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ON DENSITY OF A STABLE UNIFORMLY CONVEX NORM

BY

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Abstract. Let $(E, \|\cdot\|)$ be a uniformly convex Banach space and assume that its modulus of uniform convexity $\alpha(\cdot)$ satisfies the condition: $\alpha(\varepsilon) \geqslant \text{const}\cdot\varepsilon^n$, $n \in N$. We prove that for every stable symmetric measure μ on E the density of the distribution function $F_z(t) = \mu\{\|\cdot + z\| < t\}$, $z \in E$, is bounded on every interval $(0, T)$, $T > 0$. Under some additional assumptions we extend the conclusion to the whole half-line $(0, \infty)$.

1. Introduction. Let $(E, \|\cdot\|)$ be a separable Banach space. Consider a symmetric p-stable measure μ on E, $0 < p \leqslant 2$. Several authors have proved that for $z \in E$ the distribution function $F_z(t) = \mu\{x\colon \|x+z\| < t\}$ is absolutely continuous provided that $z \in \operatorname{supp}\mu$ (see [4], [5], [21]). Our aim is to examine whether the density of $F_z(t)$ is bounded on $(0, \infty)$.

For the Gaussian case, i.e. for $p = 2$, under various assumptions on E the answer to our question is affirmative. For example, when $E = l^q$, $q > 1$, with the standard norm, Davidov and Lifschitz [6] proved that the density is bounded. Recently their result has been generalized by Rhee and Talagrand [17] to uniformly convex Banach spaces with modulus of uniform convexity having order of power type. On the other hand, if this condition is slightly weaker, i.e., the modulus of uniform convexity has order $\varepsilon^{q(\varepsilon)}$, where $q(\varepsilon) \to \infty$ as $\varepsilon \to 0$, then the density may be unbounded. The corresponding example has been given on l^2 with a norm equivalent to the standard one (see [17]).

Properties of stable densities were investigated in a number of papers. For p-stable symmetric measures on l^2, $1 < p < 2$, the boundedness of the density was established by Pap [14] and Bentkus and Pap [2]. In the case of p-stable symmetric measures, $0 < p < 1$, on arbitrary separable Banach spaces an interesting formula for the density was proved by Lewandowski and Żak [10]. This formula was extended to the p-stable case on l^2 for $1 \leqslant p < 2$ by Żak [24]. As a consequence we have the boundedness of the density for $0 < p < 1$ established in [19]. One may wonder whether the density is always bounded. This is not so and appropriate examples were constructed in [18], [19].

In this paper we try to extend the above-mentioned result of Rhee and Talagrand to the case of p-stable measures, $0 < p < 2$. However, we have not managed to prove the boundedness on the whole $\boldsymbol{R}^+$ but we have shown it on every interval $(0, T)$, $T \in \boldsymbol{R}^+$. Assuming further conditions on the norm and p we are able to extend our conclusion to the whole $\boldsymbol{R}^+$. For example, for l^q, $q > 1$, the density of the standard norm is bounded on $\boldsymbol{R}^+$ whenever $z = 0$.

The main tool of our proofs is a representation of symmetric stable measures as mixtures of Gaussian ones. Next, we apply corresponding estimates for Gaussian measures obtained by Rhee and Talagrand. We also use some stopping time idea from [14], where it was shown that the density is bounded in l^2. To see that our assumptions cannot be weakened we construct an appropriate example in l^q, $1 < q < 2$, with norm equivalent to the standard one.

Another approach to our problem was presented by Lifschitz and Smorodina [11]. In addition to the uniform convexity conditions they assumed some strong differentiability properties of the norm on E. Then, using a Malliavin-type calculus they obtained not only the boundedness of the density but also the existence of further bounded derivatives of $F_z(t)$, the number of which depends on the number of derivatives of the norm. However, their result does not cover the case of l^q, $1 < q < 2$.

Pap investigated in [15] the dependence of the density of $F_z(t)$ on z. Under the assumptions of Lifschitz and Smorodina he obtained the following estimate of $F'_z(t)$:

$$\sup_{t>0} F'_z(t) \leqslant c(\mu)(1 + \|z\|^m), \qquad z \in E,$$

where m depends on the power of order of modulus of uniform convexity. For example, for l^2 the corresponding m equals 1 (see [2]).

We extend this result (with $m = 1$) to the class of spaces for which the power of order of modulus of uniform convexity equals 2.

The method of representing stable measures as mixtures of Gaussian ones enables us to conclude two additional regularity properties of $F_z(t)$ which are valid for Gaussian measures in general. That is, using the results of Linde [12] and Talagrand [22], respectively, we show that under our convexity assumptions the function $z \to F_z(t)$ is Gateaux differentiable at any $z \in \operatorname{supp} \mu$ and the function $t \to F'_z(t)$ is continuous on $\boldsymbol{R}^+$.

2. Preliminaries. Throughout the whole paper, $(E, \|\cdot\|)$ denotes a separable Banach space. Let μ be a symmetric p-stable measure on E, $0 < p < 2$. A nice feature of the measure μ is that it can be represented as a mixture of Gaussian ones (see [9], [13]):

PROPOSITION 1. *Let (g_i) be a sequence of i.i.d. standard normal random variables and let (α_i) be a sequence of i.i.d. random variables with the exponential distribution with expectation one. Let $\Gamma_n = \alpha_1 + \ldots + \alpha_n$. Assume that (g_i) and (α_i)*

are independent. Then there exist a sequence (V_i) of i.i.d. random vectors concentrated on the unit sphere of E independent of (α_i) and (g_i) and a positive constant $c(\mu)$ such that the series

$$Z = c(\mu) \sum_{i=1}^{\infty} \Gamma_i^{-1/p} g_i V_i \tag{1}$$

is convergent a.s. and has a law of μ.

