PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 1 (1989), p. 45–56

# ASYMPTOTIC BEHAVIOUR OF THE INTEGRAL OF A FUNCTION ON THE LEVEL SET OF A MIXING RANDOM FIELD *

BY

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*Abstract.* Let $X = \{X(t): t \in \boldsymbol{R}^2\}$ be a centered stationary real random field with a.s. differentiable paths. Let $T$ be a rectangle of $\boldsymbol{R}^2$ and let $F(f, T)$ denote the integral of the continuous function $f$ over a level curve $\mathscr{C}_x$ of $X$ for a fixed level $x$, observed in $T$. We show that if a field $X$ satisfies some mixing condition, then $F(f, T)$, adequately normalized, converges weakly to the Wiener process indexed in $T$. The limit variance has a precise expression in the Gaussian case and *-mixing case. A geometrical lemma shows cases where the higher order moments of $F(f, T)$ are finite.

## 1. INTRODUCTION.

In what follows, $X = \{X(t): t \in \boldsymbol{R}^2\}$ denotes an a.s. differentiable centered stationary random field. $X$ is said to be *isotropic* if, given any $k \in \boldsymbol{N}$ and $t_1, t_2, \ldots, t_k \in \boldsymbol{R}^2$, the joint laws of $(X(t_1), X(t_2), \ldots, X(t_k))$ and $(X(Qt_1), X(Qt_2), \ldots, X(Qt_k))$ are the same when $Q$ is any isometry in $\boldsymbol{R}^2$. $X$ is said to be *affine*, when it is equal in law to $\{Y(At): t \in \boldsymbol{R}^2\}$, where $Y$ is isotropic and $A: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a linear self adjoint transformation. The angle $\theta_0$ defining the eigenvectors directions $(\cos\theta_0, \sin\theta_0)$ and $(-\sin\theta_0, \cos\theta_0)$ and the respective eigenvalues $\lambda_1$, $\lambda_2$ specify the affinity $A$. There is no loss of generality in assuming $\lambda_1 \geqslant \lambda_2 > 0$, $\lambda_1 \lambda_2 = 1$. Cabaña [2] has proposed estimators of the *affinity parameters* $k = (1-\lambda_1^2/\lambda_2^2)^{1/2}$ and $\theta_0$ based on the shape of the level curve of $X$, corresponding to a given level $x$.

The gradient process $\dot{X}(t) = \|\dot{X}(t)\| \, (\cos\Theta(t), \sin\Theta(t))$ is determined by the two real stationary processes $\|\dot{X}\|$ and $\Theta$. If $X$ has a.s. differentiable Jacobian and $T$ is an open rectangle of $\boldsymbol{R}^2$, the set $\mathscr{C}_x = \{t \in T: X(t) = x, \dot{X}(t) \neq 0\}$, for a fixed $x$, is a.s. a $C^1$ one-dimensional manifold, as it results from applying the implicit Function Theorem. For $f: (-\pi, \pi] \to \boldsymbol{R}$

* Some of the results presented here are part of the author's Doctoral dissertation at Universidad Central de Venezuela.

continuous and bounded we write

$$F(f, T) = |T|^{-1} \int_{\mathscr{C}_x} f(\Theta(t))\, dS(t),$$

where $|T|$ is the area of $T$, $dS$ is the surface measure of $\mathscr{C}_x$, and the integral is defined for every $\omega \in \Omega$ as the usual integral over a differentiable manifold [3]. $F(f, T)$ is a random variable with respect to the $\sigma$-algebra generated by $\{X(t): t \in T\}$. We write

$$\mathscr{L}(T) = F(1, T), \quad \mathscr{C}(T) = F(\cos 2\theta, T), \quad \mathscr{S}(T) = F(\sin 2\theta, T).$$

Cabaña [3] and Wschebor [10] have proved by different methods the formulae, known as *Rice's Formulae*, for the moments of the variable $F(f, T)$ (Theorem 3). Making use of these formulae we obtain

$$\tan 2\theta_0 = \mathrm{E}\mathscr{S}/\mathrm{E}\mathscr{C}, \quad g(k) = \frac{\sqrt{(\mathrm{E}\mathscr{C})^2 + (\mathrm{E}\mathscr{S})^2}}{\mathrm{E}\mathscr{L}},$$

where $g$ is a certain function of the parameter $k$. These equations led Cabaña to propose the following estimators for the affinity parameters $\theta_0$ and $k$:

$$\theta_0 = \frac{1}{2}\arg(\mathscr{C}(\lambda T), \mathscr{S}(\lambda T)), \quad k = g^{-1}\left(\frac{\sqrt{\mathscr{C}^2(\lambda T) + \mathscr{S}^2(\lambda T)}}{\mathscr{L}(\lambda T)}\right),$$

