PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 12, Fasc. 1 (1991), pp. 57–65

ON THE DISTRIBUTION OF A USEFUL MAXIMAL INVARIANT

BY

ADAM ĆMIEL AND ZBIGNIEW SZKUTNIK (KRAKÓW)

Abstract. The Wijsman theorem and a characterization of a quotient measure by invariance, due to Andersson, are used to describe exact distributions of some maximal invariants especially useful in the context of testing multivariate normality. Some possible applications are indicated.

1. Introduction. Let X be a (p, n)-matrix. In some statistical testing problems (cf. [8]) it is of interest to study the group G^* of transformations acting on $\boldsymbol{R}^{pn}$ according to $gX = CX + b\boldsymbol{1}_n^{\mathrm{T}}$, $C \in \mathrm{UT}(p)$ being the group of upper triangular (p, p)-matrices with positive diagonal, $b \in \boldsymbol{R}^p$, $\boldsymbol{1}_n^{\mathrm{T}} = (1, \ldots, 1) \in \boldsymbol{R}^n$. Since, under mild restrictions, each invariant test has a factorization through the so-called maximal invariant (see [6]), the construction of maximal invariants and the derivation of their distributions are important in the context of invariant testing problems. Moreover, most powerful invariant tests are maximin in the cases where the Hunt–Stein theorem is applicable. This is the case of G^*. For applications see [9].

In this paper, we construct some maximal invariants under G^*, derive their distributions and indicate some practical applications.

2. A maximal invariant and its distribution. Let $M_x = XA(XA)^{\mathrm{T}}$, where A is an $(n, n-1)$-matrix given by

$$A = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{bmatrix}.$$

If X is a random matrix with a probability distribution absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{R}^{pn}$, $n > p$, then M_x is a.s. nonsingular (see [2]). Thus the matrix $L_x \in \mathrm{UT}(p)$ satisfying $M_x = L_x L_x^{\mathrm{T}}$ is a.s. uniquely determined. Let $B_x = L_x^{-1} XA$.

PROPOSITION 1. *B_x is a maximal invariant under G^*.*

Proof. A maximal invariant under G^* can be constructed in two steps. First, note that XA is a maximal invariant under translations and that the action of $\mathrm{UT}(p)$ on X induces an action of $\mathrm{UT}(p)$ on XA. Hence it suffices to show that a maximal invariant under the action $Y \to CY$ of $\mathrm{UT}(p)$ on $\boldsymbol{R}^{p(n-1)}$, $Y_{(p,n-1)} \in \boldsymbol{R}^{p(n-1)}$, $C \in \mathrm{UT}(p)$, is $B_y = L_y^{-1} Y$, where $L_y \in \mathrm{UT}(p)$ and $YY^{\mathrm{T}} = L_y L_y^{\mathrm{T}}$. This follows easily from the following consideration. Take $Z = CY$, $C \in \mathrm{UT}(p)$. Then

$$ZZ^{\mathrm{T}} = CYY^{\mathrm{T}}C^{\mathrm{T}} = CL_y(CL_y)^{\mathrm{T}} \quad \text{and} \quad L_z = CL_y$$

by the uniqueness of the Cholesky decomposition. Hence we have $B_z = L_y^{-1}C^{-1}CY = B_y$ and B_y is an invariant. In order to show that B_y is also a maximal invariant assume that $B_y = B_z$. This implies $L_y^{-1}Y = L_z^{-1}Z$ and $Y = CZ$ with $C = L_y L_z^{-1} \in \mathrm{UT}(p)$, which completes the proof.

