B}{B}\,[\wedge E_2]$$

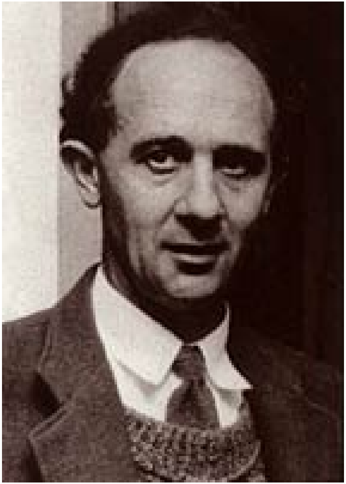
That's *all there is* to conjunction.

You don't need to give *truth conditions*,
satisfaction conditions
or any other sort of 'semantics.'

These rules tie *meaning* to *use*.

But . . . does it work?

But . . . does it work?



THE RUNABOUT INFERENCE-TICKET

By A. N. Prior

IT is sometimes alleged that there are inferences whose validity arises solely from the meanings of certain expressions occurring in them. The precise technicalities employed are not important, but let us say that such inferences, if any such there be, are analytically valid.

One sort of inference which is sometimes said to be in this sense analytically valid is the passage from a conjunction to either of its conjuncts, e.g., the inference ' Grass is green and the sky is blue, therefore grass is green '. The validity of this inference is said to arise solely from the meaning of the word ' and '. For if we are asked what is the meaning of the word ' and ', at least in the purely conjunctive sense (as opposed to, e.g., its colloquial use to mean ' and then '), the answer is said to be *completely* given by saying that (i) from any pair of statements P and Q we can infer the statement formed by joining P to Q by ' and ' (which statement we hereafter describe as ' the statement P-and-Q '), that (ii) from any conjunctive statement P-and-Q we can infer P, and (iii) from P-and-Q we can always infer Q. Anyone who has learnt to perform these inferences knows the meaning of ' and ', for there is simply nothing more *to* knowing the meaning of ' and ' than being able to perform these inferences.

A doubt might be raised as to whether it is really the case that, for any pair of statements P and Q, there is always a statement R such that given P and given Q we can infer R, and given R we can infer P and can also infer Q. But on the view we are considering such a doubt is quite misplaced, once we have introduced a word, say the word ' and ', precisely in order to form a statement R with these properties from any pair of statements P and Q. The doubt reflects the old superstitious view that an expression must have some independently determined meaning before we can discover whether inferences involving it are valid or invalid. With analytically valid inferences this just isn't so.

I hope the conception of an analytically valid inference is now at least as clear to my readers as it is to myself. If not, further illumination is obtainable from Professor Popper's paper on ' Logic without Assumptions ' in *Proceedings of the Aristotelian Society* for 1946–7, and from Professor Kneale's contribution to *Contemporary British Philosophy*, Volume III. I have also been much helped in my understanding of the notion by some lectures of Mr. Strawson's and some notes of Mr. Hare's.

I want now to draw attention to a point not generally noticed, namely that in this sense of ' analytically valid ' any statement whatever may be

But . . . does it work?

$$\frac{A}{A \text{ tonk } B}\,[\text{tonk}I] \qquad \frac{A \text{ tonk } B}{B}\,[\text{tonk}E]$$

But . . . does it work?

$$\frac{A}{A \text{ tonk } B}\,[\text{tonk}I] \qquad \frac{A \text{ tonk } B}{B}\,[\text{tonk}E]$$

$$\cfrac{\cfrac{p}{p \text{ tonk } q}\,[\text{tonk}I]}{q}\,[\text{tonk}E]$$

It would be *bad* to have tonk in your language.

But . . . does it work?

TONK, PLONK AND PLINK[1]

By Nuel D. Belnap

A. N. PRIOR has recently discussed[2] the connective *tonk*, where *tonk* is defined by specifying the role it plays in inference. Prior characterizes the role of *tonk* in inference by describing how it behaves as conclusion, and as premiss: (1) A ⊦ A-*tonk*-B, and (2) A-*tonk-B* ⊦ B (where we have used the sign ' ⊦ ' for deducibility). We are then led by the transitivity of deducibility to the validity of A ⊦ B, " which promises to banish *falsche Spitzfindigkeit* from Logic for ever."

A possible moral to be drawn is that connectives cannot be defined in terms of deducibility at all; that, for instance, it is illegitimate to define *and* as that connective such that (1) A-*and*-B ⊦ A, (2) A-*and*-B ⊦ B, and (3) A, B ⊦ A-*and*-B. We must first, so the moral goes, have a notion of what *and* means, independently of the role it plays as premiss and as conclusion. Truth-tables are one way of specifying this antecedent meaning; this seems to be the moral drawn by J. T. Stevenson.[3] There are good reasons, however, for defending the legitimacy of defining connections in terms of the roles they play in deductions.

It seems plain that throughout the whole texture of philosophy one can distinguish two modes of explanation: the analytic mode, which tends to explain wholes in terms of parts, and the synthetic mode, which explains parts in terms of the wholes or contexts in which they occur.[4] In logic, the analytic mode would be represented by Aristotle, who commences with terms as the ultimate atoms, explains propositions or judgments by means of these, syllogisms by means of the propositions which go to make them up, and finally ends with the notion of a science as a tissue of syllogisms. The analytic mode is also represented by the contemporary logician who first explains the meaning of complex sentences, by means of truth-tables, as a function of their parts, and then proceeds to give an account of correct inference in terms of the sentences occurring therein. The *locus classicus* of the application of the synthetic mode is, I suppose, Plato's treatment of justice in the *Republic*, where he defines the just man by reference to the larger context of the community. Among formal logicians, use of the synthetic mode in logic is illustrated by Kneale and Popper (cited by Prior), as well as by Jaskowski, Gentzen, Fitch, and Curry, all of these treating the meaning of connectives as

[1] This research was supported in part by the Office of Naval Research, Group Psychology Branch, Contract No. SAR/Nonr-609(16).

[2] ' The Runabout Inference-ticket ', Analysis 21.2, December 1960.

[3] ' Roundabout the Runabout Inference-ticket ', Analysis 21.6, June 1961. Cf. p. 127: " The important difference between the theory of analytic validity [Prior's phrase for what is here called the synthetic view] as it should be stated and as Prior stated it lies in the fact that he gives the meaning of connectives in terms of permissive rules, whereas they should be stated in terms of truth-function statements in a meta-language."

[4] I learned this way of looking at the matter from R. S. Brumbaugh.

Nuel Belnap: "Tonk, Plonk and Plink" *Analysis* 1962

But . . . does it work?