where $\lambda T$ is a dilation of $T$, $\theta_0$ and $k$ are consistent and asymptotically Gaussian under adequated mixing conditions. Cabaña [2] has proved this assertion when $X$ is $\delta$-dependent, and he has computed the limit variance. The aim of the present paper is to prove the asymptotical normality of $F(f, T)$ under less strict conditions of dependence. We prove, making use of a Functional Central Limit Theorem for strong mixing fields, that adequately normalized $F(f, T)$ converges weakly to a Wiener process indexed by the rectangles of $\mathscr{R}^2$. Generally, the Central Limit Theorem for mixing variables has the inconvenient that the variance is impossible to compute, but here we compute the limit variance in the Gaussian and *-mixing cases and show that this variance has the same expression that in the $\delta$-dependent case (see [2]).

Since the existence of higher order moments plays a fundamental role, we improve Wschebor's results [11], giving conditions under which the $p$-th order moment of $F(f, T)$ is finite.

## 2. FRAMEWORK

We begin with

Definition 1. Let $\xi = \{\xi(t): t \in \boldsymbol{R}^d\}$ be a stationary random field. We say that $\xi$ is *strongly mixing with coefficient* $\alpha$, or $\alpha$*-mixing*, if for any Borelian sets $U$ and $V$

$$\sup\{|\mathrm{P}(A\cap B)-\mathrm{P}(A)\mathrm{P}(B): A\in\sigma(U),\ B\in\sigma(V)\}\leqslant\alpha(p(U,V))$$

and $\alpha(p)\to 0$, $p\to\infty$. Here $p(U,V)$ is the Euclidean distance and $\sigma(U) = \sigma\{\xi(t):\ t\in U\}$. In the same manner, we say $\xi$ *satisfies a *-mixing condition with coefficient* $\psi$ if

$$\sup\left\{\left|\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A)\mathrm{P}(B)}-1\right|:\ A\in\sigma(U),\ B\in\sigma(V)\right\}\leqslant\psi(p(U,V))$$

and $\psi(p)\to 0$, $p\to\infty$.

Let $\xi=\{\xi(t):\ t\in \boldsymbol{Z}^d\}$ be a discret strong mixing stationary centered real random field and

$$S_n=\sum_{1\leqslant j\leqslant n}\xi_j,$$

where $j\leqslant n$ means $j_1\leqslant n_1,\ j_2\leqslant n_2,\ \ldots,\ j_d\leqslant n_d$. We prove

LEMMA 1. *If* $\mathrm{E}|\xi_0|^{2+\delta}<\infty$ *and the mixing satisfies*

$$\sum_{r=1}^{\infty} r^{d-1}\alpha^{\delta/(2+\delta)}(r)<\infty,$$

*then*

(i) $$\sum_{j\in\boldsymbol{Z}^d}\mathrm{E}(\xi_0\xi_j)<\infty,$$

(ii) $$|n|^{-1}\mathrm{E}S_n^2\to\sum_{j\in\boldsymbol{Z}^d}\mathrm{E}(\xi_0\xi_j)=\sigma^2\quad \textit{when } n\to\infty,$$

*where* $n\to\infty$ *means* $\min_{1\leqslant j\leqslant d} n_j\to\infty$ *and* $|n|=n_1n_2\ldots n_d$.

Proof. Lemma 3 of Billingsley [1], p. 172, and the inequality of moments for strong mixing fields [7] give the proof.

Let $T^d$ be a Cartesian product of $[0,1]\subset\boldsymbol{R}$. We will denote by $C_d$ the set of continuous functions on $T^d$ provided with the uniform metric, and $D_d$ is the Skorohod function space on $T^d$. A subset $B$ of $T^d$, $B=\{u=(u_1,\ldots,u_d):\ s_j\leqslant u_j\leqslant t_j\}$ is called a *block*, and increment $\xi(B)$ of $\xi$ around a block $B$ is given by

$$\xi(B)=\sum_{\varepsilon\in\{0,1\}^d}(-1)^{d-|\varepsilon|}\xi(s+\varepsilon(t-s)),$$

where $|\varepsilon|=\varepsilon_1+\ldots+\varepsilon_d$ if $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_d)$.

The Wiener process $W=\{W(t):\ t\in T^d\}$ on $T^d$ is characterized by

(a) $P\{W\in C_d\}=1$;

(b) if $B_1, B_2, \ldots, B_k$ are pairwise disjoint blocks in $T^d$, then the increments $W(B_1), W(B_2), \ldots, W(B_k)$ are independent normal variables with means zero and variances $|B_1|, |B_2|, \ldots, |B_k|$, where $|\cdot|$ denotes the $d$-dimensional Lebesgue measure in $T^d$.

