Formal Borel sets —
a proof-theoretic approach

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Context of talk

Domain theory

Topological models

Probabilistic models

Randomness

This talk

Proof theory

Background from domain theory

In domain theory, computations are modelled in a domain D (typically a dcpo) of possible outcomes.

The Jones/Plotkin probabilistic powerdomain $\mathcal{V}(D)$ models probabilistic computations producing outputs in D.

Concretely, $\mathcal{V}(D)$ is defined as the domain of probability valuations on D.

A probability valuation is a function $\nu\colon \mathcal{O}(D) \to [0,1]$ (where $\mathcal{O}(D)$ is the lattice of Scott-open sets) satisfying a few natural well-behavedness conditions (see later).

Background from probability and topology

In probability, events form a σ-algebra.

Probability "distributions" on a topological space X are implemented by Borel measures on X.

Recall, the Borel sets $\mathcal{B}(X)$ over a topological space X is the smallest σ-algebra containing the open sets $\mathcal{O}(X)$ (i.e. the closure of $\mathcal{O}(X)$ under complements and countable unions).

A Borel (probability) measure is then a function $\mu\colon \mathcal{B}(X) \to [0,1]$ satisfying natural conditions.

Borel measures carry more information than valuations since weights are assigned to a larger collection of sets.

Can this extra information be recovered from a valuation alone?

Theorem (Lawson 1982)   Every (continuous) valuation on a countably-based locally compact sober space extends to a Borel measure.

Subsequently extension results achieved for wider collections of sober spaces: regular spaces; general locally compact spaces, etc., by Jones, Norberg, Alvarez-Manilla, Edalat, Saheb-Djahromi, Keimel, Lawson

Open question   Does an extension result hold for all sober spaces?

N.B., the requirement of sobriety is necessary (at least in countably-based case)

Goals of talk

- Obtain a general extension theorem with no technical side-conditions ...

- ... in the more general setting of point-free topology.

A mathematical motivation for considering probability in the setting of point-free topology comes from study of radomness, see:

"The locale of random sequences" (S., 3WFTop 2007)

More generally, point-free topology is related to logic of observable properties (Abramsky, Vickers , ...) and Stone duality (Johnstone "Stone Spaces")

σ-frames

Point-free topology replaces families of open sets with lattice of formal opens given as an algebraic structure.

Definition   A σ-frame is a partially ordered set with:

- finite infima (including top element $\top$)

- countable suprema (including least element $\bot$)

Satisfying the distributive law for countable suprema:

$$x \wedge \bigvee_i y_i = \bigvee_i x \wedge y_i$$

Examples: $\mathcal{O}(X)$ for any topological space X, more generally any frame.

σ-boolean algebras

Definition   A σ-boolean algebra is a σ-frame such that for every u there exists (a necessarily unique) $\neg u$ satisfying:

$$u \vee \neg u = \top$$
$$u \wedge \neg u = \bot$$

Equivalently (and more straightforwardly), a σ-boolean algebra is a countably-complete boolean algebra

Examples: $\mathcal{B}(X)$, more generally any σ-algebra.

Formal Borels

The following proposition and associated definition play a central role in this talk

Proposition For any σ-frame F, there exists a free σ-boolean algebra $\mathcal{B}_f(F)$ over F (preserving existing σ-frame structure)

Proof Apply Freyd's adjoint fuctor theorem. $\square$

Definition We call $\mathcal{B}_f(F)$ the σ-boolean algebra of formal Borels over a σ-frame F.

N.B. One can give more informative (and constructive) proofs of the Proposition. For example, proof systems 2 & 3 introduced later in the talk, will provide explicit syntactic constructions of $\mathcal{B}_f(F)$

Valuations and measures

Definition   A (σ-continuous) valuation on a σ-frame F is a (necessarily monotonic) function $\nu \colon F \to [0, \infty]$ satisfying:

$$\nu(\bot) \ = \ 0$$

$$\nu(u \vee v) + \nu(u \wedge v) \ = \ \nu(u) + \nu(v)$$

$$\nu(\textstyle\bigvee_i u_i) \ = \ \sup_i \nu(u_i) \qquad (u_i) \text{ an ascending sequence}$$

A valuation ν is finite if $\nu(\top) < \infty$. It is σ-finite if there exists a countable family $\{u_i\}$ of elements of F with $\bigvee_i u_i = \top$ and $\nu(u_i) < \infty$ for all i.

Definition   A measure on a σ-boolean algebra H is simply a valuation on H. (This is equivalent to the standard definition using countable additivity.)

Main results

Theorem 1 Every σ-finite valuation on a σ-frame F extends to a unique (σ-finite) measure on the lattice $\mathcal{B}_f(F)$ of formal Borels.

