The Urysohn sphere is oscillation stable

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Oscillation stability for Banach spaces

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Definition

Let X be an infinite dimensional Banach space.
X is oscillation stable when for every $f : \mathbb{S}_X \longrightarrow [0,1]$ uniformly continuous, every $\varepsilon > 0$, every $Y \subset X$ closed infinite dimensional, there $Z \subset Y$ closed infinite dimensional such that:

$$\operatorname{osc}(f, Z) := \sup_{x,y \in \mathbb{S}_X \cap Z} |f(y) - f(x)| < \varepsilon.$$

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Theorem (Gowers, 91)

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Theorem (Odell-Schlumprecht, 94)

The Hilbert space ℓ_2 is not oscillation stable.

Reformulation of the problem for ℓ_2

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Definition

Let X be a metric space.
X is metrically oscillation stable if for every $f : X \longrightarrow [0, 1]$ uniformly continuous, $\varepsilon > 0$, there is $\widetilde{X}$ isometric to X such that:

$$\forall x, y \in \widetilde{X}, \quad |f(y) - f(x)| < \varepsilon.$$

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Equivalently:
Let $X = A_1 \cup \ldots \cup A_k$, $\varepsilon > 0$. There is $\widetilde{X}$ isometric to X and $i \leqslant k$ such that

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Theorem (Odell-Schlumprecht)

The unit sphere $\mathbb{S}^\infty$ of ℓ_2 is not metrically oscillation stable.

Open question

Question

Is there a proof based on the intrinsic metric structure of $\mathbb{S}^\infty$*?*

A good candidate for a better understanding

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Definition

Up to isometry, there is a unique metric space $\mathbb{S}$ with distances in $[0,1]$ which is:

1. *Complete, separable.*
2. *Ultrahomogeneous (every isometry between finite subsets of $\mathbb{S}$ extends to an isometry of $\mathbb{S}$ onto itself).*
3. *Universal for the separable metric spaces with distances in $[0,1]$.*

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- Higher dimensional Ramsey properties.
- Behaviour of $\mathrm{iso}(\mathbb{S}^\infty)$ and $\mathrm{iso}(\mathbb{S})$ as topological groups.

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Is $\mathbb{S}$ metrically oscillation stable?

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Proposition

The space $\mathbb{S}$ admits countable ultrahomogeneous dense subsets.

Question

Let $X \subset \mathbb{S}$ be countable dense ultrahomogeneous.
Is X indivisible?
ie: Let $X = A_1 \cup \ldots \cup A_k$. Is there $\widetilde{X}$ isometric to X and $i \leqslant k$ such that

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Theorem (Delhommé-Laflamme-Pouzet-Sauer)

No.

Remark

Crucial point: The distance set of X is too rich.

Second attempt: Discretization

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Definition

Up to isometry, there is a unique metric space $\mathbb{U}_m$ *with distances in* $\{1, \ldots, m\}$ *which is:*

1. *Countable.*
2. *Ultrahomogeneous.*
3. *Universal for the countable metric spaces with distances in* $\{1, \ldots, m\}$.

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1. $\mathbb{S}$ *is metrically oscillation stable.*
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Let $X \subset \mathbb{U}_m$, $f : X \longrightarrow \{1, \ldots, m\}$.
f *is Katětov over* X *when it defines a* 1*-point metric extension of* X*:*

$$\forall x, y \in X, \quad |f(x) - f(y)| \leqslant d(x, y) \leqslant f(x) + f(y).$$

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The orbit of f *in* $\mathbb{U}_m$ *is then:*

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Remark

If $\min f = p$, *then* $O(f, \mathbb{U}_m)$ *is isometric to* $\mathbb{U}_{\min(2p,m)}$.

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Definition

- *Elements of* $\mathbb{P}$*:*
  Pairs $s = (f_s, C_s)$ *where*
  1. $f_s : \mathrm{dom} f_s \longrightarrow \{1, \ldots, m\}$ *finite.*
  2. $C_s \subset \mathbb{U}_m$ *isometric to* $\mathbb{U}_m$.
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  4. f_s *is Katětov over* $\mathrm{dom} f_s$*.*
- *Relations $\leqslant$ and $\leqslant_k$:*

$$t \leqslant s \leftrightarrow (\mathrm{dom} f_s \subset \mathrm{dom} f_t \subset C_t \subset C_s \quad \text{and} \quad f_t \upharpoonright \mathrm{dom} f_s = f_s) \,.$$

$$t \leqslant_k s \leftrightarrow (t \leqslant s \quad \text{and} \quad \min f_t = \min f_s - k) \,.$$

A notion of largeness

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Definition

Let $\Gamma \subset \mathbb{U}_m$, $s \in \mathbb{P}$.
Then Γ is large relative to s when:

- *If* $\min f_s = 1$*:*

$$\forall t \leqslant_0 s \;\; (O(f_t, \mathbf{C}_t) \cap \Gamma \textit{ is infinite}) \, .$$

- *If* $\min f_s > 1$*:*

$$\forall t \leqslant_0 s \;\; \exists u \leqslant_1 t \;\;\; (\Gamma \textit{ is large relative to } u) \, .$$

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Let $(f_s, C_s) \in \mathbb{P}$. Assume that Γ is large relative to (f_s, C_s). Then there is C isometric to $\mathbb{U}_m$ such that:

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Lemma

Let $s \in \mathbb{P}$. Assume Γ is not large relative to s.
Then there is $t \leqslant_0 s$ such that $\mathbb{U}_m \smallsetminus \Gamma$ is large relative to t.

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There is a unique Banach space H into which the Urysohn space $\mathbb{U}$ embeds isometrically with dense linear span.

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