> It seems to me that the key to a solution[2] lies in observing that even on the synthetic view, we are not defining our connectives *ab initio*, but rather in terms of an *antecedently given context of deducibility*, concerning which we have some definite notions. By that I mean that before arriving at the problem of characterizing connectives, we have already made some assumptions about the nature of deducibility. That this is so can be seen immediately by observing Prior's use of the transitivity of deducibility in order to secure his ingenious result. But if we note that we already *have* some assumptions about the context of deducibility within which we are operating, it becomes apparent that by a too careless use of definitions, it is possible to create a situation in which we are forced to say things inconsistent with those assumptions.

Nuel Belnap: "Tonk, Plonk and Plink" *Analysis* 1962

But . . . does it work?

(1) We consider some characterization of deducibility, which may be treated as a formal system, *i.e.*, as a set of axioms and rules involving the sign of deducibility, '$\vdash$', where '$A_1, \ldots, A_n \vdash B$' is read 'B is deducible from $A_1, \ldots, A_n$.' For definiteness, we shall choose as our characterization the structural rules of Gentzen:

Axiom. $A \vdash A$

Rules.
- *Weakening:* from $A_1, \ldots, A_n \vdash C$ to infer $A_1, \ldots, A_n B \vdash C$
- *Permutation:* from $A_1, \ldots, A_i, A_{i+1}, \ldots, A_n \vdash B$ to infer $A_1, \ldots, A_{i+1}, A_i, \ldots, A_n \vdash B$.
- *Contraction:* from $A_1, \ldots, A_n, A_n \vdash B$ to infer $A_1, \ldots, A_n \vdash B$
- *Transitivity:* from $A_1, \ldots, A_m \vdash B$ and $C_1, \ldots, C_n, B \vdash D$ to infer $A_1, \ldots, A_m, C_1, \ldots, C_n \vdash D$.

In accordance with the opinions of experts (or even perhaps on more substantial grounds) we may take this little system as expressing all and only the universally valid statements and rules expressible in the given notation: it completely determines the context.

(2) We may consider the proposed definition of some connective, say *plonk*, as an *extension* of the formal system characterizing deducibility, and an extension in two senses. (a) The notion of sentence is extended by introducing A-*plonk*-B as a sentence, whenever A and B are sentences. (b) We add some axioms or rules governing A-*plonk*-B as occurring as one of the premisses or as conclusion of a deducibility-statement. These axioms or rules constitute our definition of *plonk* in terms of the role it plays in inference.

(3) We may now state the demand for the consistency of the definition of the new connective, *plonk*, as follows: the extension must be *conservative*[1]; *i.e.*, although the extension may well have new deducibility-statements, these new statements will all involve *plonk*. The extension will not have any new deducibility-statements which do not involve *plonk* itself. It will not lead to any deducibility-statement $A_1, \ldots, A_n \vdash B$ not containing *plonk*, unless that statement is already provable in the absence of the *plonk*-axioms and *plonk*-rules. The justification for unpacking the demand for consistency in terms of conservativeness is precisely our antecedent assumption that we already had *all* the universally valid deducibility-statements not involving any special connectives.

Nuel Belnap: "Tonk, Plonk and Plink" *Analysis* 1962

- "EXISTENCE": Good rules conservatively extend your prior commitments concerning consequence.

- "EXISTENCE": Good rules conservatively extend your prior commitments concerning consequence.
  - A subtlety: you could, of course *revise* your account of consequence in the original vocabulary. That's OK for Nuel.

- "EXISTENCE": Good rules conservatively extend your prior commitments concerning consequence.
  - A subtlety: you could, of course *revise* your account of consequence in the original vocabulary. That's OK for Nuel.
  - EXAMPLE: Peirce's Law, after adding Boolean negation to the rules for the material conditional.

Existence and Uniqueness

- "EXISTENCE": Good rules conservatively extend your prior commitments concerning consequence.
  - A subtlety: you could, of course *revise* your account of consequence in the original vocabulary. That's OK for Nuel.
  - EXAMPLE: Peirce's Law, after adding Boolean negation to the rules for the material conditional.
- "UNIQUENESS": The proposed rules should *fix* the concept if they are to be *definitions*.

- "EXISTENCE": Good rules conservatively extend your prior commitments concerning consequence.
  - A subtlety: you could, of course *revise* your account of consequence in the original vocabulary. That's OK for Nuel.
  - EXAMPLE: Peirce's Law, after adding Boolean negation to the rules for the material conditional.
- "UNIQUENESS": The proposed rules should *fix* the concept if they are to be *definitions*.
  - If you add $*_1$ and $*_2$ using identical rules, then they should be *equivalent*.

Existence and Uniqueness

- "EXISTENCE": Good rules conservatively extend your prior commitments concerning consequence.
  - A subtlety: you could, of course *revise* your account of consequence in the original vocabulary. That's OK for Nuel.
  - EXAMPLE: Peirce's Law, after adding Boolean negation to the rules for the material conditional.
- "UNIQUENESS": The proposed rules should *fix* the concept if they are to be *definitions*.
  - If you add $*_1$ and $*_2$ using identical rules, then they should be *equivalent*.

How far can we go, keeping *existence* and *uniqueness*?

PROPOSITIONAL LOGIC

A tonk connective doesn't pass the 'existence' test for most accounts of logical consequence.

Non-Conservative Extension

However it's a bit *complicated* . . .

Journal of Philosophical Logic (2005) 34: 217–226
DOI: 10.1007/s10992-004-7805-x

ROY T. COOK

WHAT'S WRONG WITH TONK(?)

1. TONK AND LOGIC

In "The Runabout Inference Ticket" A. N. Prior (1960) examines the idea that logical connectives can be given a meaning solely in virtue of the stipulation of a set of rules governing them, and thus that logical truth/consequence can be explicated in terms of the meanings (so understood) of the logical connectives involved. He proposes a counterexample to such a view, his notorious binary connective tonk (which I will symbolize as ⊗), whose meaning is given by the following introduction and elimination rules:

Journal of Philosophical Logic (2006) 35: 653–660
DOI: 10.1007/s10992-006-9025-z

HEINRICH WANSING

CONNECTIVES STRANGER THAN TONK

Received on 30 September 2005

ABSTRACT. Many logical systems are such that the addition of Prior's binary connective `tonk` to them leads to triviality, see [1, 8]. Since `tonk` is given by some introduction and elimination rules in natural deduction or sequent rules in Gentzen's sequent calculus, the unwanted effects of adding `tonk` show that some kind of restriction has to be imposed on the acceptable operational inferences rules, in particular if these rules are regarded as definitions of the operations concerned. In this paper, a number of simple observations is made showing that the unwanted phenomenon exemplified by `tonk` in some logics also occurs in contexts in which `tonk` *is* acceptable. In fact, in any non-trivial context, the acceptance of arbitrary introduction rules for logical operations permits operations leading to triviality. Connectives that in all non-trivial contexts lead to triviality will be called *non-trivially trivializing connectives*.

A tonk connective doesn't pass the 'existence' test for most accounts of logical consequence.

The traditional connectives fare somewhat better.

But not *quite* as well as you might think . . .