This is the promised extension result with no side-conditions on F. Instead topological side-conditions reappear as conditions under which formal Borels coincide with Borel sets.

Theorem 2 If X is a countably-based locally compact sober space then $\mathcal{B}_f(\mathcal{O}(X)) \cong \mathcal{B}(X)$.

N.B., Lawson's 1982 extension result follows from Theorems 1 and 2 combined

Acknowledgement The idea of investigating Theorem 1 was suggested to me by André Joyal.

A logical language for Borels

Let F be a σ-frame, and $B \subseteq F$ a base (i.e., every element of F arises as a countable supremum of elements of B).

We define formulas for (formal) Borels by taking elements $b \in B$ as propositional constants and closing under negation, and countable conjunctions and disjunctions:

$$\phi ::= b \mid \neg\phi \mid \bigwedge_i \phi_i \mid \bigvee_i \phi_i$$

(So we have the propositional fragment of $L_{\omega_1\omega}$)

We consider 3 proof systems, each involving sequents of the form

$$\Gamma \vdash \Delta$$

where Γ, Δ are finite sets of formulas.

System 1 (non-well-founded)

Usual sequent proof rules on left and right for each connective. E.g.

$$\frac{\Gamma, \phi_i \vdash \Delta}{\Gamma \vdash \Delta} \ \bigwedge_i \phi_i \in \Gamma \qquad \frac{\{\Gamma \vdash \phi_i, \Delta\}_i}{\Gamma \vdash \Delta} \ \bigwedge_i \phi_i \in \Delta$$

Also include atomic cuts:

$$\frac{\Gamma, b \vdash \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash \Delta}$$

A basic entailment, written $C \Rightarrow D$, is given by a finite $C \subseteq B$ and countable $D \subseteq B$ such that: $\bigwedge C \leq \bigvee D$ in F.

An infinite branch $(\Gamma_i \vdash \Delta_i)$ in a rule tree is justified if there exist $C \subseteq \bigcup_i \Gamma_i$ and $D \subseteq \bigcup_i \Delta_i$ such that $C \Rightarrow D$.

A rule tree is a proof if every infinite branch is justified.

Example proof (System 1)

Proof of $\bigwedge_{i \geq 0}(0, 2^{-i}) \vdash$, where the σ-frame is $\mathcal{O}(\mathbb{R})$ and B is the basis of rational open intervals.

$$
\cfrac{
\cfrac{
\cfrac{(0, 2^{-1}), (2^{-1}, 1) \Rightarrow}{(0, 2^{-1}), (2^{-1}, 1) \vdash}}{\psi, (2^{-1}, 1) \vdash}
\quad
\cfrac{
\cfrac{(0, 2^{-2}), (2^{-2}, 1) \Rightarrow}{(0, 2^{-2}), (2^{-2}, 1) \vdash}}{\psi, (2^{-2}, 1) \vdash}
\quad
\cfrac{
\cfrac{(0, 1) \Rightarrow (2^{-1},1),(2^{-2},1),(2^{-3},1)\dots \quad \vdots}{(0, 1), \psi \vdash (2^{-1}, 1), (2^{-2}, 1)}}{(0, 1), \psi \vdash (2^{-1}, 1)} \text{(cut)}
}{\cfrac{(0, 1), \psi \vdash}{\bigwedge_{i \geq 0}(0, 2^{-i}) \vdash}} \;(\textstyle\bigwedge \mathrm{L})
$$

using abbreviation $\psi := \bigwedge_{i \geq 0}(0, 2^{-i})$.

System 1: Soundness and completeness

System 1 captures inclusion between Borel sets of topological spaces.

Suppose $F = \mathcal{O}(X)$ for some topological space X.

Formulas are interpreted as Borel sets in the obvious way.

Soundness Theorem   If $\Gamma \vdash \Delta$ has a proof then $\bigcap \Gamma \subseteq \bigcup \Delta$ in $\mathcal{B}(X)$.

Completeness Theorem   Suppose X is sober and countably based. If $\bigcap \Gamma \subseteq \bigcup \Delta$ in $\mathcal{B}(X)$ then $\Gamma \vdash \Delta$ has a proof.

Proof of completeness is by a search tree construction.

System 2 (non-well-founded)

The proof rules are as for System 1, but there is a stronger requirement on being a proof.

A rule tree is a proof if there exists a countable set of basic entailments such that every infinite branch is justified by a basic entailment from the set. (N.B., the tree may nonetheless have uncountably many infinite branches.)

Clearly the example proof on slide 14 is also a proof in system 2.

System 2: soundness and completeness

System 2 captures implications between formal Borels.

Let F be any σ-frame and $B \subseteq F$ any base.