The $(p, n-1)$-matrix B_x forms a part of an $(n-1, n-1)$-orthogonal matrix. Denote by ν the probabilistic Haar measure on the group $\mathrm{SO}(n-1)$ of orthogonal matrices with determinant 1. Each element of $\mathrm{SO}(n-1)$ can be identified with a point of an $[(n-1)(n-2)/2]$-dimensional Riemannian manifold $\mathfrak{M}_0$, and each matrix B_x can be identified with a point of a $[p(2n-p-3)/2]$-dimensional Riemannian submanifold $\mathfrak{M}$ of $\mathfrak{M}_0$. Let t be a transformation $\mathfrak{M}_0 \to \mathfrak{M}$ given by

$$\mathfrak{M}_0 \ni [B_1^{\mathrm{T}}:\ldots:B_{n-1}^{\mathrm{T}}]^{\mathrm{T}} = B \xrightarrow{t} B_{(p)} = [B_1^{\mathrm{T}}:\ldots:B_p^{\mathrm{T}}]^{\mathrm{T}} \in \mathfrak{M},$$

B_i being row vectors in $\boldsymbol{R}^{n-1}$ $(i = 1, \ldots, n-1)$, and let us define a measure μ on $\mathfrak{M}$ by $\mu = t\nu$. It is clear that μ remains invariant under the transformations

$$B_{(p)} \to B_{(p)}C, \quad C \in \mathrm{SO}(n-1). \tag{1}$$

It is (up to multiplication by a constant) the unique measure on $\mathfrak{M}$ with such a property. This is a consequence of the well-known Weil theorem on the existence and uniqueness of relatively invariant measures on left-homogeneous spaces (cf. [2], Theorem 6.3 and Example 6.16, or [7], p. 138, Theorem 1).

Let P denote the distribution of the random matrix X, absolutely continuous with respect to the Lebesgue measure λ_{np} on $\boldsymbol{R}^{pn}$, $p = dP/d\lambda_{np}$, $\tilde{p}$ a density of the distribution of XA with respect to $\lambda_{(n-1)p}$, Π: $\boldsymbol{R}^{pn} \to \mathfrak{M}$ the orbit projection $\Pi(X) = B_x$, and $\Pi(P)$ the distribution of the maximal invariant.

PROPOSITION 2. *In the notation above*:

$$d\Pi(P)/d\mu = c_{np}^{-1} \int \tilde{p}(LB_x) \prod_{i=1}^{p} l_{ii}^{n-p-2+i} dL, \tag{2}$$

where the integration is performed with respect to the elements of

$$L = [l_{ij}] \in \mathrm{UT}(p) \quad \textit{and} \quad c_{np} = 2^{-p}\pi^{(1+p-2n)/4} \prod_{j=1}^{p} \Gamma\left(\frac{n-j}{2}\right).$$

Proof. It is seen from the proof of Proposition 1 that B_x is a maximal invariant for UT(p) acting on the space of matrices $Y = XA$. The modular function of UT(p) is

$$\Delta_u(L) = \prod_{i=1}^{p} l_{ii}^{2i-p-1},$$

$\lambda_{(n-1)p}$ is relatively invariant with multiplier $(\det L)^{n-1}$ under the action of UT(p) on $R^{p(n-1)}$: $Y \to LY$, and α defined by

$$d\alpha(L) = \prod_{i=1}^{p} l_{ii}^{i-p-1} dL$$

is a left Haar measure on UT(p). Using the Wijsman theorem ([1], [11]) we get easily

$$\frac{d\Pi(P)}{d\lambda/\beta} = \prod_{i=1}^{p} (L_x)_{ii}^{2i+n-p-2} \int_{UT(p)} \tilde{p}(TY)(\det T)^{n-1} d\alpha(T)$$

$$= \prod_{i=1}^{p} (L_x)_{ii}^{2i+n-p-2} \int_{UT(p)} \tilde{p}(TL_x B_x) \prod_{i=1}^{p} t_{ii}^{n-2+i-p} dT$$