Suppose that in the vocabulary p, q, . . . the *only* proofs are identities.

A [*Id*]

But not *quite* as well as you might think . . .

Suppose that in the vocabulary p, q, . . . the *only* proofs are identities. Now add Gentzen's *conjunction*.

$$A \ [Id] \qquad \frac{A \qquad B}{A \wedge B}\,[\wedge I] \qquad \frac{A \wedge B}{A}\,[\wedge E_l] \quad \frac{A \wedge B}{B}\,[\wedge E_r]$$

But not *quite* as well as you might think . . .

Suppose that in the vocabulary p, q, . . . the *only* proofs are identities. Now add Gentzen's *conjunction*.

$$A \;\; [Id] \qquad \frac{A \qquad B}{A \wedge B}\,[\wedge I] \qquad \frac{A \wedge B}{A}\,[\wedge E_l] \quad \frac{A \wedge B}{B}\,[\wedge E_r]$$

Now we have a proof from p and q to p.

$$\cfrac{\cfrac{p \qquad q}{p \wedge q}\,[\wedge I]}{p}\,[\wedge E]$$

Suppose that in the vocabulary p, q, . . . the *only* proofs are identities. Now add Gentzen's *conjunction*.

$$A \ [Id] \qquad \frac{A \qquad B}{A \wedge B}\,[\wedge I] \qquad \frac{A \wedge B}{A}\,[\wedge E_l] \quad \frac{A \wedge B}{B}\,[\wedge E_r]$$

Now we have a proof from p and q to p.

$$\dfrac{\dfrac{p \qquad q}{p \wedge q}\,[\wedge I]}{p}\,[\wedge E]$$

We didn't have one before. The addition is non-conservative.

Why people haven't noticed this,

Why people haven't noticed this,

- We have a proof whose premises are *among* p, q, and whose conclusion is p – just not a proof with those premises *exactly*.

Why people haven't noticed this,

- We have a proof whose premises are *among* p, q, and whose conclusion is p – just not a proof with those premises *exactly*.
- Most people think that the argument from p, q to p is *valid*, despite not having a (normal) proof with those premises and that conclusion.

Why people haven't noticed this, and what we can do to fix it

- We have a proof whose premises are *among* p, q, and whose conclusion is p – just not a proof with those premises *exactly*.
- Most people think that the argument from p, q to p is *valid*, despite not having a (normal) proof with those premises and that conclusion.

- We have a proof whose premises are *among* p, q, and whose conclusion is p – just not a proof with those premises *exactly*.
- Most people think that the argument from p, q to p is *valid*, despite not having a (normal) proof with those premises and that conclusion.
- Accept primitive *weakening* proofs, like this: $\dfrac{p \quad q}{p}\,[K]$

- We have a proof whose premises are *among* p, q, and whose conclusion is p – just not a proof with those premises *exactly*.
- Most people think that the argument from p, q to p is *valid*, despite not having a (normal) proof with those premises and that conclusion.

- Accept primitive *weakening* proofs, like this: $\dfrac{p \quad q}{p}\,[K]$
- Reject weakening as *invalid*, and hence reject $[\wedge I]$ or $[\wedge E]$.

- We have a proof whose premises are *among* p, q, and whose conclusion is p – just not a proof with those premises *exactly*.
- Most people think that the argument from p, q to p is *valid*, despite not having a (normal) proof with those premises and that conclusion.

- Accept primitive *weakening* proofs, like this: $\dfrac{p \quad q}{p}\,[K]$
- Reject weakening as *invalid*, and hence reject $[\wedge I]$ or $[\wedge E]$.
- Put up with the mismatch between *validity* (the argument from p, q to p is *valid*) and *proofs* (there is no proof from p, q to p) in the basic language.

This means *paying attention* to the context of deducibility.

This means *paying attention* to the context of deducibility.

Let's look at some of the assumptions we've been making.

Choice of proof structure

$$\dfrac{\dfrac{p \land (q \lor r)}{q \lor r}[\land E] \qquad \dfrac{\dfrac{\dfrac{p \land (q \lor r)}{p}[\land E] \quad [q]}{p \land q}[\land I]}{(p \land q) \lor (p \land r)}[\lor I] \qquad \dfrac{\dfrac{\dfrac{p \land (q \lor r)}{p}[\land E] \quad [r]}{p \land r}[\land I]}{(p \land q) \lor (p \land r)}[\lor I]}{(p \land q) \lor (p \land r)}[\lor E]$$

Gentzen proofs have premises and a conclusion.

$$\dfrac{\dfrac{p \wedge (q \vee r)}{q \vee r}\,[\wedge E] \qquad \dfrac{\dfrac{\dfrac{p \wedge (q \vee r)}{p}\,[\wedge E] \qquad [q]}{p \wedge q}\,[\wedge I]}{(p \wedge q) \vee (p \wedge r)}\,[\vee I] \qquad \dfrac{\dfrac{\dfrac{p \wedge (q \vee r)}{p}\,[\wedge E] \qquad [r]}{p \wedge r}\,[\wedge I]}{(p \wedge q) \vee (p \wedge r)}\,[\vee I]}{(p \wedge q) \vee (p \wedge r)}\,[\vee E]$$

Gentzen proofs have premises and a conclusion.

1	(1)	$p \wedge (q \vee r)$	A
1	(2)	p	1,$\wedge E$
1	(3)	$q \vee r$	1,$\wedge E$
4	(4)	q	A
5	(5)	r	A
1,4	(6)	$p \wedge q$	2,4,$\wedge I$
1,4	(7)	$(p \wedge q) \vee (p \wedge r)$	6,$\vee I$
1,5	(8)	$p \wedge r$	2,5,$\wedge I$
1, 5	(9)	$(p \wedge q) \vee (p \wedge r)$	8,$\vee I$
1	(10)	$(p \wedge q) \vee (p \wedge r)$	3,4,5,7,9,$\vee E$

So do Lemmon proofs.

Choice of proof structure

1	$p \wedge (q \vee r)$	A
2	p	1,$\wedge E$
3	$q \vee r$	1,$\wedge E$
4	q	A
5	$p \wedge q$	2,5,$\wedge I$
6	$(p \wedge q) \vee (p \wedge r)$	5,$\vee I$
7	r	A
8	$p \wedge r$	2,7,$\wedge I$
9	$(p \wedge q) \vee (p \wedge r)$	8,$\vee I$
10	$(p \wedge q) \vee (p \wedge r)$	3,4-6,7-9,$\vee E$

And so do Fitch proofs.

