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WEIGHTED SQUARE SUMMABLE AND GENERALIZED HARMONIZABLE SEQUENCES*

BY

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Abstract. It is shown that a weighted square summable process (sequence) with weights related to a Stieltjes moment sequence is *generalized harmonizable* (i.e., it is represented by a Borel vector-valued measure on the complex plane). An explicit formula for a normal dilation of such a process is presented. An example of a generalized harmonizable process which does not admit any representing measure on a compact set is given. It is proved that a process which is generalized harmonizable on a compact set always has a representing measure supported on at most two circles centered at the origin. The question of the existence and summability of densities of representing measures of such a process is investigated.

Introduction and preliminaries. Ideas of our paper go back to the works [5] by Niemi and [7] by Rozanov. Namely, in [7] Rozanov have given the well-known definition of a harmonizable process as that which has an integral representation with some vector measure on the unit circle (this measure need not be orthogonally scattered as in the case of a stationary process). Niemi in [5] has considered square summable sequences and has shown that those sequences have stationary dilations and, consequently, are harmonizable. The method of Niemi's proof is based on the construction of some density function which is vector valued and has Fourier coefficients equal to a given square summable sequence. In [8] Salehi and Słociński have generalized Rozanov's definition of a harmonizable process (sequence) to that which has an integral representation with some vector measure on a compact subset of the complex plane C and have extended Niemi's result to the case of weighted square summable sequences with weights being powers of some fixed real.

In our paper we extend Niemi's idea of looking for some vector density function to the case of weighted square summable sequences with weights related to strong Stieltjes moment sequences. In the case of a moment sequence with compact support (i.e., with representing measure supported on a compact

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subset of the interval $(0, \infty)$) we prove that a related weighted square summable sequence has an integral representation with vector measure supported on a compact subset of $C\backslash\{0\}$. This shows that such sequences are harmonizable in the sense as in [8]. Our method is more general, namely it allows us to get an integral representation even for weighted square summable sequences with weights related to moment sequences which have representing measures not necessarily compactly supported in the open interval $(0, \infty)$. In this case the representing measure of the process does not have the compact support, thus in our paper we use a little more general definition from that used in [8]. Results obtained in this general frame are presented in Section 2.

In Section 3 we consider generalized harmonizable sequences with compact support (i.e. generalized harmonizable in the sense of [8]). We prove that in this case we can choose some representing measure supported on at most two circles in the complex plane C. This result allows us to give an easily verifiable characterization of generalized harmonizable sequences. Moreover, we show that we have some freedom in choosing the circles on which the measure is concentrated. The extent of this freedom is considered. Some positive answers and some open problems are presented. Section 3 deals with the existence and summability of densities of representing measures of generalized harmonizable sequences with compact support.

In what follows H denotes a separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Denote by $C^* = C\backslash\{0\}$ and by $\mathscr{B}(C^*)$ the σ-algebra of all Borel subsets of C^*. For a finite and positive measure μ we use the usual definition of $L^2(\mu, H)$ as the Hilbert space of all square summable functions with values in H and inner product $\langle f, g\rangle = \int (f(z), g(z))d\mu(z)$. For a given vector-valued measure ξ (with values in H) we use the definition of variation

$$|\xi|(\sigma) = \sup_{\|x\| \leqslant 1} |\xi_x|(\sigma),$$

where $|\xi_x|$ denotes the semivariation of the complex measure $\xi_x(\sigma) = (\xi(\sigma)x, x)$. For this and other topics on integration with respect to vector-valued functions see [1].

2. Weighted square summable sequences. Our main goal in the present section is to show that weighted square summable sequences with weights related to Stieltjes moment sequences are generalized harmonizable. For this we need some definitions.

Let S stand for Z or Z^+ (all integers or nonnegative integers). Extending Definition 1.10 in [8] we call the sequence $\{x_n\}_{n \in S} \subset H$ *generalized harmonizable* iff there exists a measure

$$\xi\colon \mathscr{B}(C^*) \to H \quad (\xi\colon \mathscr{B}(C) \to H \text{ for } S = Z^+)$$

such that

(2.1) $\quad e_n$ is ξ integrable for any $n \in S$, and

(2.2) $\quad x_n = \int e_n d\xi,$

where $e_n(z) = z^n$, $z \in C$.

