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MINIMAX SEQUENTIAL ESTIMATION BASED ON RANDOM FIELDS

BY

ROMAN RÓŻAŃSKI (WROCŁAW)

Abstract. The problem of minimax sequential estimation for random fields is considered. A theorem which is a generalization of the results of Dvoretzky, Kiefer and Wolfowitz [2], Rhiel [7], Wilczyński [15], Trybuła [14], and Różański [9]–[12] is proved. This theorem is applicable to the case of a Poisson random field when the cost of observation depends on a state of the process.

1. Introduction. Dvoretzky, Kiefer and Wolfowitz were the first who considered the problem of minimax sequential decisions based on continuous stochastic processes. They proved that for the Poisson and Wiener processes a fixed-time sequential plan is a minimax sequential plan in the class of all sequential plans. These results were generalized by Magiera [6] who considered the exponential class of processes. Analogous results were obtained by Franz [4] for the multidimensional exponential class of processes and by Różański [8] for the Ornstein–Uhlenbeck process. In the paper of Trybuła [14], an inverse minimax sequential plan for estimation of the inverse of an intensity parameter of a Poisson homogeneous process was constructed, when the cost of observation depends on a state of the process. Similar and even more general results concerned with minimax sequential estimation for multinomial and gamma processes were obtained by Wilczyński [15]. He considered the case where the cost of observation may simultaneously depend on the time of observation and the state of a process. Other aspects and methods were developed by Rhiel [7]. Recently Różański [9]–[12] has obtained some results concerned with minimax sequential estimation for random fields. He proved that if the loss incurred by a statistician is due to the error of estimation and to the cost of observation of a random field on some compact set, then a simple plan is a minimax sequential plan. As an example one can consider Poisson, Wiener or Ornstein–Uhlenbeck random fields.

In this paper we consider the case where the cost of observation depends simultaneously on a compact set of observation of a random field and on a state of this random field. A general theorem on minimax estimation is proved that is applicable to minimax sequential estimation of the inverse of intensity of a Poisson random field.

2. Preliminaries and notation. Let V be a set of realizations of a random field X_z, $z=(s,t)\in R^2$. By μ_θ we denote a measure corresponding to this random field and defined on (V, F), where F is a σ-algebra of subsets of V generated by cylindrical sets and $\theta\in\Theta\subseteq R$ is a parameter. Let $\mathscr{K}$ be a family of compact subsets K of R^2. Denote by $\delta(K)$ the diameter of K. In the paper we shall assume that the family $\mathscr{K}$ satisfies the following

CONDITION 1. *There exists a countable family of compact sets $P_i(n)$, $i, n\in N$, such that*

$$\sup_i\{\delta(P_i(n))\}\to 0 \quad \text{as } n\to\infty$$

and for each $K\in\mathscr{K}$ there exists a finite covering $C_n\in\mathscr{K}$ of K by some sets among $P_i(n)$, $i\in N$, for which

$$C_{n+1}\subseteq C_n, \quad \bigcap_{n=1}^{\infty} C_n = K.$$

Remark 1. Let $R_+^2=[0,\infty)\times[0,\infty)$. In this set we consider the partial order

$$z_1=(s_1,t_1)\leqslant z_2=(s_2,t_2) \quad \text{iff} \quad s_1\leqslant t_1 \text{ and } s_2\leqslant t_2.$$

Let $z_0\in R_+^2$; then by R_{z_0} we denote the rectangle $\{z\in R_+^2:\ z\leqslant z_0\}$. Now let $\mathscr{K}=\{R_z:\ z\in R_+^2\}$. If we take

$$P_i(n)=[s_{k_i}^n, s_{k_i+1}^n]\times[t_{l_i}^n, t_{l_i+1}^n], \quad (s_{k_i}^n, t_{l_i}^n)=(k_i/2^n, l_i/2^n),$$
$$k_i, l_i\in\{0,1,2,\ldots\},\ i\in N,$$

then we can see that Condition 1 is satisfied.