For convenience we assume that the sequences (g_i) and (Γ_i, V_i) are defined on probability spaces (Ω_1, P_1) and (Ω_2, P_2), respectively, and E_i denotes the expectation with respect to P_i, $i = 1, 2$. It is clear that for almost all $\omega \in \Omega_2$ the series Z is convergent P_1-a.s. and the limit has a Gaussian law. We denote it by γ_ω.

Let H be a measurable subset of E. Then the following 0-1 law due to Sztencel [21] holds:

$$P_2\{\omega\colon \gamma_\omega(H) > 0\} = 1 \quad \text{whenever } \mu(H) > 0. \tag{2}$$

In the sequel we will use the following properties of Gaussian measures:

PROPOSITION 2. *Let γ be a Gaussian measure on E and let $z \in \operatorname{supp} \mu$.*

(i) *The mapping $x \to G_x(t) = \gamma\{\|x + \cdot\| < t\}$ is Gateaux differentiable at z* (Linde [12]).

(ii) *The density $G_z'(t)$ of $G_z(t)$ is continuous for $t > 0$* (Talagrand [22]).

Let $0 < \varepsilon < 2$ and let

$$\alpha(\varepsilon) = 1 - \sup\left\{\left\|\frac{x+y}{2}\right\|\colon \|x\|,\ \|y\| \leqslant 1;\ \|x-y\| \geqslant \varepsilon\right\}.$$

We say that $(E, \|\cdot\|)$ is *uniformly convex* if $\alpha(\varepsilon) > 0$ for $\varepsilon > 0$, and $\alpha(\varepsilon)$ is called a *modulus of uniform convexity* of E.

Since now we always assume, if it is not stated otherwise, that $(E, \|\cdot\|)$ is uniformly convex with modulus of uniform convexity satisfying the following condition:

There exist a positive constant D and $n \in N$ such that

$$\alpha(\varepsilon) \geqslant D\varepsilon^n. \tag{3}$$

For example, l^q with standard norm satisfies (3) with $n = 2$ for $1 < q \leqslant 2$ and $n = q \in N$ or $n = [q]+1$ if $q \notin N$ for $q \geqslant 2$.

3. The Gaussian case. In this section we follow the method of Rhee and Talagrand [17] to give an estimate of the distribution function of a Gaussian norm. This estimate is basic in the sequel, where properties of the distribution of a stable norm are considered.

First we introduce some notation. Let $x_1, \ldots, x_n \in E$ be a collection of linearly independent vectors with $\|x_i\| = 1$, $1 \leqslant i \leqslant n$. Let $E_1 = \operatorname{span}\{x_1, \ldots, x_n\}$.

By $\|\cdot\|_2$ we denote the Euclidean norm on E_1 such that $x_1, \ldots, x_n$ is an orthonormal system. Since E_1 is n-dimensional, there exists a positive β such that

$$\beta\|x\|_2 \leqslant \|x\|, \quad x \in E_1. \tag{4}$$

For $0 < s < t$ and $y \in E$ let

$$A_y(s, t) = \{x \in E_1 : s < \|x+y\| \leqslant t\}$$

and

$$N(y) = \inf\{\|x+y\|, x \in E_1\}.$$

Next, by λ we denote the Lebesgue measure on E_1. Let (g_i) be a sequence of i.i.d. standard normal random variables and let $0 < \lambda_n \leqslant \lambda_{n-1} \leqslant \ldots \leqslant \lambda_1$.

Our main purpose is to estimate the following probability:

$$P\{s < \|X+y\| \leqslant t\} = P\{X \in A_y(s, t)\},$$

where

$$X = \sum_{i=1}^{n} \lambda_i g_i x_i.$$

To begin with we recall two lemmas from [17]. For the proof of the second lemma see also [4].

LEMMA 3. *Let $N(y) < t$ and let $x_y \in E_1$ with $\|x_y+y\| = N(y)$. Then for $x \in A_y(s, t)$ we have*

$$\|x-x_y\| \leqslant Ct(1-t^{-1}N(y))^{1/n}, \tag{5}$$

where $C = D^{-1/n}$ and D appears in (3).

LEMMA 4. *Let $N(y) < s$ and let $z \in E_1$ with $\|z\|_2 = 1$. Let $B_z = \{u \in R^+ : x_y + uz \in A_y(s, t)\}$ and let $u_1 = \inf B_z$, $u_2 = \sup B_z$. Then*

$$u_2 - u_1 \leqslant u_1 \frac{t-s}{s-N(y)}. \tag{6}$$

Originally, Rhee and Talagrand assumed that $\|z\| = 1$, but for us it is more convenient to consider polar coordinates with respect to the Euclidean unit sphere in E_1, i.e. $S = \{\|z\|_2 = 1\}$.

LEMMA 5. *Let $N(y) < t$. Then*

$$\lambda\{A_y(s, t)\} \leqslant 2(C\beta^{-1})^n s_n t^{n-1}(t-s), \tag{7}$$

where s_n is equal to the measure of the surface of the Euclidean unit ball in R^n.

Proof. First assume that $2s-t \leqslant N(y) < t$. By (4) and (5) we have

$$A_y(s, t) \subset \{x \in E_1 : \|x-x_y\| < Ct(1-t^{-1}N(y))^{1/n}\}$$
$$\subset \{x \in E_1 : \|x-x_y\|_2 < C\beta^{-1}t(1-t^{-1}N(y))^{1/n}\}.$$

Therefore

$$\lambda\{A_y(s,t)\} \leqslant m_n(C\beta^{-1})^n t^{n-1}(t-N(y)) \leqslant 2m_n(C\beta^{-1})^n t^{n-1}(t-s),$$

where $m_n = \lambda\{x \in E_1 \colon \|x\|_2 \leqslant 1\} \leqslant s_n$.