These proofs match *sequents* with premises and a conclusion

These proofs match *sequents* with premises and a conclusion

$$A_1, \ldots, A_n \vdash B$$

These proofs match *sequents* with premises and a conclusion

$$A_1, \ldots, A_n \vdash B$$

The natural rules for the conditional in this context are *incomplete* for classical logic.

$$\frac{X, A \vdash B}{X \vdash A \supset B}\,[\supset R] \qquad \frac{X \vdash A \qquad Y, B \vdash C}{X, Y, A \supset B \vdash C}\,[\supset L]$$

These proofs match *sequents* with premises and a conclusion

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

The natural rules for the conditional in this context are *incomplete* for classical logic.

$$\frac{X, A \vdash B, Y}{X \vdash A \supset B, Y}\,[\supset R] \qquad \frac{X \vdash A, W \qquad Y, B \vdash Z}{X, Y, A \supset B \vdash Z, W}\,[\supset L]$$

But if we allow conclusions, the rules become complete for classical logic.

These proofs match *sequents* with premises and a conclusion

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

The natural rules for the conditional in this context are *incomplete* for classical logic.

$$\frac{X, A \vdash B, Y}{X \vdash A \supset B, Y}\,[\supset R] \qquad \frac{X \vdash A, W \qquad Y, B \vdash Z}{X, Y, A \supset B \vdash Z, W}\,[\supset L]$$

But if we allow conclusions, the rules become complete for classical logic.

Are there any *proofs* that look like *that*?

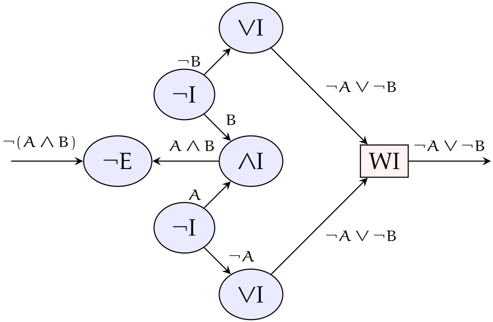
Well, yes

Proofs with *restart* do. $\dfrac{A}{B}\,[restart]$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p]^1}{p \vee (q \supset p)}\,[\vee I]}{q}\,[restart]}{p \supset q}\,[\supset I^1]}{p \vee (p \supset q)}\,[\vee I]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{p \vdash p \vee (p \supset q)}\,[\vee R]}{p \vdash p \vee (p \supset q), q}\,[KR]}{\vdash p \vee (p \supset q), p \supset q}\,[\supset R]}{\vdash p \vee (p \supset q), p \vee (p \supset q)}\,[\vee R]}{\vdash p \vee (p \supset q)}\,[WR]$$

And so do circuits.



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \neg A}}{\vdash A, \neg A \vee \neg B} \qquad \dfrac{\dfrac{A \vdash A}{\vdash B, \neg B}}{\vdash B, \neg A \vee \neg B}}{\vdash A \wedge B, \neg A \vee \neg B, \neg A \vee \neg B}}{\neg(A \wedge B) \vdash \neg A \vee \neg B, \neg A \vee \neg B}}{\neg(A \wedge B) \vdash \neg A \vee \neg B}$$

How do we Choose?

Different *contexts of deducibility* motivate different logics.

- No weakening? Relevant logic.

How do we Choose?

Different *contexts of deducibility* motivate different logics.

- No weakening? Relevant logic.
- Two uses differs from one? linear or other contraction-free logics.

Different *contexts of deducibility* motivate different logics.

- No weakening? Relevant logic.
- Two uses differs from one? linear or other contraction-free logics.
- Single conclusions? Intuitionistic logic.

How do we Choose?

Different *contexts of deducibility* motivate different logics.

- No weakening? Relevant logic.
- Two uses differs from one? linear or other contraction-free logics.
- Single conclusions? Intuitionistic logic.
- Multiple conclusions? Classical logic.

How do we Choose?

Different *contexts of deducibility* motivate different logics.

- No weakening? Relevant logic.
- Two uses differs from one? linear or other contraction-free logics.
- Single conclusions? Intuitionistic logic.
- Multiple conclusions? Classical logic.

So how do we pick?

Different *contexts of deducibility* motivate different logics.

- No weakening? Relevant logic.
- Two uses differs from one? linear or other contraction-free logics.
- Single conclusions? Intuitionistic logic.
- Multiple conclusions? Classical logic.

So how do we pick?

It depends, of course, on what a proof is *for*.

I’ve been through this before . . .

Multiple Conclusions

Greg Restall*

Philosophy Department, The University of Melbourne
restall@unimelb.edu.au

Abstract. I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with *multiple* premises and *multiple* conclusions. Gentzen, Gerhard’s multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for *classical* logic as it does for *intuitionistic* logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us und erstand and adjudicate debates between proponents of classical and non-classical logics.

LMPS, Oviedo, 2003

Multiple Conclusions

Greg Restall*

Philosophy Department, The University of Melbourne
restall@unimelb.edu.au

Abstract. I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with *multiple* premises and *multiple* conclusions. Gentzen, Gerhard's multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for *classical* logic as it does for *intuitionistic* logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us und erstand and adjudicate debates between proponents of classical and non-classical logics.

LMPS, Oviedo, 2003

- A proof from X to A rules out the (assertion of the Xs and denial of A).

I’ve been through this before . . .

Multiple Conclusions

Greg Restall*

Philosophy Department, The University of Melbourne
restall@unimelb.edu.au

Abstract. I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with *multiple* premises and *multiple* conclusions. Gentzen, Gerhard’s multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for *classical* logic as it does for *intuitionistic* logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us und erstand and adjudicate debates between proponents of classical and non-classical logics.

LMPS, Oviedo, 2003

- A proof from X to *A* rules out the (assertion of the Xs and denial of *A*).
- This works *just* as well with multiple conclusions: a proof from X to Y rules out the (assertion of the Xs and denial of the Ys).

I’ve been through this before . . .

Multiple Conclusions

Greg Restall*

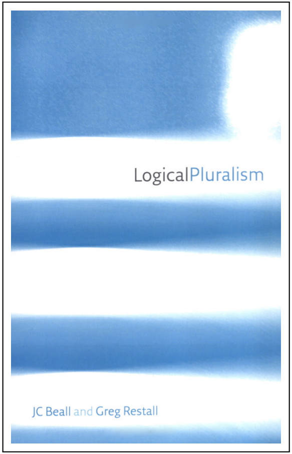
Philosophy Department, The University of Melbourne
restall@unimelb.edu.au

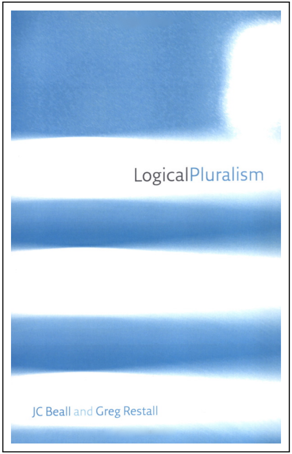
Abstract. I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with *multiple* premises and *multiple* conclusions. Gentzen, Gerhard’s multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for *classical* logic as it does for *intuitionistic* logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us und erstand and adjudicate debates between proponents of classical and non-classical logics.