By F_K we denote a σ-algebra of subsets of V generated by cylindrical sets $\{v:\ (v(z_1), v(z_2),\ldots, v(z_n))\in B_{R^n}\}$, $B\in B_{R^n}$, $z_i\in K$, $i\in\{1,2,\ldots,n\}$, and by μ_θ^K the restriction of μ_θ to the σ-algebra F_K.

DEFINITION 1 (see [10], [11]). A *Markov stopping set* τ is a mapping $\tau:\ V\to\mathscr{K}$ such that, for every $K\in\mathscr{K}$, $\{v:\ \tau(v)\subseteq K\}\in F_K$.

To each Markov stopping set τ there corresponds a σ-algebra F_τ of sets $U\in F$ such that, for every $K\in\mathscr{K}$, $\{v:\ \tau(v)\subseteq K\}\cap U\in F_K$. Denote by μ_θ^τ the measure μ_θ restricted to the σ-algebra F_τ. If $\mathscr{K}$ satisfies Condition 1 and, for each $K\in\mathscr{K}$, μ_θ^K is absolutely continuous with respect to $\mu_{\theta_0}^K$ with the density function

$$\frac{d\mu_\theta^K}{d\mu_{\theta_0}^K}(v)=g_{\theta_0}(K, v, \theta),$$

where g_{θ_0} is such that for each sequence $K_n\searrow K$, $K_n, K\in\mathscr{K}$,

$$g_{\theta_0}(K_n, v, \theta)\to g_{\theta_0}(K, v, \theta)\ \mu_{\theta_0}\text{-almost surely,}$$

then μ_θ^τ is absolutely continuous with respect to $\mu_{\theta_0}^\tau$ and

$$\frac{d\mu_\theta^\tau}{d\mu_{\theta_0}^\tau}(v) = g_{\theta_0}(\tau(v), v, \theta)$$

(see [10], [12]).

In the sequel we shall assume that the density function $d\mu_\theta^K/d\mu_{\theta_0}^K$ is given by the formula

$$\frac{d\mu_\theta^K}{d\mu_{\theta_0}^K}(v) = g_{\theta_0}(Q(K), S(K, v), \theta),$$

where $S: \mathscr{K} \times V \to R$ is such that, for every $K \in \mathscr{K}$, $S(K, \cdot)$ is F_K-measurable and $S(K_n, v) \to S(K, v)$ μ_{θ_0}-almost surely whenever $K_n \searrow K$, $K, K_n \in \mathscr{K}$, Q is a set function from $\mathscr{K}$ into R such that for each sequence $K_n \searrow K$, $K_n, K \in \mathscr{K}$, $Q(K_n) \to Q(K)$.

Thus we can conclude that the statistic $(Q(\tau), S(\tau))$ is sufficient for the parameter θ, and therefore we can restrict ourselves to the estimators of the form $f(Q(\tau), S(\tau))$. By a *sequential plan* we mean any pair $\delta = (\tau, f)$, where τ is a Markov stopping set and f is an estimator of the parameter θ. Let $L(f, \theta)$ denote the loss incurred by a statistician if θ is a true value of the parameter and f is an estimator he uses.

Let $c(h(K, u))$ be the cost function representing the cost of observation of the random field X on the set K. We assume that the cost function depends upon the set K and the value u of the statistic $(Q(K), S(K))$ through a function $h: \mathscr{K} \times R \to R$. Then the risk function is given by

$$R(\delta, \theta) = \mathrm{E}_\theta\{L(f, \theta) + c(h(\tau, (Q(\tau), S(\tau))))\}.$$

In the sequel we assume that $R(\delta, \theta) < \infty$.