To consider the case $N(y) < 2s-t$ we use the polar coordinates with centre x_y. Let $S = \{z \in E_1 \colon \|z\|_2 = 1\}$ and let $\bar{s}$ be the surface measure on S. Clearly, $\bar{s}(S) = s_n$. By inequalities (4)-(6) we obtain the following estimates:

$$\begin{aligned}\lambda\{A_y(s,t)\} &= \int_S \int_{u_1(z)}^{u_2(z)} u^{n-1}\,du\,\bar{s}(dz)\\ &\leqslant \int_S (u_2(z))^{n-1}(u_2(z)-u_1(z))\,\bar{s}(dz)\\ &\leqslant \int_S (u_2(z))^n (t-s)(s-N(y))^{-1}\,\bar{s}(dz)\\ &\leqslant (C\beta^{-1})^n t^{n-1}(t-N(y))(s-N(y))^{-1}(t-s)\,s_n\\ &\leqslant 2(C\beta^{-1})^n s_n t^{n-1}(t-s),\end{aligned}$$

which completes the proof.

Now we are able to estimate $P\{X \in A_y(s,t)\}$. Let

$$R_n(\beta) = 2\big(C(\beta\sqrt{2\pi})^{-1}\big)^n s_n.$$

COROLLARY 6. *If* $\|y\| < \frac{1}{2}s$, *then*

(8) $$P\{X \in A_y(s,t)\} \leqslant (\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} t^{n-1} \exp\{-s^2(8n)^{-1}\lambda_1^{-2}\}\, R_n(\beta)(t-s).$$

If $y \geqslant \frac{1}{2}s$, *then*

(9) $$P\{X \in A_y(s,t)\} \leqslant (\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} t^{n-1} R_n(\beta)(t-s).$$

Proof. Denote by B the linear operator $B\colon E_1 \to E_1$, $Bx_i = \lambda_i^{-1}x_i$. Then

(10) $$P\{X \in A_y(s,t)\} = \big(\lambda_1 \cdot \ldots \cdot \lambda_n (2\pi)^{n/2}\big)^{-1} \int_{A_y(s,t)} \exp\{-\tfrac{1}{2}\|Bx\|_2^2\}\,\lambda(dx).$$

Assume that $\|y\| < \frac{1}{2}s$. Since $\|x\| < n^{1/2}\|x\|_2$, $x \in E_1$, we get for $x \in A_y(s,t)$

$$\|Bx\|_2 \geqslant \|x\|(n^{1/2}\lambda_1)^{-1} \geqslant (\|x+y\| - \|y\|)(n^{1/2}\lambda_1)^{-1} \geqslant (2\lambda_1 n^{1/2})^{-1}s.$$

This, the estimate (7) and formula (10) imply (8). Inequality (9) follows directly from (7) and (10).

COROLLARY 7. *Let Y be an E-valued random vector independent of X. Then*

(11) $$\begin{aligned}&P\{s < \|X+Y\| \leqslant t\}\\ &\leqslant (\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} R_n(\beta)\, t^{n-1}\big\{\exp\{-s^2(8n)^{-1}\lambda_1^{-2}\} + P\{\|Y\| \geqslant \tfrac{1}{2}s\}\big\}(t-s).\end{aligned}$$

Proof. Let μ_Y denote the law of Y. By Fubini's theorem we have

$$P\{s < \|X+Y\| \leqslant t\} = \int_{\|y\| < (1/2)s} P\{X \in A_y(s, t)\} \mu_Y(dy) + \int_{\|y\| \geqslant (1/2)s} P\{X \in A_y(s, t)\} \mu_Y(dy).$$

Applying the estimates (8) and (9) to the first and second integrals, respectively, we obtain (11).

The next proposition is a generalization of the results of Paulauskas [16] and Davidov, Lifschitz [6] concerning the behaviour of the density of the distribution of the norm for a shifted Gaussian random vector in l^p, L^p $(p > 1)$.

PROPOSITION 8. *If Z is an E-valued symmetric Gaussian vector which is not concentrated on a finite-dimensional subspace, then for evey $z \in E$ the density of the distribution function $G_z(t)$ of $\|Z+z\|$ admits the following estimate:*

$$\sup_{t>0} G_z'(t) \leqslant C_1 + C_2 \|z\|^{n-1}, \tag{12}$$

where the constants C_1, C_2 depend only on the covariance of Z.

Proof. It is well known that we may assume

$$Z = \sum_{i=1}^{\infty} \lambda_i x_i g_i,$$

where (λ_i) is a decreasing positive sequence of reals and x_i are linearly independent and of norm one (see, e.g., [3]). Then writing

$$X = \sum_{i=1}^{n} \lambda_i x_i g_i \quad \text{and} \quad Y = \sum_{i>n} \lambda_i x_i g_i + z,$$

by Corollary 7 and Chebyshev's inequality we get

$$\begin{aligned} G_z'(t) &\leqslant (\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} R_n(\beta) t^{n-1} \{\exp\{-t^2(8n)^{-1}\lambda_1^{-2}\} + P\{\|Y\| \geqslant \tfrac{1}{2}t\}\} \\ &\leqslant C_3(\lambda_1, \ldots, \lambda_n, \beta, n) + 2^{n-1}(\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} R_n(\beta) E\|Y\|^{n-1} \\ &= C_3 + (\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} 2^{n-1} R_n(\beta) E\Big\|\sum_{i>n} \lambda_i x_i g_i + z\Big\|^{n-1} \\ &\leqslant C_3 + (\lambda_1 \cdot \ldots \cdot \lambda_n)^{-1} R_n(\beta) 4^{n-1}(E\|Z\|^{n-1} + \|z\|^{n-1}), \end{aligned}$$

where C_3 is some positive constant depending on $\lambda_1, \ldots, \lambda_n$, β and n. By Fernique's theorem, $E\|Z\|^{n-1} < \infty$ and this completes the proof.