$$= \int_{UT(p)} \tilde{p}(LB_x) \prod_{i=1}^{p} l_{ii}^{n-p-2+i} dL,$$

where λ is a measure on $R^{p(n-1)}$ such that

$$d\lambda(Y) = \prod_{i=1}^{p} (L_y)_{ii}^{2-2i-n+p} d\lambda_{(n-1)p}(Y),$$

β is a right Haar masure on UT(p), and λ/β is the so-called quotient measure. In view of our previous remarks on the measure μ, to show the proportionality of λ/β and μ it suffices to prove that λ/β remains invariant under transformations (1). This may easily be deduced from the results contained in Section 5 of [1]. Consider the group $K = HG$ with $G = \mathrm{UT}(p)$ and $H = \mathrm{SO}(n-1)$ acting on the space $R^{p(n-1)}$ of $(p, n-1)$ real matrices according to $kY = AYB$, $A \in \mathrm{UT}(p)$, $B \in \mathrm{SO}(n-1)$, $k = (B, A) \in K$. Since the actions of H and G commute, the automorphism Φ_h: $g \to hgh^{-1}$ is the identity mapping and $\mathrm{mod}\,\Phi_h = 1$. Elementary calculations show that λ is relatively invariant under the action of K with multiplier Δ_u^{-1}. By virtue of Proposition 2 in [1] this is equivalent to the invariance of λ/β under the action (1) of $H = \mathrm{SO}(n-1)$. In order to find c_{np} take

$$\tilde{p}(Z) = (2\pi)^{-p(n-1)/2} \exp\{-0.5 \operatorname{Tr} ZZ^T\},$$

the density of the multivariate $(p, n-1)$ normal distribution, and integrate the right-hand side of (2) over $\mathfrak{M}$ with respect to μ. We have $B_z = L_z^{-1} Z$, where $ZZ^T = L_z L_z^T$, $L_z \in \mathrm{UT}(p)$ and $\operatorname{Tr} LB_z(LB_z)^T = \operatorname{Tr} LL^T$, since $B_z B_z^T = I$. Consequently,

$$c_{np}^{-1}\mu(\mathfrak{M})(2\pi)^{-p(n-1)/2}\int \exp\{-0.5\operatorname{Tr} LL^{\mathrm{T}}\}\prod_{i=1}^{p} l_{ii}^{n-p-2+i}dL = 1.$$

Making use of the equality $\mu(\mathfrak{M}) = 1$ and computing the integral we get the value of c_{np}. A more explicit form of c_{np} is given in Section 4.

3. The normal case. Let X be distributed as $N(M, \Sigma\otimes I_n)$. Because of the invariance we may assume $M = 0$ and $\Sigma = I_p$. Then

$$\tilde{p}(Y) = (2\pi)^{-p(n-1)/2}(\det \Lambda)^{p/2}\exp\{-0.5\operatorname{Tr} Y\Lambda Y^{\mathrm{T}}\},$$

where $\Lambda^{-1} = [\lambda_{ij}]$, $\lambda_{ij} = 1$ for $i\neq j$ and $\lambda_{ii} = 2$. Since $\det\Lambda = n^{-1}$ and $\Lambda = I-n^{-1}1_n1_n^{\mathrm{T}}$, we get, using $B_xB_x^{\mathrm{T}} = I$,

$$d\Pi(P)/d\mu$$
$$= c_{np}(2\pi)^{-p(n-1)/2}n^{-p/2}\int \exp\{-0.5\operatorname{Tr} L(I-n^{-1}bb^{\mathrm{T}})L^{\mathrm{T}}\}\prod_{i=1}^{p} l_{ii}^{n-p-2+i}dL,$$

where $b = [b_1, \ldots, b_p]^{\mathrm{T}} = B_x1_{n-1}$. Define $L_0\in \mathrm{UT}(p)$ by $I-n^{-1}bb^{\mathrm{T}} = L_0L_0^{\mathrm{T}}$. Taking $T = LL_0$ as a new variable in the integral with

$$\partial L/\partial T = \prod_{i=1}^{p}(L_0)_{ii}^{-i}$$

we have the following

COROLLARY 1. *If X is distributed as $N(M, \Sigma\otimes I_n)$, then*

$$d\Pi(P)/d\mu = n^{-p/2}\prod_{i=1}^{p}(L_0)_{ii}^{-(n+2i-p-2)}.$$

If $p = 2$, then

$$d\Pi(P)/d\mu = n^{-1}(1-n^{-1}(b_1^2+b_2^2)^{-(n-2)/2})(1-n^{-1}b_2^2)^{-1}.$$

For $p > 2$ the formula becomes more complicated. It still depends, however, only on the vector b.