The measure ξ will be called a *representing measure of the process* $\{x_n\}_{n \in S}$. If the closed support of $|\xi|$ is compact in C^* (C for $S = Z^+$), then the sequence will be shortly called *harmonizable on a compact set.*

It is obvious that if $\{x_n\}_{n \in S}$ is harmonizable on a compact set, then the condition (2.1) is satisfied automatically and, consequently, in that case we have exactly the same definition as that used in [8].

The sequence $\{\beta_n\}_{n \in Z}$ of reals is called a *bilateral Stieltjes moment sequence* (shortly, a *moment sequence*, see [4]) if there exists a finite positive measure μ on the open interval $(0, \infty)$ such that $e_n \in L^2(\mu)$ and $\beta_n = \int t^{2n} d\mu(t)$ for $n \in Z$. The measure μ is called the *representing measure* of the sequence $\{\beta_n\}_{n \in Z}$. A moment sequence $\{\beta_n\}_{n \in Z}$ is called *nondegenerate* iff $\beta_n > 0$ for any n in Z. For a given finite positive measure μ on the interval $(0, \infty)$ we define a new measure (the rotation measure) μ_{rot} on the Borel subsets of the complex plane as follows:

$$\mu_{\text{rot}}(\sigma) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \chi_\sigma(re^{i\theta}) d\theta \, d\mu(r),$$

where χ_σ denotes the indicator function of the Borel set $\sigma \subset C$. Evidently, μ_{rot} is a positive and finite measure such that $\mu_{\text{rot}}(\{0\}) = 0$. It is easy to see that μ_{rot} corresponds (via the polar coordinates) to the product measure of μ and normalized Lebesgue measure on the interval $[0, 2\pi)$.

The following theorem gives some sufficient conditions for a process to be generalized harmonizable.

THEOREM 1. *Suppose μ is the representing measure of a nondegenerate moment sequence $\{\beta_n\}_{n \in Z}$. Let $x_n \in H$ for $n \in Z$.*

(i) *The following conditions are equivalent*:

(2.3) $\quad \sum_{k=-\infty}^{\infty} \|x_k\|^2 \frac{\beta_{n+k}}{\beta_k^2} < \infty$ *for any* $n \in Z$;

(2.4) *there exists* $h \in L^2(\mu_{\text{rot}}, H)$ *such that*

(a) $h = Ph$, *where P denotes the orthogonal projection of $L^2(\mu_{\text{rot}}, H)$ onto the closed span of* $\bigcup_{n \in Z} \bar{e}_n H$,

(b) $e_n h \in L^2(\mu_{\text{rot}}, H)$ *for any* $n \in Z$,

(c) $x_n = \int e_n h \, d\mu_{\text{rot}}$.

Moreover, under the assumptions (a), (b) *and* (c) *the function h is unique and takes the form*

(2.5) $\quad h = \sum_{k=-\infty}^{\infty} \bar{e}_k \beta_k^{-1} x_k.$

(ii) *If additionally the closed support of* μ *is compact in* $(0, \infty)$, *then the condition* (2.3) *is equivalent to the following one*:

$$(2.6)\qquad \sum_{k=-\infty}^{\infty} \beta_k^{-1} \|x_k\|^2 < \infty.$$

Proof. Assume that (2.3) holds. Let $g_k^n = e_n \bar{e}_k x_k \beta_k^{-1}$. Observe first that for any fixed integer n the sequence g_k^n, $k \in \mathbf{Z}$, consists of pairwise orthogonal vectors in $L^2(\mu_{\mathrm{rot}}, H)$. Indeed,

$$\begin{aligned}\langle g_k^n, g_l^n\rangle &= \int |e_n|^2 \bar{e}_k e_l (x_k, x_l) d\mu_{\mathrm{rot}} \beta_k^{-1} \beta_l^{-1} \\ &= \int_{(0,\infty)} r^{k+l+2n} d\mu(r)(x_k, x_l)\beta_k^{-1}\beta_l^{-1} \int_0^{2\pi} e^{i(l-k)\theta} \frac{d\theta}{2\pi} \\ &= \delta_0(k-l)\|x_k\|^2 \frac{\beta_{k+n}}{\beta_k^2}.\end{aligned}$$