Remark 9. Paulauskas [16] has shown that in l^n $(n = 2, 3, \ldots)$ the power $n-1$ in (12) is the best possible.

4. Regularity properties of the distribution of the norm for a stable vector. In this section we apply the estimates from the previous section to deduce some properties of the distribution of the norm for a stable vector. As a main tool we use the series representation described in Proposition 1. For the notation used in this section we refer to Section 2.

Let Z be a symmetric p-stable random vector. With the notation of Proposition 1 we can write

$$Z = c\sum_{i=1}^{\infty} \Gamma_i^{-1/p} g_i V_i.$$

For the sake of simplicity we take $c = 1$.

LEMMA 10. *Assume that the linear span of* $\operatorname{supp}\mathscr{L}(V_1)$ *is at least* n*-dimensional. Then there exist positive* ε, β *such that*

$$P_2\{\inf\{\|r_1 V_1+\ldots+r_n V_n\|,\ \bar{r}\in R^n,\ \|\bar{r}\|_2 = 1\} \geqslant \beta\} = \varepsilon > 0. \tag{13}$$

Proof. Since $V_1, \ldots, V_n$ are independent, we assume that V_1 is defined on a probability space (Ω_3, P_3) and $V_2, \ldots, V_n$ are defined on a probability space (Ω_4, P_4). Let

$$A_n = \{V_1, \ldots, V_n \text{ are linearly dependent}\}$$

and

$$A_{n-1} = \{V_2, \ldots, V_n \text{ are linearly dependent}\}.$$

It is clear that (13) is equivalent to $P_2(A_n) < 1$. By Fubini's theorem we get

$$P_2(A_n) = \int_{A_{n-1}} P_3(A_n)\,dP_4 + \int_{A_{n-1}^c} P_3(A_n)\,dP_4$$

$$\leqslant P_4(A_{n-1}) + \int_{A_{n-1}^c} P_3\{V_1\in \operatorname{span}\{V_2, \ldots, V_n\}\}\,dP_4.$$

Now, by assumption the last integrand is less than one, and therefore

$$P_2(A_n) < P_4(A_{n-1}) + P_4(A_{n-1}^c) = 1,$$

which completes the proof.

Remark 11. If $V_1, \ldots, V_n$ satisfy the condition under probability in (13), then on the linear space spanned by $V_1, \ldots, V_n$ we have $\beta\|\cdot\|_2 < \|\cdot\|$, where $\|\cdot\|_2$ is defined at the beginning of Section 3 for $x_i = V_i$.

Now, we are ready to formulate and prove the main result. Let $z\in E$ and let $F_z(t)$ denote the distribution function of $\|Z+z\|$.

THEOREM 12. *Under the assumption of Lemma* 10 *the density* $F_z'(t)$ *of* $F_z(t)$ *is bounded on every interval* $(0, T)$, $T > 0$. *In the following cases it is bounded on* R^+:

(a) *E is of stable type* p [1] *and* $z = 0$.

(b) $n = 2$ *and* $1 \leqslant p < 2$. *Moreover, in this case there exist constants* C_4 *and* C_5 *depending on* Z *such that*

$$\sup_{t>0} F'_z(t) \leqslant C_4 + C_5 \|z\|.$$

Remark 13. It was shown in [18] that under the assumption (a) the density $F'_0(t)$ is bounded on every interval (T, ∞), $T > 0$, without requiring uniform convexity properties of E. Moreover, for $0 < p < 1$ it is known that $F'_0(t)$ is always bounded on $\boldsymbol{R}^+$ (see [19]). For example, if $E = l^q$, $q \geqslant 2$, then E satisfies (a), but unfortunately we cannot drop the condition $z = 0$. In the case (b) we need $1 \leqslant p < 2$ since then the value of $1 - F_z(t)$ is small enough at infinity. It is worthwhile to notice that we do not assume any differentiability properties of the norm $\|\cdot\|$. So (b) covers the case of l^q-spaces, $1 < q < 2$, which does not follow from the papers of Lifschitz and Smorodina [11] and Pap [15].

Proof of Theorem 12. Let $m_k = (k-1)n$, $k \geqslant 1$, and let

$$\tau = \inf\{k:\ \inf\{\|V_{m_k+1} r_1 + \ldots + V_{m_{k+1}} r_n\|,\ \|\bar{r}\|_2 = 1\} \geqslant \beta\}.$$

It is clear that τ has a geometric distribution, and if ε, β are as in Lemma 10, then

$$\boldsymbol{P}_2\{\tau = k\} = (1-\varepsilon)^{k-1}\varepsilon.$$

Let

$$X_k = \sum_{i=m_k+1}^{m_{k+1}} \Gamma_i^{-1/p} g_i V_i, \qquad Y_k = Z + z - X_k,\ k \geqslant 1.$$

Since for fixed (Γ_i) and (V_i) the vectors X_k and Y_k are independent Gaussian, we can apply Corollary 7. First, let us notice that if $\tau = k$, then

$$\|V_{m_k+1} r_1 + \ldots + V_{m_{k+1}} r_n\| \geqslant \beta\sqrt{r_1^2 + \ldots + r_n^2}, \qquad r_1, \ldots, r_n \in \boldsymbol{R}.$$