LMPS, Oviedo, 2003

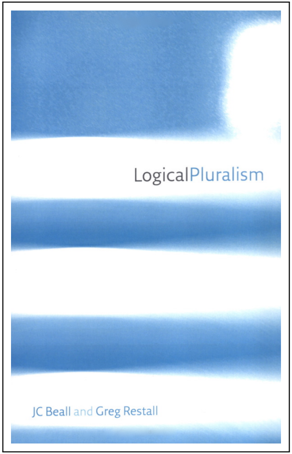
- A proof from X to A rules out the (assertion of the Xs and denial of A).
- This works *just* as well with multiple conclusions: a proof from X to Y rules out the (assertion of the Xs and denial of the Ys).
- Proofs provide normative statuses of combinations of assertions and denials.

Of course, there are others

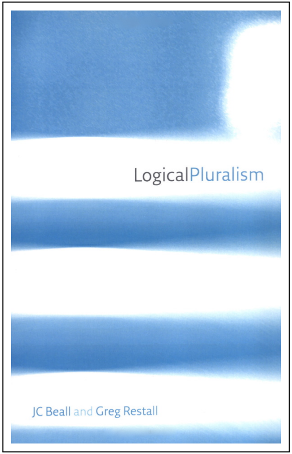




- Proofs as *functions* converting warrants for premises into warrant for a conclusion?

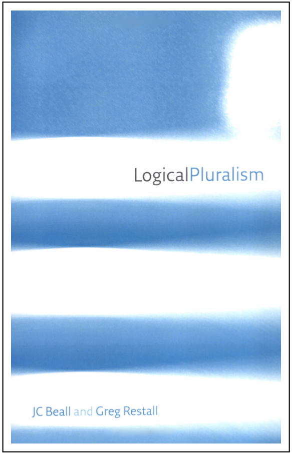


- Proofs as *functions* converting warrants for premises into warrant for a conclusion? *Intuitionistic logic*.

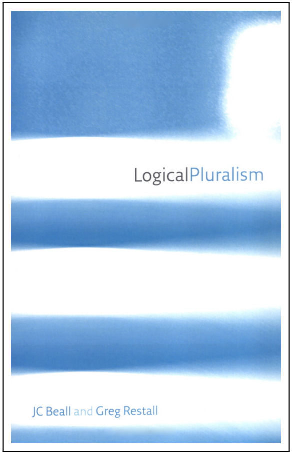


- Proofs as *functions* converting warrants for premises into warrant for a conclusion? *Intuitionistic logic*.
- Proofs keeping track of *use*?

Of course, there are others



- Proofs as *functions* converting warrants for premises into warrant for a conclusion? *Intuitionistic logic*.
- Proofs keeping track of *use*? *Relevant logic*.



- Proofs as *functions* converting warrants for premises into warrant for a conclusion? *Intuitionistic logic*.
- Proofs keeping track of *use*? *Relevant logic*.
- Different contexts of deducibility track different normative statuses. There is no need to choose *one* as the WHOLE STORY.

Belnap provides us with two criteria to evaluate our rules:

Once you have a context of deducibility ...

Belnap provides us with two criteria to evaluate our rules:

- CONSERVATIVENESS: don't add new inferences to the old vocabulary.

Once you have a context of deducibility . . .

Belnap provides us with two criteria to evaluate our rules:

- CONSERVATIVENESS: don't add new inferences to the old vocabulary.
- UNIQUENESS: do they *define*, or merely *describe*?

Belnap provides us with two criteria to evaluate our rules:

- CONSERVATIVENESS: don't add new inferences to the old vocabulary.
  - This can be supplied by a cut-elimination or normalisation theorem.
- UNIQUENESS: do they *define*, or merely *describe*?

Belnap provides us with two criteria to evaluate our rules:

- CONSERVATIVENESS: don't add new inferences to the old vocabulary.
  - This can be supplied by a cut-elimination or normalisation theorem.
  - We have this for classical propositional logic (and circuits or proofs with restart, and many other systems).
- UNIQUENESS: do they *define*, or merely *describe*?

Once you have a context of deducibility …

Belnap provides us with two criteria to evaluate our rules:

- CONSERVATIVENESS: don't add new inferences to the old vocabulary.
  - This can be supplied by a cut-elimination or normalisation theorem.
  - We have this for classical propositional logic (and circuits or proofs with restart, and many other systems).
- UNIQUENESS: do they *define*, or merely *describe*?
  - This is supplied by a simple argument for each rule:

Once you have a context of deducibility . . .

Suppose we have *two* conjunctions $\wedge$ and $\&$, both satisfying the usual rules.

Suppose we have *two* conjunctions $\wedge$ and $\&$, both satisfying the usual rules.

We have the following proofs

$$\dfrac{\dfrac{A \,\&\, B}{A}\,[\&E] \qquad \dfrac{A \,\&\, B}{B}\,[\&E]}{A \wedge B}\,[\wedge I] \qquad\qquad \dfrac{\dfrac{A \wedge B}{A}\,[\wedge E] \qquad \dfrac{A \wedge B}{B}\,[\wedge E]}{A \,\&\, B}\,[\& I]$$

So $\wedge$ is interchangeable with $\&$ as a premise or a conclusion in any argument. They are *equivalent*.

$$\&\text{-}IS:\ \frac{\Gamma \rightarrow \Theta, \mathfrak{A} \qquad \Gamma \rightarrow \Theta, \mathfrak{B}}{\Gamma \rightarrow \Theta, \mathfrak{A} \,\&\, \mathfrak{B}},$$

$$\&\text{-}IA:\ \frac{\mathfrak{A}, \Gamma \rightarrow \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \rightarrow \Theta} \qquad \frac{\mathfrak{B}, \Gamma \rightarrow \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \rightarrow \Theta},$$

$$\vee\text{-}IA:\ \frac{\mathfrak{A}, \Gamma \rightarrow \Theta \qquad \mathfrak{B}, \Gamma \rightarrow \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \rightarrow \Theta},$$

$$\vee\text{-}IS:\ \frac{\Gamma \rightarrow \Theta, \mathfrak{A}}{\Gamma \rightarrow \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \rightarrow \Theta, \mathfrak{B}}{\Gamma \rightarrow \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

Gentzen's rules for classical propositional connectives satisfy existence and uniqueness in this context of deducibility.

QUANTIFICATION

$$\frac{(\forall x)A(x)}{A(t)}\,[\forall E] \qquad \frac{A(c)}{(\forall x)A(x)}\,[\forall I]$$

These rules seem straightforward,

$$\frac{(\forall x)A(x)}{A(t)}\,[\forall E] \quad \text{(for any term } t\text{)} \qquad \frac{A(c)}{(\forall x)A(x)}\,[\forall I] \quad \text{(for any constant } c \text{ not in the premises)}$$

These rules seem straightforward, but things are subtle.

$$\frac{(\forall x)A(x)}{A(t)}\ [\forall E] \quad \text{(for any term } t\text{)} \qquad \frac{A(c)}{(\forall x)A(x)}\ [\forall I] \quad \text{(for any constant } c \text{ not in the premises)}$$

These rules seem straightforward, but things are subtle.