4. Parametrization by Euler's angles. Let R_{ij} $(1\leqslant i\leqslant n-1, 1\leqslant j\leqslant n-1, i<j)$ be a rotation matrix from $\mathrm{SO}(n-1)$ defined as follows:

$$(R_{ij})_{ii} = (R_{ij})_{jj} = \cos\theta_{ij},\quad (R_{ij})_{ij} = \sin\theta_{ij},$$
$$(R_{ij})_{ji} = -\sin\theta_{ij},\quad (R_{ij})_{kk} = 1$$

for $k\neq i$ and $k\neq j$ and all the remaining elements are equal to zero. It is easy to check (cf. [4] and [10]) that for every matrix $G\in\mathrm{SO}(n-1)$ the following decomposition is valid:

(3) $$G = G^{(n-2)}\ldots G^{(1)},$$

where $G^{(i)} = R_{i,n-1} \ldots R_{i,i+1}$ with suitably chosen θ_{kl}. Denote by e_i $(i = 1, \ldots, n-1)$ the i-th vector of the usual canonical basis of R^{n-1}. The vector of Euler's angles

$$E_G = (\theta_{12}, \ldots, \theta_{1,n-1}, \theta_{23}, \ldots, \theta_{2,n-1}, \ldots, \theta_{n-2,n-1})$$

can be interpreted in the following way: $\theta_{12}, \ldots, \theta_{1,n-1}$ are spherical coordinates of

$$G^{-1}e_1 = (\cos\theta_{1,n-1} \ldots \cos\theta_{12}, \ldots, \cos\theta_{1,n-1}\sin\theta_{1,n-2}, \sin\theta_{1,n-1})^{\mathrm{T}},$$

where $0 \leqslant \theta_{12} < 2\pi$, $-\pi/2 \leqslant \theta_{1k} \leqslant \pi/2$, $3 \leqslant k \leqslant n-1$.

In the same way the angles $\theta_{i,i+1}, \ldots, \theta_{i,n-1}$ are spherical coordinates of

$$G^{(i-1)} \ldots G^{(1)} G^{-1} e_i$$
$$= (0, \ldots, 0, \cos\theta_{i,n-1} \ldots \cos\theta_{i,i+1}, \ldots, \cos\theta_{i,n-1}\sin\theta_{i,n-2}, \sin\theta_{i,n-1})^{\mathrm{T}},$$

where $0 \leqslant \theta_{i,i+1} < 2\pi$, $-\pi/2 \leqslant \theta_{ik} \leqslant \pi/2$, $i+2 \leqslant k \leqslant n-1$.

This interpretation of Euler's angles indicates an easy way of obtaining E_G for a given matrix $G \in \mathrm{SO}(n-1)$. Passing from E_G to G may easily be performed according to (3).

Note that inequalities for θ_{kl} given above determine them uniquely and we have a 1-1 correspondence between matrices $G \in \mathrm{SO}(n-1)$ and vectors E_G. We will denote by the same letter ν the Haar measure on $\mathrm{SO}(n-1)$ and the corresponding measure on the space of Euler's angles. Taking into account the above representation of $G \in \mathrm{SO}(n-1)$ we are able to express the density of ν with respect to the Lebesgue measure λ_{E_G} on the space of Euler's angles in the form

$$d\nu/d\lambda_{E_G} = \prod_{j=1}^{n-2} A_{nj} \prod_{i=j}^{n-2} \cos^{i-j}\theta_{j,i+1}, \quad \text{where } A_{nj} = \Gamma[(n-j)/2]/(2\pi^{(n-j)/2}).$$

There is also a 1-1 correspondence between matrices $B_x \in \mathfrak{M}$ and subvectors $E_x = (\theta_{12}, \ldots, \theta_{1,n-1}, \ldots, \theta_{p,p+1}, \ldots, \theta_{p,n-1})$ of Euler's angles and we will again denote by the same letter μ the measure $t\nu$ and the corresponding measure on the space of vectors E_x.