Therefore, by Corollary 7, for $0 < s < t$ we have

$$(14)\qquad \boldsymbol{E}_1 \boldsymbol{1}_{\{s < \|X_k + Y_k\| \leqslant t\}} \boldsymbol{1}_{\{\tau = k\}}$$

$$\leqslant \boldsymbol{1}_{\{\tau=k\}} \Big(\prod_{i=m_k+1}^{m_{k+1}} \Gamma_i\Big)^{1/p} R_n(\beta)\, t^{n-1} \{\exp\{-s^2(8n)^{-1}\Gamma_{m_k+1}^{2/p}\}$$

$$+ \boldsymbol{P}_1\{\|Y_k\| \geqslant \tfrac{1}{2}s\}\}(t-s).$$

Since $\exp\{-s^2(8n)^{-1}\Gamma_{m_k+1}^{2/p}\} + \boldsymbol{P}_1\{\|Y_k\| \geqslant \frac{1}{2}s\} \leqslant 2$, in the general case we obtain, by Fubini's theorem,

[1] For the definition of "stable type" see [23].

$$(t-s)^{-1}\big(F_z(t)-F_z(s)\big) = (t-s)^{-1}\sum_{k=1}^{\infty} E_1\times E_2 \mathbf{1}_{\{s<\|X_k+Y_k\|\leqslant t\}}\mathbf{1}_{\{\tau=k\}}$$

$$\leqslant 2R_n(\beta)\,t^{n-1}\sum_{k=1}^{\infty} E_2\Big(\prod_{i=m_k+1}^{m_{k+1}}\Gamma_i\Big)^{1/p}E_2\mathbf{1}_{\{\tau=k\}}.$$

Since

$$E_2\Big(\prod_{i=m_k+1}^{m_{k+1}}\Gamma_i\Big)^{1/p} = O(k^{n/p}) \quad \text{as } k\to\infty,$$

there exists a positive constant $C_6 = C_6(n, \varepsilon, \beta, p)$ such that

$$(t-s)^{-1}\big(F_z(t)-F_z(s)\big)\leqslant C_6\sum_{k=1}^{\infty} k^{n/p}(1-\varepsilon)^{k-1}t^{n-1}<\infty.$$

This proves the boundedness on every interval $(0, T)$, $T>0$. Moreover, $F_z'(t) = O(t^{n-1})$ as $t\to 0$.

Since the case (a) was completely explained in Remark 13, we proceed to prove (b). Therefore, we assume that $n = 2$ and $1<p<2$. We do not consider $p = 1$ because this case is technically a bit more complicated but the idea is the same. By (14) we have

$$(t-s)^{-1}\big(F_z(t)-F_z(s)\big)$$

$$\leqslant tR_2(\beta)\sum_{k=1}^{\infty} E_2(\Gamma_{2k-1}\Gamma_{2k})^{1/p}\exp\{-s^2 16^{-1}\Gamma_{2k-1}^{2/p}\}\mathbf{1}_{\{\tau=k\}}$$

$$+tR_2(\beta)\sum_{k=1}^{\infty} E_2(\Gamma_{2k-1}\Gamma_{2k})^{1/p}\mathbf{1}_{\{\tau=k\}}P_1\{\|Y_k\|>\tfrac{1}{2}s\} = \mathrm{I}+\mathrm{II}.$$

We try to estimate I. Since

$$t\exp\{-s^2 16^{-1}\Gamma_{2k-1}^{2/p}\}\leqslant 4t\big(s(2e)^{1/2}\Gamma_{2k-1}^{1/p}\big)^{-1},$$

we get

$$\mathrm{I}\leqslant 2R_2(\beta)\,ts^{-1}\sum_{k=1}^{\infty} E_2\Gamma_{2k}^{1/p}P_2\{\tau=k\} \tag{15}$$

$$\leqslant C_7(\varepsilon, \beta, p)\sum_{k=1}^{\infty} k^{1/p}(1-\varepsilon)^{k-1}ts^{-1},$$

where C_7 is some positive constant.

Now, we deal with II. By Chebyshev's inequality and by Anderson's type inequality for Gaussian measures [3], we obtain

$$P_1\{\|Y_k\|\geqslant\tfrac{1}{2}s\}\leqslant 2s^{-1}E_1\|Y_k\|\leqslant 2s^{-1}(E_1\|Z\|+\|z\|).$$

Next, using Hölder's inequality with $1 < p' < p$ and $q' = (p'-1)^{-1}p'$, we can estimate

$$\text{(16)}\qquad \begin{aligned} \mathrm{II} &\leqslant 2R_2(\beta)\,t\sum_{k=1}^{\infty} E_2\Gamma_{2k}^{2/p}1_{\{\tau=k\}}P_1\{\|Y_k\| \geqslant \tfrac{1}{2}s\} \\ &= 2R_2(\beta)\,ts^{-1}\sum_{k=1}^{\infty} E_2\times E_1\Gamma_{2k}^{2/p}1_{\{\tau=k\}}(\|Z\|+\|z\|) \\ &\leqslant \Big(2R_2(\beta)\,ts^{-1}\sum_{k=1}^{\infty} E_2\Gamma_{2k}^{2/p}P_2\{\tau=k\}\Big)\|z\| \\ &\quad + 2R_2(\beta)\,ts^{-1}(E\|Z\|^{p'})^{1/p'}\sum_{k=1}^{\infty}\big(E_2(1_{\{\tau=k\}}\Gamma_{2k}^{2/p})^{q'}\big)^{1/q'} \\ &\leqslant \Big(ts^{-1}C_8(\varepsilon,\beta,p)\sum_{k=1}^{\infty} k^{2/p}(1-\varepsilon)^{k-1}\Big)\|z\| \\ &\quad + (ts^{-1})\,C_9(p,p',\beta,\varepsilon)\sum_{k=1}^{\infty}(1-\varepsilon)^{(k-1)/q'}k^{2/p}(E\|Z\|^{p'})^{1/p'}, \end{aligned}$$

where C_8 and C_9 are positive and finite and $E\|Z\|^{p'} < \infty$ by de Acosta's result [1]. Also, all the series appearing in (15) and (16) are convergent so, when $s \to t$, inequalities (15) and (16) give the desired result.