Moreover, we get

$$\|g_k^n\|^2 = \frac{\beta_{k+n}}{\beta_k^2}\|x_k\|^2 \quad \text{for any } n, k \in \mathbf{Z}.$$

Thus for any $n \in \mathbf{Z}$ the series $\sum_{k=-\infty}^{\infty} g_k^n$ converges in $L^2(\mu_{\mathrm{rot}}, H)$ because by the assumption (2.3) so does the series

$$\sum_{k=-\infty}^{\infty} \|g_k^n\|^2 = \sum_{k=-\infty}^{\infty} \frac{\beta_{k+n}}{\beta_k^2}\|x_k\|^2.$$

In particular, the function h given by (2.5) is well defined. Consequently, h belongs to the range of the projection P and the property (a) is proved.

For a complex polynomial p, we denote by M_p the operator of multiplication by p with the domain

$$\mathscr{D}_{M_p} = \{f \in L^2(\mu_{\mathrm{rot}}, H) \colon pf \in L^2(\mu_{\mathrm{rot}}, H)\}.$$

It is the well-known fact that M_p is closed. Thus, since the series $\sum_{k=-\infty}^{\infty} g_k^n$ converges in $L^2(\mu_{\mathrm{rot}}, H)$, it is easy to show that

$$e_n h = M_{e_n} h = \sum_{k=-\infty}^{\infty} g_k^n.$$

Consequently, $e_n h$ belongs to $L^2(\mu_{\mathrm{rot}}, H)$, and the property (b) is proved. Moreover, we have

$$\int e_n h d\mu_{\mathrm{rot}} = \sum_{k=-\infty}^{\infty} \int e_n \bar{e}_k \beta_k^{-1} x_k d\mu_{\mathrm{rot}} = \int_0^{\infty} r^{2n} d\mu(r) \beta_n^{-1} x_n = x_n,$$

which shows (c). This completes the proof of the implication (2.3)$\Rightarrow$(2.4).

Now we will show the uniqueness of h. Suppose that $g \in L^2(\mu_{\mathrm{rot}}, H)$ has the properties (a), (b) and (c). Let P_n denote the orthogonal projection of $L^2(\mu_{\mathrm{rot}}, H)$

onto $H_n = \bar{e}_n H$. Evidently, the spaces H_n, $n \in \boldsymbol{Z}$, are pairwise orthogonal. Thus we get $Pg = \sum_{k=-\infty}^{\infty} P_k g$. As an element of H_n, $P_n g$ takes the form $P_n g = \bar{e}_n g_n$ with some $g_n \in H$. Then for $x \in H$ we have

$$\begin{aligned}(g_n, x) &= \beta_n^{-1} \langle \bar{e}_n g_n, \bar{e}_n x \rangle = \beta_n^{-1} \langle P_n g, \bar{e}_n x \rangle \\ &= \beta_n^{-1} \langle g, P_n(\bar{e}_n x) \rangle = \beta_n^{-1} \langle g, \bar{e}_n x \rangle \\ &= \beta_n^{-1} \int e_n (g, x) d\mu_{\mathrm{rot}} = \beta_n^{-1} \left\langle \int e_n g d\mu_{\mathrm{rot}}, x \right\rangle.\end{aligned}$$

This shows that $P_n g = \bar{e}_n g_n = \beta_n^{-1} \bar{e}_n \int e_n g d\mu_{\mathrm{rot}}$. It follows from (c) that $P_n g = \beta_n^{-1} \bar{e}_n x_n$ and, consequently, $g = \sum_{k=-\infty}^{\infty} P_k g = h$, which shows that (2.4) implies (2.5).