As a consequence of the above considerations we get finally for $p < n-1$

$$d\mu = \prod_{j=1}^{p} A_{nj} \prod_{i=j}^{n-2} \cos^{i-j}\theta_{j,i+1}\, d\theta_{j,i+1}$$

which can be expressed in a more explicit form as

$$d\mu = c_{np} \prod_{j=1}^{p} \prod_{i=j}^{n-2} \cos^{i-j}\theta_{j,i+1}\, d\theta_{j,i+1}, \tag{4}$$

where

$$c_{np} = \begin{cases} 2^{-p/2}(2\pi)^{p(2-2n-p)/4}(n-p-1)!(n-p+1)!\dots(n-3)!, & p \text{ even},\\ 2^{(1-p-n)/2}(2\pi)^{[(p+1)^2-2np]/4}\Gamma^{-1}(n/2)(n-p-1)!(n-p+1)!\dots(n-2)!, & p \text{ odd}.\end{cases}$$

Note that exactly the same constant c_{np} occurs in Proposition 2. It is clear that E_x also forms a maximal invariant under G^*. A density of the distribution of E_x with respect to the Lebesgue measure on $\boldsymbol{R}^{p(2n-p-3)/2}$ containing E_x is determined by (2) and (4). To find its value the matrix B_x corresponding to E_x is needed. Such a B_x can easily be obtained according to (3) with $E_G = (E_x, 0, \dots, 0)$ after applying the transformation t to the resulting matrix G.

5. Another maximal invariant and its distribution. In some cases it is useful to consider another maximal invariant for G^* defined in [8]. Let $S_x = (X-\bar{X})(X-\bar{X})^{\mathrm{T}}$, where $\bar{X} = n^{-1}X\boldsymbol{1}_n\boldsymbol{1}_n^{\mathrm{T}}$. If X is a random matrix with a probability distribution absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{R}^{pn}$ and $n > p$, then S_x is nonsingular almost surely and we may define $\tilde{L}_x \in \mathrm{UT}(p)$ by

$$S_x = \tilde{L}_x\tilde{L}_x^{\mathrm{T}} \quad \text{and} \quad \tilde{B}_x = \tilde{L}_x^{-1}(X-\bar{X}).$$

$\tilde{B}_x$ is another maximal invariant for G^*.

Let $\Lambda = I - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n^{\mathrm{T}}$ and D be an orthogonal (n, n)-matrix with the last row of the form $(n^{-1/2}, \dots, n^{-1/2})$. The first $n-1$ rows of D form a matrix $\tilde{D}$. Choose and fix such a $\tilde{D}$ and note that $\tilde{D}^{\mathrm{T}}\tilde{D} = \Lambda$, $X-\bar{X} = X\Lambda$ and $\tilde{D}$ is a full-rank matrix. This implies that there exists a unique matrix $U_{(p,n-1)}$, namely $U = X\tilde{D}^{\mathrm{T}}$, such that $X-\bar{X} = U\tilde{D}$. This equality establishes a 1-1 transformation from the space of (p, n)-matrices with rows orthogonal to $\boldsymbol{1}_n$ to the space of $(p, n-1)$-matrices. Denote by $\tilde{B}_u$ the matrix constructed from U in the same way as $\tilde{B}_x$ was constructed from $X-\bar{X}$.

The uniqueness of the Cholesky decomposition and the fact that Λ is idempotent imply that $\tilde{L}_x = \tilde{L}_u$ and we have

$$\tilde{B}_x = \tilde{B}_u\tilde{D}, \tag{5}$$

which establishes a 1-1 correspondence between $\tilde{B}_x$ and $\tilde{B}_u$. This enables us to apply the results of Section 2 and describe the distribution of $\tilde{B}_x$ indirectly through the distribution of $\tilde{B}_u$ and the transformation (5). The distribution of $\tilde{B}_u$ is given by Proposition 2 with the replacement of B_x by $\tilde{B}_u$ and of $p(\cdot)$ by the density of the distribution of U.