The rest of this section is devoted to studying other regularity properties of $F_z(t)$. Namely, we prove that $F_z(t)$, as a function of z, is Gateaux differentiable at all $z \in \operatorname{supp}\mu$, and moreover the density $F_z'(t)$ is continuous as a function of $t > 0$. Let us recall that μ is the law of Z. As before, we use the series representation of Z and appropriate results for Gaussian measures.

Let $G_z(\omega, t) = \gamma_\omega\{x\colon \|x+z\| < t\}$, $\omega \in \Omega$, $z \in E$, where γ_ω are Gaussian measures defined in Section 2. From the first part of the proof of Theorem 12 (see (14)), one can observe that the density $G_z'(\omega, t)$ can be estimated in the following way:

$$\text{(17)}\qquad G_z'(\omega, t) \leqslant L(\omega)\,t^{n-1}, \qquad \omega \in \Omega_2,$$

and $L(\omega)$ is an integrable random variable.

PROPOSITION 14. *Let $z \in \operatorname{supp}\mu$. Then:*

(i) *The mapping $E \ni x \to F_x(t)$ is Gateaux differentiable at z.*

(ii) *The density $F_z'(t)$ is continuous on R^+.*

Proof. Observe that by property (2) for almost all $\omega \in \Omega_2$ we have $\operatorname{supp}\mu = \operatorname{supp}\gamma_\omega$. Denote by $DG_z(\omega, t)$ the Gateaux derivative of $G_z(\omega, t)$ at z, which exists for almost all $\omega \in \Omega_2$ by Proposition 2 (i).

Let $h \in R$, $y \in E$ and put

$$I(h, y) = [t - \|hy\|,\ t + \|hy\|].$$

Then

$$(18)\qquad \left|h^{-1}\big(F_{z+hy}(t)-F_z(t)\big)\right| \leqslant \left|h^{-1}\big(F_z(t+\|hy\|)-F_z(t-\|hy\|)\big)\right| \leqslant 2 \sup_{\xi\in I(h,y)} F_z'(\xi)\,\|y\|.$$

Analogously, for almost all ω we have

$$\left|h^{-1}\big(G_{z+hy}(\omega,\, t)-G_z(\omega,\, t)\big)\right| \leqslant 2 \sup_{\xi\in I(h,y)} G_z'(\omega,\, \xi)\,\|y\| \leqslant 2L(\omega)(t+\|hy\|)^{n-1},$$

where the last step follows from (17). Since $L(\omega)$ is integrable, by the Lebesgue theorem we obtain

$$I_z(y) = \lim_{h\to 0} h^{-1}\big(F_{z+hy}(t)-F_z(t)\big)$$

$$= \int \lim_{h\to 0} h^{-1}\big(G_{z+hy}(\omega,\, t)-G_z(\omega,\, t)\big)\, dP_2(\omega) = \int \langle DG_z(\omega,\, t),\, y\rangle\, dP_2(\omega).$$

It is obvious that $I_z(y)$ is a linear functional, and inequality (18) yields its continuity. The proof of (i) is complete.

To prove (ii) let us notice that $F_z'(t) = \int G_z'(\omega,\, t)\, dP_2(\omega)$. Next, $G_z'(\omega,\, t)$ is continuous for almost all ω by Proposition 2 (ii). Using again (17) and the Lebesgue theorem we conclude (ii).

5. Example. It is a natural question whether weaker conditions imposed on the modulus of uniform convexity $\alpha(\cdot)$ would imply the conclusion of Theorem 12. To show that this is not possible we provide appropriate examples on l^p-spaces, $1 < p < 2$. We follow the ideas of Rhee and Talagrand [17], where a similar example was given for a Gaussian measure on the Hilbert space l^2.

PROPOSITION 15. *Let $1 < p < 2$. There exists $\varepsilon(p) > 0$ such that for any decreasing sequence $a_n \to 0$, $a_n < \varepsilon(p)$, one can construct a norm $q(\cdot)$ on l^p which is equivalent to the standard norm and the modulus of uniform convexity $\alpha_q(\cdot)$ of $(l^p,\, q(\cdot))$ satisfies $\alpha_q(\varepsilon) \geqslant \varepsilon^n$ for $n \geqslant 18$ and $a_n < \varepsilon \leqslant \varepsilon(p)$. Moreover, there exists a p-stable measure μ on l^p such that the density $F'(t)$ of $F(t) = \mu\{q < t\}$ is unbounded in any neighbourhood of the origin.*

One of the crucial points of the example in [17] is to find an auxiliary norm on l^2 satisfying certain conditions. We need to construct such a norm on l^p, $1 < p < 2$.

Let $(B,\, \|\cdot\|)$ be a uniformly convex Banach space and let

$$l^q(B) = \Big\{(x_i),\, x_i\in B\colon \sum_{i=1}^{\infty} \|x_i\|^q < \infty\Big\}, \qquad q > 1.$$

Let us consider the norm

$$|||(x_i)||| = \Big(\sum_{i=1}^{\infty} \|x_i\|^q\Big)^{1/q} \quad \text{on } l^q(B).$$

Then, by the result of Day [7] the Banach space $(l^q(B), |||\cdot|||)$ is uniformly convex. Moreover, from his proof it is possible to find out how the modulus of uniform convexity behaves. For example, if $B = l^p$, $1 < p < 2$, then there exists $M = M(p) \geqslant 1$ such that

$$\alpha_{|||\cdot|||}(\varepsilon) \geqslant (\varepsilon M^{-1})^{(q+2)^2}. \tag{19}$$

Now, we are ready to carry out our construction.