They depend on an *analysis* of formulas, identifying constituents as *terms*, and defining the appropriate notion of *substitution*.

$$\frac{(\forall x)A(x)}{A(t)}\,[\forall E]\quad \text{(for any term } t\text{)} \qquad\qquad \frac{A(c)}{(\forall x)A(x)}\,[\forall I]\quad \text{(for any constant } c \text{ not in the premises)}$$

These rules seem straightforward, but things are subtle.

They depend on an *analysis* of formulas, identifying constituents as *terms*, and defining the appropriate notion of *substitution*.

Remember: multi-sorted predicate logic.

Existence (conservative extension)

The usual cut-elimination or normalisation process an show that proofs with the universal quantifier conservatively extend proofs without it.

The usual cut-elimination or normalisation process an show that proofs with the universal quantifier conservatively extend proofs without it.

$$\dfrac{\dfrac{\begin{matrix} X \\ \vdots\ \pi \\ A(c) \end{matrix}}{(\forall x)A(x)}\,[\forall I]}{A(t)}\,[\forall E]$$

The usual cut-elimination or normalisation process an show that proofs with the universal quantifier conservatively extend proofs without it.

$$\dfrac{\dfrac{\begin{matrix} X \\ \vdots\ \pi \\ A(c) \end{matrix}}{(\forall x)A(x)}\,[\forall I]}{A(t)}\,[\forall E] \quad\Longrightarrow\quad \begin{matrix} X \\ \vdots\ \pi_{[c \mapsto t]} \\ A(t) \end{matrix}$$

The result π^c_t is a proof from the same premises since

- The constant c does not appear in X, premises of π.
- Any rule is closed under the substitution of terms for constants.

Unique Definition (equivalence)?

We need to do more to prove uniqueness of the universal quantifier.

We need to do more to prove uniqueness of the universal quantifier.

But uniqueness can *fail*.

Unique Definition (equivalence)?

We need to do more to prove uniqueness of the universal quantifier.

But uniqueness can *fail*.

We can have *two* disjoint categories of terms, two sets of quantifiers — two-sorted first-order logic.

Uniqueness, relative to an *analysis*

However, if the two quantifiers are defined using the *same* class of terms, and the *same* notion of substitution, then uniqueness follows:

Uniqueness, relative to an *analysis*

However, if the two quantifiers are defined using the *same* class of terms, and the *same* notion of substitution, then uniqueness follows:

$$\dfrac{\dfrac{(\mathsf{U}x)A(x)}{A(c)}\,[\mathsf{U}E]}{(\forall x)A(x)}\,[\forall I] \qquad \dfrac{\dfrac{(\forall x)A(x)}{A(c)}\,[\forall E]}{(\mathsf{U}x)A(x)}\,[\mathsf{U}I]$$

However, if the two quantifiers are defined using the *same* class of terms, and the *same* notion of substitution, then uniqueness follows:

$$\dfrac{\dfrac{(\mathsf{U}x)A(x)}{A(c)}\,[\mathsf{U}E]}{(\forall x)A(x)}\,[\forall I] \qquad \dfrac{\dfrac{(\forall x)A(x)}{A(c)}\,[\forall E]}{(\mathsf{U}x)A(x)}\,[\mathsf{U}I]$$

These proofs work only when a term substituted using one quantifier may be substituted using the other.

However, if the two quantifiers are defined using the *same* class of terms, and the *same* notion of substitution, then uniqueness follows:

$$\dfrac{\dfrac{(\cup x)A(x)}{A(c)}\,[\cup E]}{(\forall x)A(x)}\,[\forall I] \qquad \dfrac{\dfrac{(\forall x)A(x)}{A(c)}\,[\forall E]}{(\cup x)A(x)}\,[\cup I]$$

These proofs work only when a term substituted using one quantifier may be substituted using the other.

This analysis of the vocabulary is part of the context of deducibility.

As for names, so for predicates?

Normalisation for the universal quantifier appealed to a closure property concerning the constant c.

Normalisation for the universal quantifier appealed to a closure property concerning the constant c.

$$\dfrac{\dfrac{\begin{matrix} X \\ \vdots\ \pi \\ A(c) \end{matrix}}{(\forall x)A(x)}\,[\forall I]}{A(t)}\,[\forall E] \quad \Longrightarrow \quad \begin{matrix} X \\ \vdots\ \pi_{[c \mapsto t]} \\ A(t) \end{matrix}$$

In any inference c be everywhere replaced by t.

Normalisation for the universal quantifier appealed to a closure property concerning the constant c.

$$\dfrac{\dfrac{\begin{array}{c}X\\ \vdots\ \pi\\ A(c)\end{array}}{(\forall x)A(x)}\,[\forall I]}{A(t)}\,[\forall E] \quad\Longrightarrow\quad \begin{array}{c}X\\ \vdots\ \pi_{[c\mapsto t]}\\ A(t)\end{array}$$

In any inference c be everywhere replaced by t.

Do *predicates* satisfy this condition?

In the rules for first-order logic, no predicates (except for identity) are singled out for special treatment.

Predicate 'variables'?

In the rules for first-order logic, no predicates (except for identity) are singled out for special treatment.

In the general case we want either:

In the rules for first-order logic, no predicates (except for identity) are singled out for special treatment.

In the general case we want either:

- Rules to be closed under substitution of predicates.

Predicate ‘variables’?

In the rules for first-order logic, no predicates (except for identity) are singled out for special treatment.

In the general case we want either:

- Rules to be closed under substitution of predicates.
- A special class of predicates (predicate ‘variables’) that satisfy this closure condition.

