Independence in structures and finite satisfiability

Vera Djordjević

1 INTRODUCTION

Let G denote the (generic) random graph.

(1) G has the property that every sentence which is true in G is true in a finite subgraph of G;

we call this the *finite submodel property*;

this is a consequence of the 0-1 law, the proof of which uses a probabilistic argument

The probabilistic argument relies on the fact that whenever A is a substructure of G and we remove or add an edge in A to get A', then A' is also (isomorphic to) a finite substructure of G.

In this sense edges are *independent* of each other.

$Th(G)$ is $\aleph_0$-categorical, supersimple with SU-rank 1 and has trivial forking.

We would like to generalize the result (1) about G to structures M (not necessarily graphs) which are $\aleph_0$-categorical, have *finite* SU-rank and trivial forking.

It appears like, in order to carry out a probabilistic argument which proves the generalization, we need to assume that some notion of independence between definable relations holds for M.

2 PRELIMINARIES

We assume that T is a ***countable complete simple*** theory with elimination of hyperimaginaries.

REMARK: A simple theory which is $\aleph_0$-categorical or supersimple has elimination of hyperimaginaries.

Elements, sequences, sets and models/structures that we talk about come from a very saturated "monster model" of T.

DEFINITION: We say that a structure M has the ***finite submodel property*** if whenever $M \models \varphi$, then there is a finite substructure $N \subseteq M$ such that $N \models \varphi$.

REMARK: (1) If $M \models T$ and M has the finite submodel property, then T is not finitely axiomatizable.

(2) Suppose that $M \models T$, T is enumerated as $(\varphi_i : i < \aleph_0)$ and that M has the finite submodel property.

Then, for each k, there is a finite model N_k of $\varphi_0 \wedge \ldots \wedge \varphi_k$. From a first-order perspective the sequence $N_0, N_1, N_2, \ldots,$ can be seen as better and better finite approximations of M.

3 INDEPENDENT SYSTEMS OF ALGEBRAICALLY CLOSED SETS

NOTATION: For every $n < \aleph_0$, n also denotes the set $\{0, \ldots, n-1\}$ (or $\emptyset$ if $n = 0$).
Let $\mathcal{P}(n)$ denote the power set of n and let $\mathcal{P}^-(n) = \mathcal{P}(n) - \{n\}$.

DEFINITION: We call $\{A_s : s \in \mathcal{P}^-(n)\}$ an ***independent system of algebraically closed sets*** if for all $s, t \in \mathcal{P}^-(n)$:

(1) A_s is algebraically closed.

(2) If $s \subseteq t$ then $A_s \subseteq A_t$.

(3) $A_s \underset{A_{s\cap t}}{\downarrow} A_t$.

(4) If $|s| > 1$ then $A_s = \mathrm{acl}\left(\bigcup_{t \subset s} A_t\right)$.

4 THE n-EMBEDDING OF TYPES PROPERTY

DEFINITION:

(i) **Let** $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(n)\}$ **and** $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(n)\}$ **be two independent systems of algebraically closed sets. We say that** $\{f_w : w \in \mathcal{P}^-(n)\}$ **is an** ***elementary map*** **of** $\mathcal{A}$ **onto** $\mathcal{B}$ **if, for every** $w \in \mathcal{P}^-(n)$, f_w **is an elementary map from** A_w **onto** B_w **and if** $v \subseteq w$ **then** f_w **extends** f_v**.**

(ii) **We say that** T **has the** n***-embedding of types property*** **if (1) - (4) implies (5), where**

(1) $\mathcal{A} = \{A_w : w \in \mathcal{P}^-(n)\}$ **and** $\mathcal{B} = \{B_w : w \in \mathcal{P}^-(n)\}$ **are independent systems of algebraically closed sets,**

(2) $\{f_w : w \in \mathcal{P}^-(n)\}$ **is an elementary map of** $\mathcal{A}$ **onto** $\mathcal{B}$**,**

(3) $\mathrm{rng}(\bar{a}) \cap \mathrm{acl}\Big(\bigcup_{w\in\mathcal{P}^-(n)} A_w\Big) = \emptyset$**,**