LEMMA 16. *Let $0 < \eta \leqslant 1$, $1 < p < 2$. There exists $\varepsilon(p) > 0$ such that for any sequence $a_n \searrow 0$, $a_n \leqslant \varepsilon(p)$, we can find a norm $\bar{q}$ on l^p and a sequence $b_n > 0$ with the following properties:*

(a) *For $x \in l^p$, $\|x\|_p \leqslant \bar{q}(x) \leqslant (1+\eta)\|x\|_p$, where $\|\cdot\|_p$ denotes the standard norm on l^p.*

(b) *If $\alpha_{\bar{q}}(\cdot)$ is the modulus of uniform convexity of $(l^p, \bar{q})$, then*

$$\alpha_{\bar{q}}(\varepsilon) \geqslant \varepsilon^n \quad \textit{for } a_n \leqslant \varepsilon \leqslant \varepsilon(p),\ n \geqslant 18.$$

(c) *For $x = (x_1, \ldots, x_n, 0, \ldots)$ and $y = (0, \ldots, y_{n+1}, 0, \ldots)$ with $\bar{q}(y) = 1$ and $\bar{q}(x) \leqslant b_n$ we have*

$$1 \leqslant \bar{q}(x+y) \leqslant 1 + \|x\|_p^n.$$

Proof. We need the following estimate of the modulus of uniform convexity of $(l^p, \|\cdot\|_p)$:

$$\alpha(\varepsilon) > C(p)\varepsilon^2, \quad 0 < \varepsilon < 2. \tag{20}$$

Let $\varepsilon(p) = \max\{C(p)/8, (4M)^{-1}\}$, where $M = M(p)$ appeared in (19). Suppose that $a_k \leqslant \varepsilon(p)$. Define n_k as the largest positive integer so that $(n_k+2)^2 \leqslant \frac{1}{2}k$, $k \geqslant 18$. Now, if $\varepsilon \leqslant \varepsilon(p) \leqslant 1$, then

$$\varepsilon^{k-(n_k+2)^2}(4M)^{(n_k+2)^2} \leqslant (4\varepsilon M)^{(1/2)k} \leqslant 1.$$

Therefore, for $\varepsilon \leqslant \varepsilon(p)$ and $1 \leqslant n \leqslant n_k$ we have

$$1 - (\varepsilon/(4M))^{(n+2)^2} \leqslant 1 - \varepsilon^k. \tag{21}$$

Next, let (r_k) be a sequence of positive numbers, $r_k < 1$, such that

$$C(p)(\varepsilon/2)^2 > 1 - r_k(1-\varepsilon^k), \quad a_k < \varepsilon \leqslant a_{k-1} \leqslant \varepsilon(p),\ k \geqslant 18. \tag{22}$$

Obviously, r_k can be chosen increasing to one and $(1+\eta)^{-1} r_k^{-1} \leqslant 1$. Next, let $\beta_n = r_k^{-n}$ for $n_k < n \leqslant n_{k+1}$ and let

$$q_n(x) = \Big(\tfrac{1}{2}\big(\sum_{i \neq n} |x_i|^p\big)^{n/p} + \beta_n |x_n|^n\Big)^{1/n},$$

and

$$\bar{q}(x) = \sup\{\|x\|_p,\ q_n(x),\ n \geqslant 2\}.$$

Now, let us observe that

$$\tag{23} \tfrac{1}{2}\|x\|_p \leqslant q_n(x) \leqslant \beta_n^{1/n}\|x\|_p \leqslant (1+\eta)\|x\|_p \leqslant 2\|x\|_p$$

and

$$\tag{24} \|x\|_p \leqslant \bar{q}(x) \leqslant (1+\eta)\|x\|_p \leqslant 2\|x\|_p.$$

Assume that $a_k < \varepsilon \leqslant a_{k-1}$, $k \geqslant 18$. Let $x, y \in l^p$ with $\bar{q}(x) \leqslant 1$, $\bar{q}(y) \leqslant 1$ and $\bar{q}(x-y) \geqslant \varepsilon$. Then, by (24), $\|x\|_p \leqslant 1$, $\|y\|_p \leqslant 1$, and $\|x-y\|_p \geqslant \varepsilon/2$.

First suppose that there exists $n \geqslant 2$ such that

$$\|x+y\|_p \geqslant 2\beta_n^{-1/n}(1-\varepsilon^k).$$

Therefore, by (20),

$$C(\varepsilon/2)^2 \leqslant \alpha(\|x-y\|_p) \leqslant 1 - \|2^{-1}(x+y)\|_p < 1-\beta_n^{-1/n}(1-\varepsilon^k).$$

Because of (22) and the definition of β_n we have $n \leqslant n_k$.

Next, let us notice that (l^p, q_n) is isometric to a subspace of $(l^n(l^p), |||\cdot|||)$. Hence by (19) the modulus of uniform convexity $\alpha_{q_n}(\cdot)$ of (l^p, q_n) satisfies

$$\tag{25} \alpha_{q_n}(\varepsilon) \geqslant (\varepsilon/M)^{(n+2)^2}.$$

Now, by (23) we have $q_n(x) \leqslant 1$, $q_n(y) \leqslant 1$ and $q_n(x-y) \geqslant \varepsilon/4$. Therefore (21) and (25) yield

$$q_n\big((x+y)/2\big) \leqslant 1-\big(\varepsilon/(4M)\big)^{(n+2)^2} \leqslant 1-\varepsilon^k.$$

If n is such that $\|x+y\|_p \leqslant 2\beta_n^{-1/n}(1-\varepsilon^k)$, then once again by (23) we get

$$q_n\big((x+y)/2\big) \leqslant \beta_n^{1/n}\|(x+y)/2\|_p \leqslant 1-\varepsilon^k.$$

Finally,

$$\|(x+y)/2\|_p \leqslant 1-C(p)(\varepsilon/2)^2 \leqslant 1-\varepsilon^k.$$

The proof of (a) and (b) is completed. The statement (c) can be proved in the same way as the Proposition in [17].