A sequential plan $\delta_0 = (\tau_0, f_0)$ is said to be *minimax* if

$$\sup_\theta R(\delta_0, \theta) = \inf_\delta \sup_\theta R(\delta, \theta).$$

Let $\Phi(\theta)$ be a prior distribution on (Θ, B_Θ). If $R(\delta, \theta)$ is a B_Θ-measurable function, then for each sequential plan δ the Bayes risk with respect to the prior distribution Φ is given by

$$r(\delta, \Phi) = \int_\Theta R(\delta, \theta) d\Phi(\theta).$$

We say that a sequential plan $\delta' = (\tau', f')$ is a *Bayes plan* with respect to Φ if

$$r(\delta', \Phi) = \inf_\delta r(\delta, \Phi).$$

Let us define a probability measure Π_Φ on $(V \times \Theta, F \times B_\Theta)$ by the following formula:

$$\Pi_\Phi(U \times B) = \int_B \mu_\theta(U) d\Phi(\theta) \quad \text{for each } B \in B_\Theta,\ U \in F.$$

Note that

$$\Pi_{\Phi}(V\times B) = \int_{B} \mu_\theta(V)\,d\Phi(\theta) = \Phi(B)$$

and

$$\Pi_{\Phi}(U\times\Theta) = \int_{\Theta} \mu_\theta(U)\,d\Phi(\theta) \stackrel{\mathrm{df}}{=} \mu_{\Phi}(U), \qquad U\in F.$$

It is known (see [1]) that for each Markov stopping set τ there exists a transition probability measure $\Psi_{\tau,\Phi}(v,\cdot)$ such that

$$\Pi_{\Phi}(U\times B) = \int_{U} \Psi_{\tau,\Phi}(v, B)\,d\mu_{\Phi}(v), \qquad U\in F_\tau,\ B\in B_\Theta.$$

Another expression for $\Psi_{\tau,\Phi}(v,\cdot)$ will be useful. Namely,

$$\Psi_{\tau,\Phi}(v, B) = \big(\Pi_{\Phi}(V\times B)\ |\ F_\tau\times\{\emptyset, \Theta\}\big)\ \mu_{\Phi}\text{-almost surely}.$$

We define the measure $\Psi_{\tau,\Phi}(v,\cdot)$ as a posterior probability of θ having observed the realization v on the set τ. As in the stochastic process case (see [7]) the mapping $Y_{K,\Phi}\colon V\to R_+$, $K\in\mathscr{K}$, for which

$$Y_{K,\Phi}(v) = \inf_f\Big[\int_{\Theta} L(f,\theta)\Psi_{K,\Phi}(v, d\theta)\Big] + c\Big(h\big(K, (Q(K), S(K, v))\big)\Big),$$

is called a *stochastic decision process.*

3. Sequential minimax decisions based on random fields.

LEMMA 1. *If there exists an estimator $f'\big(Q(K), S(K)\big)$ such that*

$$Y_{K,\Phi}(v) = \int_{\Theta} L\big(f'(Q(K), S(K, v)), \theta\big)\Psi_{K,\Phi}(v, d\theta)$$
$$+c\big(h(K, Q(K), S(K, v))\big)\ \mu_{\Phi}\textit{-almost surely for } K\in\mathscr{K},$$

then for each Markov stopping set τ

$$\mathrm{E}_{\mu_{\Phi}}(Y_{\tau,\Phi}) = r(\delta', \Phi) = \inf_{\delta=(\tau,f)} r(\delta, \Phi),$$

where $\delta' = \big(\tau, f'(Q(\tau), S(\tau))\big)$.

Proof. A proof of this lemma goes along the same lines as that in [9]. Therefore we omit it.

Now we prove a theorem which is closely related to the previous results of Dvoretzky [2], Rhiel [7], Trybuła [13], Wilczyński [15], Różański [9], [11].