Defining Identity

Identity and harmony

STEPHEN READ

1. *Harmony*

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if 'tonk' has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of 'p tonk q' is given by Prior's rule:

$$\frac{p}{p \text{ tonk } q}\ \text{tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \text{ tonk } q \quad \begin{matrix}(p)\\ r\end{matrix}}{r}\ \text{tonk-E}$$

that is, given 'p tonk q', and a derivation of r from p (the ground for asserting 'p tonk q'), we can infer r, discharging the assumption p. We can state the rule more simply as follows:

$$\frac{q \text{ tonk } q}{p}$$

For if we may infer whatever, r, we can infer from p, we can infer p and then proceed to infer r, that is, what we can infer from p. Prior's mistake was to give a rule

ANALYSIS 64.2, April 2004, pp. 113–19. © Stephen Read

Stephen Read: "Identity and Harmony" *Analysis* 2004

$$\frac{a = b \quad C(a)}{C(b)}\,[{=}E] \qquad \frac{\begin{array}{c}[Fa]\\ \vdots\,\pi\\ Fb\end{array}}{a = b}\,[{=}I]$$

(F not in the other premises of π)

$$\dfrac{a = b \quad C(a)}{C(b)}\,[{=}E] \qquad \dfrac{\begin{array}{c}[Fa] \\ \vdots\,\pi \\ Fb\end{array}}{a = b}\,[{=}I]$$

(F not in the other premises of π)

To normalise:

$$\dfrac{\dfrac{\begin{array}{c}[Fa] \\ \vdots\,\pi \\ Fb\end{array}}{a = b}\,[{=}I] \qquad C(a)}{C(b)}\,[{=}E]$$

$$\frac{a = b \quad C(a)}{C(b)}\,[{=}E] \qquad \frac{\begin{array}{c}[Fa]\\ \vdots\ \pi\\ Fb\end{array}}{a = b}\,[{=}I]$$

(F not in the other premises of π)

To normalise:

$$\frac{\dfrac{\begin{array}{c}[Fa]\\ \vdots\ \pi\\ Fb\end{array}}{a = b}\,[{=}I] \qquad C(a)}{C(b)}\,[{=}E] \quad \Longrightarrow \quad \begin{array}{c}C(a)\\ \vdots\ \pi_{[Fx \mapsto Cx]}\\ C(b)\end{array}$$

$$\frac{a = b \quad C(a)}{C(b)}\,[{=}E] \qquad \frac{\begin{array}{c}[Fa]\\ \vdots\ \pi\\ Fb\end{array}}{a = b}\,[{=}I] \quad \text{(F not in the other premises of } \pi\text{)}$$

To normalise:

$$\frac{\dfrac{\begin{array}{c}[Fa]\\ \vdots\ \pi\\ Fb\end{array}}{a = b}\,[{=}I] \qquad C(a)}{C(b)}\,[{=}E] \quad \Longrightarrow \quad \begin{array}{c}C(a)\\ \vdots\ \pi_{[Fx \mapsto Cx]}\\ C(b)\end{array}$$

(Replacing the F with C in π yields a proof from the same premises.)

$$\frac{(\forall X)A(X)}{A(C)}\,[\forall^2 E] \qquad \frac{A(F)}{(\forall X)A(X)}\,[\forall^2 I] \quad \text{(F not in the premises of the proof of } A\text{)}$$

X is a bound predicate variable of the same arity as the variable F and context C.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\forall X)(Xb \supset Xa)}{b = b \supset a = b}\,[\forall^2 E] \qquad \dfrac{[Fb]^1}{b = b}\,[=I^1]}{a = b}\,[\supset E] \qquad [Fa]^2}{Fb}\,[=E]}{Fa \supset Fb}\,[\supset I^2]}{(\forall X)(Xa \supset Xb)}\,[\forall^2 I]$$

(In the $[\forall^2 E]$ step, Xy is instantiated to $y = b$.)

$$\dfrac{\dfrac{\dfrac{(\forall X)(Xb \supset Xa)}{b = b \supset a = b}[\forall^2 E] \quad \dfrac{[Fb]^1}{b = b}[{=}I^1]}{a = b}[\supset E] \quad [Fa]^2}{\dfrac{\dfrac{Fb}{Fa \supset Fb}[\supset I^2]}{(\forall X)(Xa \supset Xb)}[\forall^2 I]}[{=}E]$$

(In the $[\forall^2 E]$ step, Xy is instantiated to $y = b$.)

This is second order logic. In multiple conclusion consequence, it's *classical* second order logic.

Existence and Uniqueness

We have existence and uniqueness in the usual way.

We have existence and uniqueness in the usual way.

- For existence, we appeal to the usual cut-elimination or normalisation proof. (In the case of second-order logic, these results are more difficult, but they still hold.)

We have existence and uniqueness in the usual way.

- For existence, we appeal to the usual cut-elimination or normalisation proof. (In the case of second-order logic, these results are more difficult, but they still hold.)
- For uniqueness (relative to a single analysis of statements, again), we reason as follows:

We have existence and uniqueness in the usual way.

- For existence, we appeal to the usual cut-elimination or normalisation proof. (In the case of second-order logic, these results are more difficult, but they still hold.)
- For uniqueness (relative to a single analysis of statements, again), we reason as follows:

$$\dfrac{\dfrac{(\text{U}X)A(X)}{A(F)}\,[\text{U}^2E]}{(\forall X)A(X)}\,[\forall^2 I] \qquad \dfrac{\dfrac{(\forall X)A(X)}{A(F)}\,[\forall^2 E]}{(\text{U}X)A(X)}\,[\text{U}^2 I]$$

We have existence and uniqueness in the usual way.

- For existence, we appeal to the usual cut-elimination or normalisation proof. (In the case of second-order logic, these results are more difficult, but they still hold.)
- For uniqueness (relative to a single analysis of statements, again), we reason as follows:

$$\dfrac{\dfrac{(\mathsf{U}X)A(X)}{A(F)}\,[\mathsf{U}^2E]}{(\forall X)A(X)}\,[\forall^2 I] \qquad \dfrac{\dfrac{(\forall X)A(X)}{A(F)}\,[\forall^2 E]}{(\mathsf{U}X)A(X)}\,[\mathsf{U}^2 I]$$

By Belnap's criteria (in this context of deducibility) second order quantification is properly logical.

None of this requires appealing to sets as semantic values for predicate variables.

If the axiom of choice is true, then in every (standard) model of second-order logic, it holds:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\exists f)(\forall x)Xxf(x))$$

(We can define function quantification in terms of predicate quantification or give separate rules, if you prefer.)

If the axiom of choice is true, then in every (standard) model of second-order logic, it holds:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\exists f)(\forall x)Xxf(x))$$

(We can define function quantification in terms of predicate quantification or give separate rules, if you prefer.)

However, it has no proof…

If the axiom of choice is true, then in every (standard) model of second-order logic, it holds:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\exists f)(\forall x)Xxf(x))$$

(We can define function quantification in terms of predicate quantification or give separate rules, if you prefer.)

However, it has no proof, so far, at least.

Take a model of ZF without choice, and define a model for second order quantification "internally" in *that* model. This is closed under each of our inference rules, but choice fails.

MATHEMATICS

Proving Choice

Proving Choice

- OPTION 1: ϵ (a choice quantifier – indefinite description.)

Proving Choice

- OPTION 1: ϵ (a choice quantifier – indefinite description.)
- OPTION 2: Assume the existence of a well ordering.

Proving Choice

- OPTION 1: ϵ (a choice quantifier – indefinite description.)
- OPTION 2: Assume the existence of a well ordering.

Both are problematic.

With ϵ, we can derive choice:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\forall x)Xx((\epsilon y)Xxy))$$

ϵ – choice

With ϵ, we can derive choice:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\forall x)Xx((\epsilon y)Xxy))$$

But how do we define ϵ?