If $\mathscr{A}$ is a class of sets, the Wiener process indexed in $\mathscr{A}$, $W = \{W(A)\colon A \in \mathscr{A}\}$, is the Gaussian centered process with covariances $\operatorname{cov}(W(A), W(B)) = |A \cap B|$ for $A, B \in \mathscr{A}$. If $\mathscr{A}$ is the class of the blocks in $T^d$, denoted by $\mathscr{R}^d$, $\{W(B)\colon B \in \mathscr{R}^d\}$ is the increments process of $\{W(t)\colon t \in T^d\}$ around the blocks $B$'s. It results that $W = \{W(B)\colon B \in \mathscr{R}^d\}$ is continuous [8].

## 3. MAIN THEOREM

Let $T = [-a, a) \times [-b, b)$ be a half-open rectangle in $R^2$. We divide the plane in a grid $\{T_j\colon j \in \mathbf{Z}^2\}$, where $T_j$'s are all rectangles with the same dimension as $T$, such that if $j = (j_1, j_2)$, $T_j$ has center $C_j = (2aj_1, 2bj_2)$. For $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2$ we define the following dilation of $T$

$$\lambda T = \begin{cases} T & \text{if } \lambda_1 = \lambda_2 = 1, \\ \bigcup\limits_{j \in Q_\lambda} T_j & \text{if } \lambda_1 > 1 \text{ or } \lambda_2 > 1, \end{cases}$$

where $Q_\lambda = \{j \in \mathbf{Z}^2\colon -(\lambda - 1) \leqslant j \leqslant \lambda - 1\}$. Note that $|\lambda T| = |Q_\lambda|\,|T|$, where $|Q_\lambda| = \operatorname{card}(Q_\lambda) = (2\lambda_1 - 1)(2\lambda_2 - 1)$ and

$$F(f, T) = |Q_\lambda|^{-1} \sum_{j \in Q_\lambda} F(f, T_j).$$

THEOREM 1. *Suppose that the field $X$ is strongly mixing and*

(i) *the mixing satisfies* $\sum\limits_{k=1}^{\infty} k^{(q+[q/2])-1} \alpha^{\delta/(q+\delta)}(k)$;

(ii) $\mathrm{E}[F(f, T)]^{q+\delta} < \infty$ *for some* $q \geqslant 2$ *and* $\delta > 0$;

(iii) *the variance defined by Lemma 1 has the form*

$$\sigma^2 = |T|^{-1} \int_{R^2} I(0, t, f)\, dt > 0,$$

*where*

$$I(s, t, f) = \int_{-\pi}^{\pi} f(\theta)\, d\theta \int_{-\pi}^{\pi} f(\varphi) H_{st}(x, \theta, \varphi)\, d\varphi$$

*with*

$$\begin{aligned} H_{st}(x, \theta, \varphi) = {} & \mathrm{P}_{X(s), X(t), \Theta(s), \Theta(t)}(x, x, \theta, \varphi) \times \\ & \times \mathrm{E}[\|\dot{X}(s)\|\,\|\dot{X}(t)\|/X(s) = X(t) = x, \Theta(s) = \theta, \Theta(t) = \varphi] - \\ & - \mathrm{P}_{X(s), \Theta(s)}(x, \theta)\, \mathrm{E}[\|\dot{X}X(s)\|/X(s) = x, \Theta(t) = \theta] \times \\ & P_{X(t), \Theta(t)}(x, \varphi)\, \mathrm{E}[\|\dot{X}(t)\|/X(t) = x, \Theta(t) = \varphi], \end{aligned}$$

*where $P_{X,\Theta}$ denotes the joint density of $X$ and $\Theta$.*

*Then the field* $\{Z_\lambda(T):\ T\in\boldsymbol{R}^2\}$, *defined by*

$$Z_\lambda(T)=\left[\frac{|Q_\lambda|\,|T|}{\sigma^2}\right]^{1/2}[F(f,\lambda T)-\mathrm{E}F(f,T)],$$

*converges weakly to the Wiener process indexed in* $\mathscr{R}^2$, $\{W(T):\ T\in\mathscr{R}^2\}$.