THEOREM 1. *Assume that*

$$Y_{K,\Phi}(v) = H_{\Phi}\Big(h\big(K, (Q(K), S(K, v))\big)\Big) + c\big(h(K, Q(K), S(K, v))\big),$$

where the function $H_{\Phi}(\cdot)+c(\cdot)$ attains its minimum at the point y_{Φ} belonging to the range W of the function $w\colon V\times\mathscr{K}\to R$ such that

$$w(v\cdot K) = h\big(K, (Q(K), S(K, v))\big) \quad \textit{for each } v\in V,\ K\in\mathscr{K}.$$

Let us suppose that there exists a sequence of prior distributions Φ_n such that

$$\lim_{n\to\infty} (H_{\Phi_n}(y)+c(y)) = H(y)+c(y) \quad \textit{for each } y\in W$$

and there exists $y_0 \in W$ such that

$$(1)\qquad H(y_0)+c(y_0) = \min_{y\in W}[H(y)+c(y)] = \min_{y\in W} \lim_{n\to\infty} [H_{\Phi_n}(y)+c(y)]$$
$$= \lim_{n\to\infty} \min_{y\in W}[H_{\Phi_n}(y)+c(y)].$$

We also assume that for every $y\in W$ there exists a Markov stopping set τ_y defined by

$$\tau_y(v) = \min\{K\in\mathscr{K}:\ h(K, (Q(K), S(K, v))) = y\}, \qquad v\in V.$$

If the sequential plan $\delta_0 = (\tau_{y_0}, f(Q(\tau_{y_0}), S(\tau_{y_0})))$ fulfils the inequality

$$\sup_\theta R(\delta_0, \theta) \leqslant H(y_0)+c(y_0),$$

then this plan is a minimax sequential plan in the class of all sequential plans $(\tau, f(Q(\tau), S(\tau)))$, where τ is a Markov stopping set with respect to $(\{F_K\}_{K\in\mathscr{K}}, \mathscr{K})$.

Proof. It is easy to see that for each sequential plan $(\tau, f(Q(\tau), S(\tau)))$ we have

$$r(\delta, \Phi) \geqslant H(y_\Phi)+c(y_\Phi).$$

Thus by Lemma 1 we conclude that $(\tau_{y_\Phi}, f'(Q(\tau_{y_\Phi}), S(\tau_{y_\Phi})))$ is a Bayes sequential plan. By the assumptions we have

$$\sup_\theta R(\delta_0, \theta) \leqslant H(y_0)+c(y_0) = \min_{y\in W}[H(y)+c(y)]$$
$$= \min_{y\in W} \lim_{n\to\infty} [H_{\Phi_n}(y)+c(y)] = \lim_{n\to\infty} \min_{y\in W}[H_{\Phi_n}(y)+c(y)] = r(\delta_{y_{\Phi_n}}, \Phi_n),$$

where

$$\delta_{y_{\Phi_n}} = (\tau_{y_{\Phi_n}}, f'(Q(\tau_{y_{\Phi_n}}), S(\tau_{y_{\Phi_n}}))).$$

Thus by the Ferguson theorem [3] we infer that δ_0 is a minimax sequential plan in the class of all sequential plans, where τ is a Markov stopping set with respect to $(\{F_K\}_{K\in\mathscr{K}}, \mathscr{K})$.

4. Application to the estimation of the intensity parameter of the Poisson random field.

DEFINITION 2. By $B^b_{R^2}$ we denote the family of all bounded Borel subsets of R^2. Let us consider the family $\{N(B),\ B\in B^b_{R^2}\}$ of random variables for which the following conditions are satisfied:

1° for arbitrary, disjoint, bounded Borel subsets $B_1, B_2, \ldots, B_n$ of R^2 the random variables $N(B_1), N(B_2), \ldots, N(B_n)$ are independent;

2° $P(N(B_i) = k) = (\theta|B_i|)^k \exp(-\theta|B_i|)/k!$, where $|B_i|$ denotes the area of B_i.