In the normal case, the distribution of X being $N(M, \Sigma \otimes I_n)$, we may take, because of invariance, $M = 0$ and $\Sigma = I_p$. Then the distribution of U is $N(0, I_p \otimes I_{n-1})$. An inspection of the last part of the proof of Proposition 2 leads to the following

COROLLARY 2. *If the distribution of X is $N(M, \Sigma \otimes I_n)$, then the distribution of $\tilde{B}_u$ is $\mu = tv$ with the density given by* (4).

The computation of $\tilde{B}_x$ given Euler's angles of the corresponding $\tilde{B}_u$ is performed in two steps:

1. compute $\tilde{B}_u$ given Euler's angles as described at the end of Section 4 in the context of B_x;
2. compute $\tilde{B}_x$ given $\tilde{B}_u$ according to $\tilde{B}_x = \tilde{B}_u \tilde{D}$.

6. Some special cases and possible applications. In Section 5 we described the distribution of the maximal invariant $\tilde{B}_x$ for normal X through the marginal distribution of the subvector E_x of E_G, with E_G being distributed according to the probabilistic Haar measure. In this section the distribution of $\tilde{B}_x$ for $p = 2$ and two other distributions of X will be given.

Denote by $\mathscr{P}_E$ the transformation family of distributions of $X = UY + m1_n^{\mathrm{T}}$, where $U \in \mathrm{UT}(2)$, $m \in \boldsymbol{R}^2$ and the columns of $(2, n)$ random matrix Y are independently and identically distributed according to the probability density function $\psi(\zeta_1, \zeta_2) = \exp\{-(\zeta_1 + \zeta_2)\}$ for $\zeta_1, \zeta_2 \geqslant 0$ and zero otherwise. Analogously we define the family of distributions $\mathscr{P}_U$ taking $\psi(\cdot, \cdot)$ to be an indicator function of the unit square. Because of the invariance, the distribution of $\tilde{B}_x$ does not depend on the particular choice of $P_E \in \mathscr{P}_E$. The same is true for $P_U \in \mathscr{P}_U$. So we can take P_E and P_U corresponding to $U = I$ and $m = (0, 0)^{\mathrm{T}}$. Put $\mu_E = \Pi(P_E)$, $\mu_U = \Pi(P_U)$ and recall that $\mu = \Pi(P_N)$, $P_N = N(M, \Sigma \otimes I_n)$.

In [9] two functions $I_E(\cdot)$ and $I_U(\cdot)$ were found such that

$$d\mu_E/d\mu = c_E I_E(\tilde{B}_x)|b_{2\min}|^{1-n} \quad \text{and} \quad d\mu_U/d\mu = c_U I_U(\tilde{B}_x)(b_{2\max} - b_{2\min})^{1-n},$$

where $b_{2\min}$ and $b_{2\max}$ are the minimal and maximal elements, respectively, of the second row of $\tilde{B}_x$. This and the results of Section 5 yield the distributions of $\tilde{B}_x$ when the distribution of X belongs to the family $\mathscr{P}_E$ or $\mathscr{P}_U$.

The constants c_E and c_U are not given explicitly in [9] but can easily be derived and are of the form

$$c_E = \frac{[(n-2)!]^2 (2\pi)^{n-1}}{(n-2)(n^n)^2} \quad \text{and} \quad c_U = \frac{(2\pi)^{n-1}}{n^2(n-1)^2(n-2)}.$$

The results obtained in this paper can be applied to the analysis of small sample behaviour of G^*-invariant tests for multinormality which are functions of $\tilde{B}_x$ (see [8] and [9]). This includes, e.g., finding α-critical values, say c_α, for tests of the form $\phi(X) = I\{T(\tilde{B}_x) < c\}$, where I is the indicator function, and T denotes any of G^*-invariant test statistics studied in [8] and [9]. This is equivalent to solving with respect to c_α the equation

$$\int I\{T[\tilde{B}_x(\bar{\theta})] < c_\alpha\} d\mu(\bar{\theta}) = \alpha, \tag{6}$$

where μ is given by (4).