Proof. We can define the following discret random field: for every $j\in\boldsymbol{Z}^2$, $Y_j=F(f,T_j)-\mathrm{E}F(f,T_j)$. It is not difficult to see that if the field $X$ is strongly mixing, then so is $Y=\{Y_j:\ j\in\boldsymbol{Z}^2\}$, and the mixing coefficient of $Y$ is less than $\alpha(k-h)$, where $\alpha$ is the strong mixing coefficient of $X$ and $h$ is the length of the diagonal of $T$. In fact, if $N\in\boldsymbol{Z}^2$, because $F(f,T)$ is measurable we have

$$\sigma\{Y_j:\ j\in N\}\subset\sigma\{X(t):\ t\in\bigcup_{j\in N}T_j\}$$

and, therefore,

$$\sup\{|\mathrm{P}(A\cap B)-\mathrm{P}(A)\mathrm{P}(B)|:\ A\in\sigma(N),\ B\in\sigma(N')\}\leqslant\alpha\big(p(\bigcup_{j\in N}T_j,\ \bigcup_{j\in N'}T_j)\big).$$

We conclude that $Y$ is strong mixing and its mixing coefficient satisfies (i).

On the other hand,

$$Z_\lambda(T)=\frac{|T|^{1/2}}{(|Q_\lambda|\,\sigma^2)^{1/2}}\sum_{j\in Q_\lambda}Y_j=(|n|\,\sigma^2)^{-1/2}\sum_{-[nt]\leqslant j\leqslant[nt]}Y_j.$$

The last equation results from the change $\lambda-1=[nt]$. Accordingly, this suggests making use of a Functional Central Limit Theorem for mixing multiparametric processes.

Proposition 1. *Suppose* $Y=\{Y_j:\ j\in\boldsymbol{Z}^d\}$ *is a strongly mixing centered stationary real random field such that, for some even number* $q\geqslant 2$ and $\delta>0$,

(i) $$\sum_{k=1}^{\infty}k^{d(3/2)q-1}\alpha^{\delta/(q+\delta)}(k)<\infty,$$

(ii) $$\mathrm{E}\,|Y_0|^{q+\delta}<\infty.$$

*If* $\sigma^2$, *defined by Lemma* 1, *is positive, then, for* $t\in T^d$,

$$Z_n(t)=(|n|\,\sigma^2)^{-1/2}\sum_{0\leqslant j\leqslant[nt]}Y_j$$

*converges weakly to the d-parameter Wiener process* $\{W(t):\ t\in T^d\}$.

Proof. The technics used in this proof are essentially those of Billingsley [1] (Theorem 20.1) in the generalization made by Deo [5] for multiparametric processes. The conditions $\mathrm{E}Z_n(t)\to 0$ and $\mathrm{E}Z_n^2(t)\to|t|$ as $n\to\infty$ are trivially seen to be satisfied. It remains, therefore, to be proved the uniform integrability of $Z_n^2(t)$ and its tightness.

In order to prove that we will make use of the following property: if the assumptions of Proposition 1 hold, then

$$\mathrm{E}\,\Big|\sum_{j\in A} Y_j\Big|^q \leqslant K\,|A|^{q/2} \tag{1}$$

for some even integer, where $K$ is a finite constant only depending on $q$ and $d$, the moments of $Y$ and $\alpha$ [9].

The uniform integrability of $Z_n^2(t)$ is a consequence of the previous property and the following inequality:

$$\mathrm{E}\,(S_n^2 I_{|S_n^2|>a}) \leqslant a^{(2-q)/2}\,|n|^{-q/2}\,\mathrm{E}\,|S_n|^q.$$

The tightness condition comes from the following Oscillation Lemma, due to Doukhan-Portal [6]. This lemma is more powerful than the one we require because it gives a precise rate for oscillation, and we need to modify it a little, since originally it is referred for empirical process.

LEMMA 2. *With the same assumption of Proposition* 1, *for $p\in N$ and some $0<\delta<1$ such that $p(1-\delta)>d$ we get*

$$\mathrm{P}\,\{\sup\{|Z_n(s)-Z_n(t)|:\ \|s-t\|\leqslant n^{\beta-1}\}\geqslant kn^{-\theta}\}\leqslant kn^{-\theta},$$

*where $\theta=[p(1-\beta(1-\delta))-d(1-\beta)]/(2p+1)$ and $k$ is a constant depending only on $p$ and $d$, the moments of $Y$ and $\alpha$. Here $\|s\|=\sum_{1\leqslant j\leqslant d}|s_j|$.*

Proof. The proof comes from the inequality

$$\mathrm{E}\,|S_{[nt]}-S_{[ns]}|^q\leqslant \mathrm{E}\,\Big|\sum_{[nt]\leqslant j\leqslant[ns]} Y_i\Big|^q\leqslant K\,|n|^{q/2}\,\|t-s\|^{q/2}$$

that is a consequence of (1) and the proper definition of $Z_n(t)$. Lemma III.4 of Doukhan-Portal [6] holds trivially in this case and the rest of the proof follows without change.