Then the random field $N_z = N(R_z)$, $z \in R_+^2$, is called a *Poisson random field.*

Let $\mathscr{K} = \{R_z\}_{z \in R_+^2}$. We want to estimate $1/\theta$ with

$$L(f, \theta) = \frac{(f-1/\theta)^2}{1/\theta^2} = \theta^2(f-1/\theta)^2$$

and

$$c(h(K, Q(K), S(K, v))) = c(N_z),$$

where $c(\cdot)$ is a nonnegative continuous function such that $c(\infty) = \infty$. By [5] the measure $\mu_\theta^{R_z}$ corresponding to the random field N_u, $u \in R_z$, is absolutely continuous with respect to the measure $\mu_1^{R_z}$ corresponding to the Poisson random field with $\theta = 1$ and

$$d\mu_\theta^{R_z}/d\mu_1^{R_z} = \theta^{N_z} \exp(-\theta|R_z|).$$

Let us choose a sequence of prior distributions of the parameter θ given by the density functions

$$\phi_n(\theta) = \begin{cases} \dfrac{\gamma^{1/n}}{\Gamma(1/n)} \theta^{1/n-1} \exp(-\gamma\theta), & \theta > 0, \\ 0, & \theta \leqslant 0. \end{cases}$$

The density of posterior distributions of the parameter having observed the realization v on the set R_z takes the form

$$\psi_{n,R_z} = \frac{(d\mu_\theta^{R_z}/d\mu_1^{R_z})\phi_n(\theta)}{\int_0^\infty (d\mu_\theta^{R_z}/d\mu_1^{R_z})\phi_n(\theta)d\theta}$$

$$= \frac{(|R_z|+\gamma)^{N_z+1/n}}{\Gamma(N_z+1/n)} \theta^{N_z+1/n-1} \exp(-\theta(|R_z|+\gamma)).$$

Thus the estimator f' (see Lemma 1) is given by

$$f' = \frac{|R_z|+\gamma}{N_z+1/n+1} \quad \text{and} \quad Y_{R_z,\Phi_n}(v) = \frac{1}{N_z+1/n+1} + c(N_z).$$

We see that

$$\lim_{n\to\infty}\left(H_{\Phi_n}(y)+c(y)\right)=\frac{1}{y+1}+c(y)=H(y)+c(y).$$

It is easy to see that (1) is satisfied. Define an inverse Markov stopping time by

$$\nu_{y_0}=\inf\{t\colon\ N(R_{(t,t)})=y_0\}.$$

Thus $\tau_{y_0}=R_{\nu_{y_0}}$ is a Markov stopping set with respect to $(\{F_K\}_{K\in\mathscr{K}},\mathscr{K})$. Let

$$\delta_0=\left(\tau_{y_0},f\left(Q(\tau_{y_0}),\,S(\tau_{y_0})\right)\right)=\left(\tau_{y_0},\,\frac{|\tau_{y_0}|}{y_0+1}\right).$$

We have (see [13])

$$R(\delta_0,\theta)=\mathrm{E}_\theta\left[\frac{\theta|\tau_{y_0}|}{y_0+1}-1\right]^2=\frac{1}{y_0+1}.$$

Finally, using Theorem 1 we have proved the following

THEOREM 2. *The sequential plan*

$$\delta_0=\left(\tau_{y_0},\,\frac{|\tau_{y_0}|}{y_0+1}\right)$$

is a minimax sequential plan for the estimation of the parameter $1/\theta$ in the class of all Markov stopping sets with respect to $(\{F_K\}_{K\in\mathscr{K}},\ \mathscr{K}=\{R_z\}_{z\in R_+^2})$ with the loss $L(f,\theta)=(f-\theta^{-1})^2/\theta^{-2}$ in the Poisson random field case, when the cost function $c\big(h(K,Q(K),S(K))\big)$ is $c(N_z)$.

Remark 2. The case $c\big(h(K,Q(K),S(K))\big)=c(|K|)$ was considered in [9] and [11].

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Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland

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