With ϵ, we can derive choice:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\forall x)Xx((\epsilon y)Xxy))$$

But how do we define ϵ? We want rules like these:

$$\frac{F(a)}{F(\epsilon x F x)}[\epsilon I] \qquad \frac{F(\epsilon x F x) \quad \begin{matrix}[Fc] \\ \vdots\ \pi \\ C\end{matrix}}{C}[\epsilon E]$$

(c occurs in no other premise in π.)

With ϵ, we can derive choice:

$$(\forall X)((\forall x)(\exists y)Xxy \supset (\forall x)Xx((\epsilon y)Xxy))$$

But how do we define ϵ? We want rules like these:

$$\frac{F(a)}{F(\epsilon x F x)}\,[\epsilon I] \qquad \frac{F(\epsilon x F x) \quad \begin{array}{c}[Fc] \\ \vdots\, \pi \\ C\end{array}}{C}\,[\epsilon E]$$

(c occurs in no other premise in π.)

These rules *don't* define ϵ uniquely.

Given a model with two different choice functions f and f' for every nonempty extension, the indefinite descriptions ϵ and ϵ' would both satisfy these rules, yet be inequivalent.

One could *define* $\epsilon x F x$ as the *first* object satisfying F.

Provided, of course, that you had a well-ordering lying around to help get things in line.

Ordering

One could *define* $\epsilon x F x$ as the *first* object satisfying F.

Provided, of course, that you had a well-ordering lying around to help get things in line.

If you were prepared to treat such an ordering ($\leqslant$) as a part of the context of deducibility, you can define ϵ uniquely (relative to $\leqslant$), and prove choice.

Ordering

One could *define* $\epsilon x F x$ as the *first* object satisfying F.

Provided, of course, that you had a well-ordering lying around to help get things in line.

If you were prepared to treat such an ordering ($\leqslant$) as a part of the context of deducibility, you can define ϵ uniquely (relative to $\leqslant$), and prove choice.

But who has an ordering lying around?

One could *define* $\epsilon x F x$ as the *first* object satisfying F.

Provided, of course, that you had a well-ordering lying around to help get things in line.

If you were prepared to treat such an ordering ($\leqslant$) as a part of the context of deducibility, you can define ϵ uniquely (relative to $\leqslant$), and prove choice.

But who has an ordering lying around?

Why not treat choice as a statement in logical vocabulary which, if true, is true on non-logical grounds?

One could *define* $\epsilon x F x$ as the *first* object satisfying F.

Provided, of course, that you had a well-ordering lying around to help get things in line.

If you were prepared to treat such an ordering ($\leqslant$) as a part of the context of deducibility, you can define ϵ uniquely (relative to $\leqslant$), and prove choice.

But who has an ordering lying around?

Why not treat choice as a statement in logical vocabulary which, if true, is true on non-logical grounds?

Like $(\exists x)(\exists y)x \neq y$.

MODALITY

$\Box$ and $\Diamond$ seem *semi-logical*.

Modal Operators

$\Box$ and $\Diamond$ seem *semi-logical*.

In a Kripke model, $\Box$ and $\Diamond$, depend on an *accessibility* relation,

$\Box$ and $\Diamond$ seem *semi-logical*.

In a Kripke model, $\Box$ and $\Diamond$, depend on an *accessibility* relation, and a model can have more than one.

You might think that we would have severe troubles with uniqueness.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,[\Box L] \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \Box A, \Delta}\,[\Box R] \quad (\Gamma \text{ and } \Delta \text{ are modalised})$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,[\Box L] \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \Box A, \Delta}\,[\Box R] \quad (\Gamma \text{ and } \Delta \text{ are modalised})$$

These *describe*, but do not *define*.

We don't have uniqueness.

We need more structure.

We need more structure.

In terms of assertion and denial, can see that an assertion of A doesn't always clash with a denial of A.

How to do it

We need more structure.

In terms of assertion and denial, can see that an assertion of A doesn't always clash with a denial of A.

Water is H_2O.

Now suppose we're actually in a twin-earth situation. Then, water is not H_2O. It's XYZ.

We need more structure.

In terms of assertion and denial, can see that an assertion of A doesn't always clash with a denial of A.

Water is H_2O.

Now suppose we're actually in a twin-earth situation. Then, water is not H_2O. It's XYZ.

We're *not* contradicting ourselves here.

We need more structure.

In terms of assertion and denial, can see that an assertion of A doesn't always clash with a denial of A.

Water is H_2O .

Now suppose we're actually in a twin-earth situation. Then, water is not H_2O . It's XYZ.

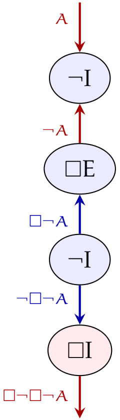
We're *not* contradicting ourselves here.

I've used this *stratification* to give a proof theory for the modal logic S5.

$$\frac{X, A \vdash Y \mid \Delta}{\Box A \vdash \mid X \vdash Y \mid \Delta}[\Box L] \qquad \frac{\vdash A \mid \Delta}{\vdash \Box A \mid \Delta}[\Box R]$$

To each modal proofnet we may associate a sequent derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}\, L\neg}{A \vdash \mid \Box\neg A \vdash}\, L\Box}{A \vdash \mid \vdash \neg\Box\neg A}\, R\neg}{A \vdash \mid \vdash \Box\neg\Box\neg A}\, R\Box}{A \vdash \Box\neg\Box\neg A}\, \mathit{merge}$$



Existence and Uniqueness

- EXISTENCE: a straightforward cut-elimination.

Existence and Uniqueness

- EXISTENCE: a straightforward cut-elimination.
- UNIQUENESS: if both $\Box$ and $\Box'$ track the one zone shift, we have uniqueness.

- EXISTENCE: a straightforward cut-elimination.
- UNIQUENESS: if both $\Box$ and $\Box'$ track the one zone shift, we have uniqueness.

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\Box' A \vdash \ \mid\ \vdash A}[\Box' L]}{\Box' A \vdash \ \mid\ \vdash \Box A}[\Box R]}{\Box' A \vdash \Box A}[merge] \qquad \dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \ \mid\ \vdash A}[\Box L]}{\Box A \vdash \ \mid\ \vdash \Box' A}[\Box' R]}{\Box A \vdash \Box A'}[merge]$$

And more . . .

And more …

- Actuality: @A is asserting A in the *actual* zone.

- Actuality: @A is asserting A in the *actual* zone.
- 2D Modal logic: Two kinds of zone shift.

Why this works

We have paid attention to the context of deducibility.

Why this works

We have paid attention to the context of deducibility.

(In this case, how assertion/denial is *stratified*.)

We have paid attention to the context of deducibility.

(In this case, how assertion/denial is *stratified*.)

We have explained the *use* of necessity talk without appealing to possible worlds.

THAT'S ALL, FOLKS!