The proof of Theorem 1 is completed by making use of Proposition 1 (taking $d=2$) and noting that $Z_\lambda(T)$ can be expressed

$$Z_\lambda(T)=(|n|\,\sigma^2)^{-1/2}\Big[\sum_{0\leqslant j\leqslant[nt]} Y_j+\sum_{[n(-t)]\leqslant j\leqslant 0} Y_j+\sum_{\substack{0\leqslant j_1\leqslant[n_1t_1]\\ [n_2(-t_2)]\leqslant j_2\leqslant 0}} Y_j+\sum_{\substack{[n_1(-t_1)]\leqslant j_1\leqslant 0\\ 0\leqslant j_2\leqslant[n_2t_2]}} Y_j$$

which converges weakly to

$$W(t_1,t_2)+W(-t_1,-t_2)-W(t_1,-t_2)-W(-t_1,t_2)=W(T).$$

## 4. APPLICATIONS STUDY OF THE VARIANCE $\sigma^2$

The expression for the variance $\sigma^2$ in Theorem 1 (iii) is a consequence of the Rice formulae for the moments of $F(f,T)$. Cabaña [3] has proved that if the field $X$ has a Jacobian a.s. Lipschitz continuous and the Lipschitz constant does not depend on $\omega$, for every $t \in T$ the probability distribution of $X(t)$ has density $P_{X(t)}$, there is a joint density of $X$ and $\dot{X}$ and it is continuous, and the function $f$ is non-negative and Lipschitz continuous, then the Rice formulae hold even if they are infinite. Moreover, if $X$ is Gaussian, instead of the first condition one can demand that $\dot{X}$ and the derivative of its covariance function satisfy a Lipschitz condition in any compact $K$.

**4.1. Gaussian case.**

LEMMA 3. *Suppose $X$ is a Gaussian field with covariance function $\Gamma$. Suppose also $X$ satisfies the assumptions of the previous section. If $\Gamma$ has two derivatives, and $\Gamma(u)$, $\dot{\Gamma}(u)$ and $\ddot{\Gamma}(u)$ tend to zero as $|u| \to \infty$, then the variance has the form* (iii) *of Theorem* 1.

Proof. Making use of Lemma 1, we have

$$|Q_\lambda| \operatorname{Var} F(f, \lambda T) = |Q_\lambda|^{-1} \mathrm{E} \Big| \sum_{j \in Q_\lambda} Y_j \Big|^2 \to \sum_{j \in \mathbf{Z}^2} \mathrm{E}(Y_0 Y_j) = \sigma^2$$

as $|\lambda| \to \infty$ and, therefore, we have to prove that

$$\big||Q_\lambda| \operatorname{Var} F(f, \lambda T) - \sigma^2\big| \tag{2}$$

$$= \Big||Q_\lambda|^{-1} |T|^{-2} \int_{\lambda T \times (\lambda T)'} ds dt \int_{-\pi}^{\pi} f(\theta) d\theta \int_{-\pi}^{\pi} f(\varphi) H_{st}(x, \theta, \varphi) d\varphi\Big|$$

tends to zero as $\lambda \to \infty$. Here $(\lambda T)'$ is the complement of $\lambda T$ in $\mathbf{R}^2$, and the equation follows from Theorem 2, the expression of the variance $\sigma^2$ and stationarity of the field.

In order to prove that we will see that

$$|\mathrm{P}_{X(s), X(t)}(x, x) - \mathrm{P}_{X(s)}(x) \mathrm{P}_{X(t)}(x)| \to 0 \quad \text{as } p(s, t) \to \infty \tag{A}$$

and

$$|\mathrm{P}\{X(s) \in I, X(t) \in J / X(s) = X(t) = x\} - \mathrm{P}\{X(s) \in I / X(s) = x\} \times \times \mathrm{P}\{X(t) \in J / X(t) = x\}| \to 0 \tag{B}$$

as $p(s, t) \to \infty$, where $I$ and $J$ are both rectangles of $\mathbf{R}^2$. The proof will follow from (A) and (B) arguing as in the last part of the proof of Lemma 4.

We can assume without loss of generality that $X$ is normalized in such a way that $\Gamma(0) = -\ddot{\Gamma}(0) = 1$. We see that (B) is less than or equal to

$$\begin{aligned} &|\mathrm{P}\{\dot{X}(s) \in I, \dot{X}(t) \in J / X(s) = X(t) = x\} - \mathrm{P}\{\dot{X}(s) \in I / X(s) = x\} \times \\ &\quad \times P\{\dot{X}(t) \in J / X(s) = X(t) = x\}| + \\ &\quad + |\mathrm{P}\{\dot{X}(s) \in I / X(s) = X(t) = x\} - \mathrm{P}\{\dot{X}(s) \in I / X(s) = x\}| + \\ &\quad + |\mathrm{P}\{\dot{X}(t) \in J / X(s) = X(t) = x\} - \mathrm{P}\{\dot{X}(t) \in J / X(t) = x\}|. \end{aligned}$$