Another very important step in the example of [17] consists in finding Gaussian measures on finite-dimensional subspaces of l^2 which are very concentrated around the unit sphere, with degree of concentration as small as required.

It turns out that a similar procedure can be carried out for p-stable measures on l^p, $1 \leqslant p < 2$. That is, by the weak law of large numbers (see, e.g., [8], p. 236) we have the following fact:

LEMMA 17. *Let* $1 \leqslant p < 2$ *and let* (θ_i) *be a sequence of i.i.d. standard* p*-stable random variables, i.e. with the characteristic function* $\exp - |t|^p$. *For any* $\varepsilon > 0$ *and* $\beta > 0$ *there exist* $n \in N$ *and* $A > 0$ *such that*

$$P\{1-\beta \leqslant A^{-1} \|\sum_{i=1}^{n} \theta_i e_i\|_p \leqslant 1+\beta\} \geqslant 1-\varepsilon,$$

where (e_i) *is the standard basis in* l^p.

The above lemma was frequently used to construct examples exhibiting pathological properties of p-stable measures for $1 \leqslant p < 2$ (see, e.g., [19], [20]).

Proof of Proposition 15. With the help of Lemmas 16 and 17 we are able to employ the reasoning of Rhee and Talagrand with minor changes, so we do not repeat it here.

Added in proof. After completing this paper it turned out that the proof of Proposition 2 (ii) in [22] (Théorème 4) had an error. A correct proof is due to T. Byczkowski: *On the density of* log *concave seminorms on vector spaces*, to appear in Studia Math. 99.2 (1991).

REFERENCES

[1] A. de Acosta, *Stable measures and seminorms*, Ann. Probab. 3 (1975), pp. 865–875.

[2] V. Bentkus and G. Pap, *On the distribution of the norm of stable random vectors taking values in Hilbert space*, Lithuanian Math. J. 26 (2) (1986), pp. 211–220.

[3] C. Borell, *Gaussian measures on locally convex spaces*, Math. Scand. 38 (1976), pp. 265–284.

[4] T. Byczkowski and K. Samotij, *Absolute continuity of stable seminorms*, Ann. Probab. 14 (1986), pp. 299–312.

[5] V. S. Cirelson, *The density of the distribution of the maximum of a Gaussian process*, Theory Probab. Appl. 20 (1975), pp. 847–855.

[6] G. A. Davydov and M. A. Lifschitz, *The fibering method in some probabilistic problems* (in Russian), Itogi Nauki i Tekhniki, Teor. Veroyatnost. Mat. Statist. Teor. Kibernet. 22 (1984), pp. 61–157.

[7] M. M. Day, *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. 47 (1941), pp. 504–507.

[8] W. Feller, *An Introduction to Probability Theory and Its Applications* 2, 2nd ed., J. Wiley, New York 1971.

[9] R. Le Page, M. Woodruffe and J. Zinn, *Convergence to a stable distribution via order statistics*, Ann. Probab. 9 (1981), pp. 624–632.

[10] M. Lewandowski and T. Żak, *On the density of the distribution of* p*-stable seminorms*, $0 < p < 1$, Proc. Amer. Math. Soc. 100 (1987), pp. 345–351.

[11] M. A. Lifschitz and N. V. Smorodina, *On the distribution of stable vectors*, Theory Probab. Appl. 34 (1989), pp. 302–313.

[12] W. Linde, *Gaussian measure of translated balls in a Banach space*, ibidem 34 (1989), pp. 349–359.

[13] M. B. Marcus and G. Pisier, *Characterizations of almost surely continuous* p*-stable random Fourier series and strongly stationary processes*, Acta Math. 152 (1984), pp. 245–301.

[14] G. Pap, *Boundedness of the density of the norm of stable random vectors in a Hilbert space*, Theory Probab. Math. Statist. (Kiev) 36 (1987), pp. 102–105.

[15] – *Density of the norm of Gaussian vectors in Banach spaces*, *Probability Theory on Vector Spaces IV*, Lecture Notes in Math. 1391 (1988), pp. 308–320.

[16] V. Paulauskas, *On the density of the norm of Gaussian vectors in Banach spaces*, ibidem 990 (1982), pp. 179–197.

[17] W. S. Rhee and M. Talagrand, *Uniform convexity and the distribution of the norm for a Gaussian measure*, Probab. Theory Related Fields 71 (1986), pp. 59–68.

[18] M. Ryznar, *Density of stable seminorms*, Bull. Polish Acad. Sci. Math. 33 (1985), pp. 431–440.

[19] – *Geometrical properties of Banach spaces and the distribution of the norm for a stable measure*, Studia Math. 91 (1988), pp. 61–71.

[20] R. Sztencel, *On the lower tail of stable seminorms*, Bull. Polish Acad. Sci. Math. 32 (1984), pp. 715–719.

[21] – *Absolute continuity of the lower tail of stable seminorms*, ibidem 34 (1986), pp. 231–234.

[22] M. Talagrand, *Sur l'intégrabilité des vecteurs gaussiens*, Z. Wahrsch. Verw. Gebiete 68 (1984), pp. 1–8.

[23] W. A. Woyczyński, *Geometry and martingales in Banach spaces*, Part II: *Independent increments*, in: J. Kuelbs (Ed.), *Advances in Probability, IV*, Dekker, New York 1978.

[24] T. Żak, *A formula for the density of the norm of stable random vectors in Hilbert spaces*, Probab. Math. Statist. 10 (1990), pp. 257–270.

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