(4) $a \in \mathrm{rng}(\bar{a})$ **and** $a \in \mathrm{acl}\Big((\mathrm{rng}(\bar{a}) - \{a\}) \cup \bigcup_{w\in\mathcal{P}^-(n)} A_w\Big)$ **implies that** $a \in \mathrm{acl}\big(\mathrm{rng}(\bar{a}) - \{a\}\big)$**,**

and

(5) **There is** $\bar{b}$ **and for every** $w \in \mathcal{P}^-(n)$ **an elementary map** $g_w : \mathrm{rng}(\bar{a}) \cup A_w \to \mathrm{rng}(\bar{b}) \cup B_w$ **which extends** f_w**.**

(iii) **We say that** T **has the** ***strong*** n***-embedding of types property*** **if (1) - (3) implies (5).**

(iv) We say that T has the *n-embedding of types property for* real *types* (or *strong n-embedding of types property for* real *types*) if whenever $\bar{a}$ is a sequence of real elements (i.e. elements of sort '$=$') and (1) - (4) hold (or (1) - (3) hold), then (5) holds.

5 EXAMPLES

(1) Every stable theory has the strong n-embedding of types property for every natural number n.

This is a consequence of the fact that every type (in a stable theory) over an algebraically closed set is stationary.

(2) The complete theory of the (generic) random graph has the strong n-embedding of types property for every natural number n.

(3) Let T_{tf} be the complete theory of the (generic) tetrahedron-free 3-hypergraph. Then T_{tf} is $\aleph_0$-categorical, simple with SU-rank 1 and has trivial forking, but does not have the n-embedding of types property for any $n \geq 3$.

6 INDEPENDENT STRUCTURES

Let T be a complete simple theory with elimination of hyperimaginaries.

Also, assume that there is $m < \aleph_0$ such that no function symbol in the language of T has arity greater than m.

DEFINITION:
(i) T is ***1-based*** if for all sets A and B, A and B are independent over $\mathrm{acl}(A) \cap \mathrm{acl}(B)$.

(ii) T has ***trivial forking*** if whenever $A \not\downarrow_B C_1C_2$, then $A \not\downarrow_B C_i$ for $i = 1$ or for $i = 2$.

DEFINITION:
(i) We say that T is ***independent*** if it is $\aleph_0$-categorical, simple with finite SU-rank, has trivial forking and has the n-embedding of types property for every $n < \aleph_0$.

(ii) A structure M is independent if its complete theory is independent.

REMARK: $\aleph_0$-categoricity, finite SU-rank and trivial forking implies 1-basedness.

7 MAIN THEOREM

THEOREM: If M is an independent structure then M has the finite submodel property.

REMARK: A stable structure M is independent if and only if it is $\aleph_0$-stable, $\aleph_0$-categorical and every strongly minimal set definable in M^{eq} has trivial pregeometry (given by the algebraic closure operator).

The class of ***stable*** independent structures has been studied by Lachlan, and it includes all stable finitely homogeneous structures, also studied and classified by Lachlan.

8 UNSTABLE EXAMPLES

(1) The random graph is an independent structure, with SU-rank 1.

(2) An example with SU-rank greater than 1:

Let $1 \leq k < \aleph_0$.

Let the vocabulary of the language L **be** $\{=, E_0, \ldots, E_k, R\}$ **and let** $\mathcal{K}$ **be the class of all finite** L**-structures** A **such that** $E_0, \ldots, E_k$ **are interpreted as equivalence relations, where** E_{i+1} **refines** E_i **for each** $i < k$, **and** R **is interpreted as a symmetric and irreflexive binary relation.**

It is easy to verify that $\mathcal{K}$ **has the hereditary property and amalgamation property, which implies that** $\mathcal{K}$ **has the joint embedding property (since no function symbols are present), so** $\mathcal{K}$ **has a so-called Fraïssé-limit** M.

CLAIM: M **is an independent structure with SU-rank** $k + 1$.