Since $X$ is Gaussian, $X$ and $\dot{X}$ conditioned by $X$ are Gaussian as well, and we can write the first term as

$$\text{(b)} \qquad |\mathrm{P}\{Z(s)\in I,\ Z(t)\in J\} - \mathrm{P}\{Z(s)\in I\}\,\mathrm{P}\{Z(t)\in J\}|,$$

where $Z(s)$ and $Z(t)$ are two Gaussian variables with parameters $m_s = -m_t = \Gamma(u)\,x/(1+\Gamma(u))$, where $u = s-t$,

$$\operatorname{Var} Z(s) = \operatorname{Var} Z(t) = 1 + \frac{\Gamma(u)\,\dot{\Gamma}^t(u)}{1-\Gamma^2(u)},$$

$$\operatorname{cov}(Z(s), Z(t)) = -\ddot{\Gamma}(u) + \frac{\Gamma(u)\,\dot{\Gamma}(u)\,\dot{\Gamma}^t(u)}{1-\Gamma^2(u)}.$$

If $\Sigma$ denotes the covariance matrix of the joint distribution of $Z(s)$ and $Z(t)$, we denote by $\Lambda$ the matrix which coincides with $\Sigma$ in the diagonal and has zeros as other elements, and by $m$ the column vector which has $m_s$ and $m_t$ as coordinates. Hence (b) can be written as

$$\left|\frac{(\det\Sigma)^{-1/2}}{2\pi}\int_{I\times J}\exp\{-(y-m)\,\Sigma^{-1}(y-m)'\}\,dy - \right.$$
$$\left. -\frac{(\det\Lambda)^{-1/2}}{2\pi}\int_{I\times J}\exp\{-(y-m)\,\Lambda^{-1}(y-m)'\}\,dy\right|$$
$$\leqslant \left|\frac{(\det\Sigma)^{-1/2}-(\det\Lambda)^{-1/2}}{2\pi}\right|\int_{I\times J}\exp\{-(y-m)\,\Lambda^{-1}(y-m)'\}\,dy +$$
$$+\frac{(\det\Sigma)^{-1/2}}{2\pi}\int_{I\times J}|1-\exp\{-(y-m)[\Sigma^{-1}-\Lambda^{-1}](y-m)^t\}|\,dy.$$

Now according to the assumption and to the form of the variance and covariance, (b) tends to zero as $|u|\to\infty$.

The other terms have a similar behaviour, since both can be thought as $|\mathrm{P}\{Z_1\in I\} - \mathrm{P}\{Z_2\in J\}|$, where $Z_1$ and $Z_2$ are two Gaussian variables with parameters

$$m_1 = \frac{\dot{\Gamma}(u)\,x}{1+\Gamma(u)}, \qquad m_2 = 0, \qquad \operatorname{Var} Z_1 = 1 + \frac{\dot{\Gamma}(u)\,\dot{\Gamma}^t(u)}{1-\Gamma^2(u)}, \qquad \operatorname{Var} Z_2 = 1.$$

Finally, (A) is deduced easily from the assumption, since the variables are centered, and the covariance matrix of $(X(s), X(t))$ has the form

$$\begin{pmatrix} 1 & \Gamma(u) \\ \Gamma(u) & 1 \end{pmatrix}$$

which tends to the identity for $|u|\to\infty$.

**4.2. The *-mixing case.**

LEMMA 4. *Suppose that $X$ satisfies the Cabaña's assumptions* (*see the beginning of section* 4) *and also*

(ii) $P_{X(t)}(x)$ *and* $E[\|\dot{X}\|/X = x]$ *are uniformly bounded in a neighbourhood $V$ of $x$; then the variance $\sigma^2$ has the form* (iii) *of Theorem* 1.

Proof. Like Lemma 3 we have to prove that expression (2) tends to zero as $|\lambda| \to \infty$. This expression is smaller than

$$K^2 |Q_\lambda|^{-1} |T|^{-2} \int_{\lambda T \times (\lambda T)'} dsdt \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H_{st}(x, \theta, \varphi)| \, d\theta d\varphi.$$

It is not difficult to see that

$$H_{st}(x, \theta, \varphi) = \lim_{\varepsilon \to 0} \varepsilon^{-4} \{E[Z_s(x, \theta, \varepsilon) Z_t(x, \varphi, \varepsilon)] - EZ_s(x, \theta, \varepsilon) EZ_t(x, \varphi, \varepsilon)\},$$

where

$$Z_s(x, \theta, \varepsilon) = \|\dot{X}(s)\| \, \mathbf{1}_{\{x \leqslant X(s) \leqslant x+\varepsilon\}} \mathbf{1}_{\{\theta \leqslant \Theta(s) \leqslant \theta+\varepsilon\}}.$$