The last part of Section 5 provides a way of computing $\tilde{B}_x(\bar{\theta})$ for a given vector $\bar{\theta}$ of Euler's angles.

The power functions of the most powerful G^*-invariant tests for binormality (see [9]) can be written in the parametric form as follows:

$$(7) \qquad \begin{aligned} x(t) &= \int I\{T[\tilde{B}_x(\bar{\theta})] < t\} d\mu(\bar{\theta}), \\ y(t) &= \int I\{T[\tilde{B}_x(\bar{\theta})] \geqslant t\} \eta(\bar{\theta}) d\mu(\bar{\theta}), \end{aligned}$$

where T, η are T_E^*, $d\mu_E/d\mu$ or T_U^*, $d\mu_U/d\mu$, respectively. The test statistics T_E^* and T_U^* are given explicitly in [9]. $x(t)$ and $y(t)$ are the size and the power, respectively, of the test corresponding to the critical value t (cf. also [5]). Calculation of these power functions is particularly interesting since G^* satisfies the Hunt–Stein assumptions. Thus the most powerful invariant tests are maximin and it is possible to construct maximin tests for approximate normality taking suitably defined neighbourhoods of the hypotheses and using results of [9] and [5].

The integrals in (6) and (7) must be computed numerically because the regions in which the indicator functions are nonzero are complicated and do not admit an analytical description. Some results of the above type will be published separately.

Such results can, of course, also be obtained by classical Monte-Carlo methods. Note, however, that finding, e.g., critical values in a Monte-Carlo simulation is, in fact, equivalent to computing by a Monte-Carlo method the value of an integral over a pn-dimensional space and that the quality of generating pseudorandom numbers from the normal and alternative distributions is equally crucial as difficult to control. Our approach reduces the dimension to $p(2n-p-3)/2$ and puts the whole problem in a more explicit form. For small sample sizes, which are interesting in the context of T_E^* and T_U^*, it is even possible to apply nonstochastic procedures of numerical integration, which makes the control of accuracy more reliable.

REFERENCES

[1] S. Andersson, *Distributions of maximal invariants using quotient measures*, Ann. Statist. 10 (1982), pp. 955–961.

[2] M. L. Eaton, *Multivariate Statistics. A Vector Space Approach*, New York 1983.

[3] — and M. D. Perlman, *The nonsingularity of generalized sample covariance matrices*, Ann. Statist. 1 (1973), pp. 710–717.

[4] W. Girko, *Theory of Random Determinants* (in Russian), Kiev 1980.

[5] R. Hafner, *Construction of least favourable pairs of distributions and of robust tests for contamination neighbourhoods*, Math. Operationsforsch. Statist. Ser. Statistics 13 (1982), pp. 47–56.

[6] E. L. Lehmann, *Testing Statistical Hypotheses*, New York 1959.

[7] L. Nachbin, *The Haar Integral*, Princeton 1965.
[8] Z. Szkutnik, *On invariant tests for multidimensional normality*, Probab. Math. Statist. 8 (1987), pp. 1–10.
[9] — *Most powerful invariant tests for binormality*, Ann. Statist. 16 (1988), pp. 292–301.
[10] N. Vilenkin, *Special Functions and the Theory of Group Representations* (in Russian), Moscow 1965.
[11] R. A. Wijsman, *Proper action in steps, with application to density ratios of maximal invariants*, Ann. Statist. 13 (1985), pp. 395–402.

Institute of Mathematics
University of Mining and Metallurgy
Al. Mickiewicza 30
30-065 Kraków, Poland

Received on 13.9.1988;
new version on 23.6.1989