9 ROUGH PROOF SKETCH OF MAIN THEOREM

Let M be independent. (We want to show that M has the finite submodel property.)

Step 1: **We find, using results from [D2], a structure N which is definable in M^{eq} (without parameters) such that (N, acl_N) is a pregeometry and $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$.**

Step 2: **We show, with the help of results from [D1], that N has the finite submodel property.**

This is where the n-embedding of types property is used as well as a probabilistic argument, which is part of the proof of the main theorems in [D1], one of which we apply here.

Step 3: **A "transfer theorem" from [D2] rougly says that if $M \subseteq \mathrm{acl}_{M^{\mathrm{eq}}}(N)$ and N has the finite submodel property then also M has it.**

Thus we can conclude, from steps 1 and 2, that M has the finite submodel property.

[D1] ***The finite submodel property and ω-categorical expansions of pregeometries*****, Ann. Pure Appl. Logic, 139 (2006) 201-229.**

[D2] ***Finite satisfiability and $\aleph_0$-categorical structures with trivial dependence*****, J. Symb. Logic, 71 (2006) 810-830.**

10 THE COMPLETE n-AMALGAMATION PROPERTY

Suppose that T is simple with elimination of hyperimaginaries.

DEFINITION (Kim, Kolesnikov, Tsuboi):
Let $\{A_w : w \in \mathcal{P}^-(n)\}$ be an independent system of algebraically closed sets. We say that $\{p_w(\bar{x}_w) : w \in \mathcal{P}^-(n)\}$, where $p_w(\bar{x}_w) \in S(A_w)$ for each $w \in \mathcal{P}^-(n)$, is a ***coherent system of types over*** $\{A_w : w \in \mathcal{P}^-(n)\}$ if the following hold:

(1) If C_w realizes p_w then $C_w \supset A_w$ (so $\bar{x}_w$ is an infinite sequence of variables).

(2) If $w \subseteq v$ then $\bar{x}_w \subseteq \bar{x}_v$ and $p_w \subseteq p_v$.

(3) For every $w \in \mathcal{P}^-(n)$ there is a bijection $f_w : C_w \to \bar{x}_w$ such that if $C_w^\emptyset = f_w^{-1} \circ f_\emptyset(C_\emptyset)$, then

(4) $C_w = \mathrm{acl}(A_w \cup C_w^\emptyset)$ and $C_w^\emptyset \underset{A_\emptyset}{\perp} A_w$ (for every $w \in \mathcal{P}^-(n)$).

DEFINITION (Kim, Kolesnikov, Tsuboi):
We say that T has the ***complete*** n***-amalgamation property*** if for every $k < n$ we have:

> If $\{A_w : w \in \mathcal{P}^-(k)\}$ be an independent system of algebraically closed sets and $\{p_w(\bar{x}_w) : w \in \mathcal{P}^-(k)\}$ is a coherent system over $\{A_w : w \in \mathcal{P}^-(k)\}$, then there is C_k which realizes every p_w and and $C_k^\emptyset \underset{A_\emptyset}{\perp} \bigcup_{i \in k} A_{\{i\}}$.

REMARK: The complete 3-amalgamation property follows from the independence theorem.

11 EXAMPLES AND A RELATIONSHIP

(1) The complete theory of the (generic) random graph has the complete n-amalgamation property for every natural number n.

(2) There is a stable theory which does not have the complete 4-amalgamation property. (See: De Piro, Kim, Young, ***The type-definable group configuation under the generalized type amalgamation.***)

THEOREM: Suppose that T is simple with SU-rank 1. Let $n \geq 3$.

(i) If T has the complete $n+1$-amalgamation property, then T has the k-embedding of types property for ***real types*** for every $k \leq n$.

(ii) If T has trivial forking and the complete $n+1$-amalgamation property, then T has the ***strong*** k-embedding of types property for every $k \leq n$.

For a study of the (complete) n-amalgamation property, see:
Kim, Kolesnikov, Tsuboi, ***Generalized amalgamation and n-simplicity.***