Since the integrands are positive, an application of Fatou's Lemma and the inequality of moments for *-mixing variables [7] give

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H_{st}(x, \theta, \varphi)| \, d\theta d\varphi$$

$$\leqslant \psi(p(s, t)) \liminf_{\varepsilon \to 0} \varepsilon^{-4} \int_{-\pi}^{\pi} E|Z_s(x, \theta, \varepsilon)| \, d\theta \int_{-\pi}^{\pi} E|Z_t(x, \varphi, \varepsilon)| \, d\varphi.$$

Making use of the conditional expectation properties and Fubini's Theorem, we obtain that each integral of the right-hand member of the previous inequality is majorated by $\varepsilon^2 CC'$, where $C$ and $C'$ are bounds of $P_{X(0)}(x)$ and $E[\|X(s)\|/X(s) = x]$ respectively, both uniform in a neighbourhood of $x$ which contains the interval $[x, x+\varepsilon]$. Hence, substituting in the last expression, the limit in $\varepsilon$ disappears and the proof will be completed if we show that

(c) $$|Q_\lambda|^{-1} |T|^{-2} \int_{\lambda T \times (\lambda T)'} \psi(p(s, t)) \, dsdt \to 0 \quad \text{as } |\lambda| \to \infty.$$

Let us denote by $m_\lambda$ and $M_\lambda$ the sets

$$m_\lambda = \{(s, t) \in \mathbf{R}^4 \colon s \in \lambda T, t \notin \lambda T, p(s, t) \leqslant \mu(\lambda)\},$$

$$M_\lambda = \{(s, t) \in \mathbf{R}^4 \colon s \in \lambda T, t \notin \lambda T, p(s, t) > \mu(\lambda)\},$$

where $\mu(\lambda) = o[(z\lambda_1 - 1)(z\lambda_2 - 1)]$ and $\mu(\lambda) \to \infty$ as $\lambda \to \infty$. If $L$ is a bound of $\psi$, then

$$\iint_{m_\lambda} \psi(p(s, t))\, ds\, dt \leqslant L|m_\lambda| = L \times o[|Q_\lambda||T|].$$

On the other hand,

$$\iint_{n_\lambda} \psi(p(s, t))\, ds\, dt = \int_{\lambda T} ds \int_{\{p(s,t) > \mu(\lambda)\}} \psi(p(s, t))\, dt = |Q_\lambda||T| \int_{\{|u| > \mu(\lambda)\}} \psi(|u|)\, du,$$

therefore (c) is less than

$$|Q_\lambda|^{-1}|T|^{-2} L \times o[|Q_\lambda||T|] + |T|^{-1} \int_{\{|u| > \mu(\lambda)\}} \psi(|u|)\, du$$

which tends to zero as $\lambda \to \infty$.

## 5. EXISTENCE OF THE MOMENT OF $F(f, T)$

Assumption (ii) in Theorem 1 requires the finiteness of the $q+\delta$ order moment. The Wschebor Theorem for the Rice formula [11] gives conditions for the validity of the formulae and also guarantees their finiteness. However the following lemma, suggested by Wschebor, shows in a geometrical way how the problem of moments existence can be studied without looking at the complicated integrals in the Rice formulae.

Let $T = T_1 \times T_2$ be the Cartesian product of two intervals of $\boldsymbol{R}$. We call *1-section* of $X$ determined by $t_1$, which will be denoted by $X^1_{t_1}$, the uniparametric process

$$X^1_{t_1}\colon T_2 \to \boldsymbol{R}, \quad t \to X^1_{t_1}(t) = X(t_1, t).$$

In the same way we can define $X^2_{t_2}$. Obviously, the 1-sections are all measurable for every $t_i$ $(i = 1, 2)$. Since $X$ is stationary, it is sufficient to consider the $i$-sections $X^i$ determined by $t_i = 0$. If $X$ is a.s. of class $C^p$, then a.s. every $i$-section is a.s. of class $C^p$.