Theorem (Soundness & Completeness)   A sequent $\Gamma \vdash \Delta$ has a proof if and only if $\bigwedge \Gamma \leq \bigvee \Delta$ in $\mathcal{B}_f(F)$.

Both directions are proved by establishing that System 2 is equivalent to yet another proof system (System 3), for which soundness and completeness are more easily established.

Before presenting System 3, we outline how Systems 1 and 2 are used to prove Theorem 2.

Proof of Theorem 2

Theorem 2   If X is a countably-based locally compact sober space then $\mathcal{B}_f(\mathcal{O}(X)) \cong \mathcal{B}(X)$.

Proof   Suppose F is $\mathcal{O}(X)$ and B is a countable basis. By the soundness/completeness theorems, we need to show that any sequent provable in System 1 is also provable in System 2. If $\Gamma \vdash \Delta$ is System-1-provable, then, in particular, its search tree is a proof. The required countable set of basic entailments for this tree is given by those of the form:

$$\{C \Rightarrow D \mid C, D \text{ finite}\} \cup \{C \Rightarrow \{b \mid b \ll \bigwedge C\} \mid C \text{ finite}\}$$

where $\ll$ is the way-below relation in the continuous lattice $\mathcal{O}(X)$. □

System 3 (well-founded)

Proof rules for logical connectives as before.

Add atomic axioms:

$$\frac{}{\Gamma \vdash \Delta} \quad \Gamma \cap \Delta \cap B \neq \emptyset$$

Replace atomic cut with (more general) rule for basic entailments:

$$\frac{\{\Gamma \vdash \Delta, c\}_{c \in C} \quad \{\Gamma, d \vdash \Delta\}_{d \in D}}{\Gamma \vdash \Delta} \quad C \Rightarrow D$$

A proof is a well-founded tree.

Example proof (System 3)

As before, proof of $\bigwedge_{i \geq 0}(0, 2^{-i}) \vdash$, where the σ-frame is $\mathcal{O}(\mathbb{R})$ and B is the basis of rational open intervals, using abbreviation
$\psi := \bigwedge_{i \geq 0}(0, 2^{-i})$:

$$
\cfrac{
\cfrac{\overline{\psi, (0,1) \vdash (0,1)}}{\psi \vdash (0,1)} (\textstyle\bigwedge \mathrm{L})
\quad
\left\{
\cfrac{
\cfrac{\overline{(0, 2^{-i}), (2^{-i}, 1) \Rightarrow}}{(0, 2^{-i}), (2^{-i}, 1) \vdash}
}{\psi, (2^{-i}, 1) \vdash} (\textstyle\bigwedge \mathrm{L})
\right\}_i
}{\bigwedge_{i \geq 0}(0, 2^{-i}) \vdash} (*)
$$

$(*)$ applies basic entailment: $(0,1) \Rightarrow (2^{-1}, 1), (2^{-2}, 1), (2^{-3}, 1) \ldots$

Equivalence of systems 2 and 3

Theorem (Equivalence)   A sequent $\Gamma \vdash \Delta$ has a proof in System 2 if and only if it has a proof in System 3.

Proof   Turning a proof in System 3 into a proof in System 1 is straightforward, by a (transfinite) recursion on proof structure (using well-foundedness). The resulting proof has only countably many infinite branches, so is a proof in System 2.

For the other direction, a non-well-founded proof in System 2 is "padded out" by systematically interleaving rules in the proof with basic entailment rules (from System 3) for each of the countably many basic entailments required to justify all infinite branches. Every infinite branch through the resulting tree eventually reaches an atomic axiom sequent. By pruning the tree at these axioms, a proof in System 3 is obtained. □

System 3: soundness and completeness

Theorem (Soundness & Completeness)   A sequent $\Gamma \vdash \Delta$ has a proof if and only if $\bigwedge \Gamma \leq \bigvee \Delta$ in $\mathcal{B}_f(F)$.

Proof   Soundness: a straightforward induction on proof structure.

For completeness, there are two components. First admissibility of cut is proved in standard syntactic way (cf. Tait for $L_{\omega_1\omega}$, Martin-Löf "Notes on constructive mathematics"). This shows provable implications between formulas define a σ-complete boolean algebra.

Second, one shows that, for negation-free formulas ϕ, ψ in which all conjunctions are finite, that $\phi \leq \psi$ in F implies $\phi \vdash \psi$ is provable. Thus the σ-complete boolean algebra defined above preserves the σ-frame structure of F. Completeness then follows from the freeness of $\mathcal{B}_f(F)$. □

Conservativity

Corollary   The canonical σ-frame map from F to $\mathcal{B}_f(F)$ is an order embedding.

Proof   If $\bigvee_i b_i \vdash \bigvee_j b'_j$ has a proof in System 3, then, by induction on the structure of the proof, one has $\bigvee_i b_i \leq \bigvee_j b'_j$ in F. By completeness of system 3, this suffices. $\qquad\qquad\square$

Remark One can give a short semantic proof of this result. By a "countably-complete ideal" construction, every σ-frame embeds in a frame. It is well known that every frame embeds in a complete boolean algebra, e.g. (Johnstone, "Stone Spaces"). Since every σ-frame thus embeds in some σ-complete boolean algebra, the canonical map to the free σ-complete boolean algebra must be an embedding.

Towards proof of Theorem 1

Let ν be a finite valuation on F.

A crescent is a formula of the form $(\bigvee_i b_i) \wedge \neg(\bigvee_j b_j')$.

By (a version of) the Smiley-Horn-Tarski theorem, ν extends to a unique finitely additive "measure" (the quotes are because finite additivity replaces countable additivity) on the boolean algebra of finite disjunctions of crescents. For a finite disjunction of crescents, ϕ, we write $\nu(\phi)$ for this extension.

A finite formula is a formula in which all disjunctions and conjunctions are finite.

N.B., every finite formula is logically equivalent to a finite disjunction of crescents.

Crucial lemma

Lemma   If the sequent

$$\phi_1, \ldots, \phi_m, \bigwedge_{i<\infty} \psi_i \vdash \phi'_1, \ldots, \phi'_n$$

is provable in System 3, where all ϕ, ψ, ϕ' formulas are finite, then, for every $\epsilon > 0$, there exists $k \geq 0$ and finitely many crescents $\theta_1, \ldots, \theta_l$ such that

$$\phi_1, \ldots, \phi_m, \bigwedge_{i<k} \psi_i \vdash \phi'_1, \ldots, \phi'_n, \theta_1, \ldots, \theta_l$$

is provable and $\nu(\theta_1) + \cdots + \nu(\theta_l) < \epsilon$.

Proof   By induction on the System 3 proof.   □

Whence Theorem 1

It follows from the lemma that ν, applied to finite formulas, is countably additive w.r.t. those disjoint suprema that exist within the boolean algebra of finite formulas. (N.B., this boolean algebra is not itself countably complete.)

Then, by a Carathéodory-type extension result, ν extends to a finite measure on the whole of $\mathcal{B}_f(F)$.

Thus every finite valuation on F extends to a finite measure on $\mathcal{B}_f(F)$. This extension is easily seen to be unique.

The full result for σ-finite measures can be derived as a consequence, cf. (Alvarez-Manilla 2002). Thus we obtain:

Theorem 1   Every σ-finite valuation on a σ-frame F extends to a unique (σ-finite) measure on the lattice $\mathcal{B}_f(F)$ of formal Borels.

Related work

- Martin-Löf, "Notes on Constructive Mathematics" (1968), uses sequent calculus to give a constructive definition of (formal) Borel sets for the special case of Cantor space.

  He also defines the uniform measure on Cantor space for a restricted class of formal Borels (the restriction being necessary for assigning real-number weights constructively).

- Coquand and Palmgren (2002) develop some constructive probability theory using the free σ-completion of a boolean algebra as a σ-algebra of events.

  This gives an alternative (equivalent) approach to formal Borel sets, which is applicable in the special case of Stone spaces (totally-disconnected compact Hausdorff spaces).

Adapting a closely related result in Halmos "Lectures on Boolean Algebras" (1963), Coquand and Palmgren prove the following theorem as a classical justification for their approach.

Theorem   If X is a separable Stone space then the free σ-completion of the boolean algebra of clopens of X is isomorphic to $\mathcal{B}(X)$.

This theorem is related to our Theorem 2. In fact, the restriction of Theorem 2 to the special case of Stone spaces follows as a straightforward consequence of the above result. (Of course, the full Theorem 2 applies to a far wider class of spaces.)

Remark   For countably-based Stone spaces, it is relatively straightforward to obtain an alternative proof of the above theorem using the completeness theorem for our proof system 1. (Exercise!)

- Coquand has a constructive (inductive) definition of Borel measure (for Cantor space) using Riesz spaces. Also related work by Coquand and Spitters (on measurable functions, constructive integration, etc.)

- Escardó and Vickers (in preparation) have directly defined integration of lower semicontinuous functions on locales w.r.t. valuations.

Further work

- It seems likely that our Theorem 1 should constructivize, but, as in the work of Coquand, one would need to deal with the unavoidable complication of replacing $[0, \infty]$ as the codomain of measures with a σ-complete Riesz space.

- The development of general (constructive) point-free probability theory is still in its infancy ...