In what follows we will take $f \equiv 1$ and write $\mathscr{L}(\mathscr{C}_x) = F(1, T)$. Let $N^i_x(T_j)$ be the number of crossing of the process $X^i$ with the level $x$ in the interval $T_j$, where $i \neq j$, $i, j = 1, 2$. Moreover, let

$$Z^i_p = \sup_{t_j \in T_j} |(X^i)^{(p)}(t_j)|.$$

LEMMA 5. *Suppose that the field $X$ and its derivative $\dot{X}$ have a.s. continuous paths. Then*

$$\mathscr{L}(\mathscr{C}_x) \leqslant \int_{T_1} N^1_x(T_2)\, dt_1 + \int_{T_2} N^2_x(T_1)\, dt_2.$$

COROLLARY 1. *With the same assumptions*

$$\{\mathrm{E}[\mathcal{L}(\mathcal{C}_x)]^p\}^{1/p} \leqslant |T_1| \{\mathrm{E}[N_x^1(T_1)]^p\}^{1/p} + |T_2| \{\mathrm{E}[N_x^2(T_2)]^p\}^{1/p}.$$

COROLLARY 2. *Suppose that X is a Gaussian field, with covariance function $\Gamma$, which is normalized $(\Gamma(0) = -\ddot{\Gamma}(0) = 1)$. If, for $u, v \in \boldsymbol{R}$,*

$$\Gamma((0, u), (0, 0)) = \mathrm{E}(X(0, u) X(0, 0)) = 1 + \frac{u^2}{2} + \frac{C|u|^3}{6} + o(u^3),$$

$$\Gamma((v, 0), (0, 0)) = \mathrm{E}(X(v, 0) X(0, 0)) = 1 + \frac{v^2}{2} + \frac{C'|v|^3}{6} + o(v^3)$$

*for u and v in a neighbourhood V of* 0 *and C and C′ positive constants, then* $\mathrm{E}[\mathcal{L}(\mathcal{C}_x)]^k < \infty$ *for every* $k = 1, 2, \ldots$

Proof. The proof follows from Lemma 5 and Theorem 4.1 in [4].

COROLLARY 3. *Suppose X has a.s. class $C^p$ paths $(p \geqslant 2)$ and*
(i) *For every $t_2 \in T_2$, $(X^2(t_1), \dot{X}^2(t_1))$ has joint density uniformly bounded by C and for every $t_1 \in T_1$, $(X^1(t_2), \dot{X}^1(t_2))$ has joint density uniformly bounded by C.*

(ii) *Let $m < 2p-2$ if there is an $r > 2m/(2p-2-m)$ such that $\mathrm{E}|Z_p^i|^r < \infty$ for every $p = 1, 2, \ldots$ and $i = 1, 2$.*

*Then*

$$\{\mathrm{E}|\mathcal{L}(\mathcal{C}_x)|^m\}^{1/m} \leqslant L_1 |T_1| \{\mathrm{E}|Z_p^2|^r\}^{1/m} + L_2 |T_2| \{\mathrm{E}|Z_p^1|^r\}^{1/m} + L_3 < \infty,$$

where $L_1$, *$L_2$ and $L_3$ depend only on $p$, $m$, $r$, $|T|$ and C.*

COROLLARY 4. *If X has a.s. class $C^\infty$ paths, if* (i) *of Corollary* 3 *holds and* $\mathrm{E}|Z_p^i| < \infty$ *for every $p = 1, 2, \ldots$ and $i = 1, 2$,* then

$$\mathrm{E}[\mathcal{L}(\mathcal{C}_x)]^m < \infty \quad \textit{for every } m = 1, 2, \ldots$$

Proofs of Corollaries 3 and 4 follow from Lemma 4 and Corollaries 3 and 4 of Wschebor [11] (p. 36).

Proof of Lemma 5. The proof is completely geometrical and the same idea can be extended for more general situations (dimension $> 2$).

Suppose $\mathcal{C}_x$ is a polygonal. Every segment of $\mathcal{C}_x$ has length less than the sum of the lengths of its projections over the coordinates axes. If we take a partition in every interval $T_1$ and $T_2$, the length of every semi-interval contributes to the sum every time the $i$-th section process determined by $t_i$ in the semi-interval, crosses the level $x$ in the interval $T_j (i \neq j)$. Therefore

$$\mathcal{L}(\mathcal{C}_x) \leqslant \sum_{k=1}^{N} N_x^1(T_2)[t_1^k - t_1^{k-1}] + \sum_{h=1}^{M} N_x^2(T_1)[t_2^h - t_2^{h-1}],$$

where $\{t_1^0, t_1^1, \ldots, t_1^N\}$ and $\{t_2^0, t_2^1, \ldots, t_2^M\}$ are both partitions of $T_1$ and $T_2$, respectively. Lemma 5 follows by taking the limit when the sizes of the partitions tend to zero.

**Acknowledgement.** I am greatly indebted to my thesis adviser, Enrique Cabaña, who formulated this problem and suggested many crucial ideas. Also, I would like to thank J. R. León, M. Wschebor and J. Ortega for many helpful discussions. The improvement in section 5 of Wschebor's results were suggested by him. I thank P. Doukhan who gave me invaluable advise in the organization of this paper.

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*Received on 24. 9. 1987*