To prove the implication (2.4) $\Rightarrow$ (2.3), fix an integer n and denote by Q (resp. Q_k) the orthogonal projection of $L^2(\mu_{\mathrm{rot}}, H)$ onto the closed span of $\bigcup_{k \in \boldsymbol{Z}} e_n \bar{e}_k H$ (resp. onto the space $e_n \bar{e}_k H$). Since the spaces $e_n \bar{e}_k H$ and $e_n \bar{e}_l H$ are orthogonal for $k \neq l$, we get

$$\sum_{k=-\infty}^{\infty} \|Q_k g\|^2 = \|Qg\|^2 < \infty \quad \text{for any } g \in L^2(\mu_{\mathrm{rot}}, H).$$

Take now $g = e_n h$, where h satisfies (a), (b) and (c). As an element of the space $e_n \bar{e}_k H$, $Q_k g$ is of the form $Q_k g = e_n \bar{e}_k g_k$ with some $g_k \in H$. Then for $x \in H$ we have

$$\begin{aligned}(g_k, x) &= \beta_{n+k}^{-1} \langle e_n \bar{e}_k g_k, e_n \bar{e}_k x \rangle \\ &= \beta_{n+k}^{-1} \langle Q_k g, e_n \bar{e}_k x \rangle = \beta_{n+k}^{-1} \langle g, e_n \bar{e}_k x \rangle \\ &= \beta_{n+k}^{-1} \langle e_n h, e_n \bar{e}_k x \rangle = \beta_{n+k}^{-1} \langle h, |e_n|^2 \bar{e}_k x \rangle.\end{aligned}$$

Since $h = Ph$, h is of the form $h = \sum_{l=-\infty}^{\infty} \beta_l^{-1} \bar{e}_l x_l$. Thus

$$\begin{aligned}(g_k, x) &= \beta_{n+k}^{-1} \left\langle \sum_{l=-\infty}^{\infty} \beta_l^{-1} \bar{e}_l x_l, |e_n|^2 \bar{e}_k x \right\rangle \\ &= \sum_{l=-\infty}^{\infty} \beta_l^{-1} \beta_{n+k}^{-1} \langle \bar{e}_l x_l, |e_n|^2 \bar{e}_k x \rangle = \beta_{n+k}^{-1} \beta_{n+k} \beta_k^{-1} (x_k, x) \\ &= (\beta_k^{-1} x_k, x),\end{aligned}$$

which shows that $Q_k g = \beta_k^{-1} e_n \bar{e}_k x_k$. Finally, we have

$$\infty > \sum_{k=-\infty}^{\infty} \|Q_k g\|^2 = \sum_{k=-\infty}^{\infty} \beta_k^{-2} \|e_n \bar{e}_k x_k\|^2 = \sum_{k=-\infty}^{\infty} \frac{\beta_{k+n}}{\beta_k^2} \|x_k\|^2,$$

which completes the proof of (i).

To prove (ii) assume that the closed support of μ is compact in $(0, \infty)$. The implication (2.3) $\Rightarrow$ (2.6) is obvious. To complete the proof it is sufficient to show the implication (2.6) $\Rightarrow$ (2.4). This can be done similarly to the proof of (2.3) $\Rightarrow$ (2.4), keeping in mind that the multiplication operators M_{e_n}, $n \in \boldsymbol{Z}$, are bounded. ■

Remark. 1. The condition (2.3) is sufficient for harmonizability. It can be generalized in the following way: choose the sequence $\{f_n\}_{n\in Z}$ such that, for any fixed n, f_n belongs to the convex hull of vectors of the form $(\bar{e}_n/\beta_n)x_n$ and $e_{-n}x_n$. It is easy to see that the vectors f_n form an orthogonal sequence in $L^2(\mu_{\rm rot}, H)$. If we assume that the series $\sum_{k=-\infty}^{\infty} \|e_n f_k\|^2$ converges for any integer n, then putting $h = \sum_{k=-\infty}^{\infty} f_k$ we get $e_n h = \sum_{k=-\infty}^{\infty} e_n f_k$ and, consequently, assuming additionally that $\beta_0 = 1$, we get $x_n = \int e_n h d\mu_{\rm rot}$. This shows that we have infinitely many possibilities of obtaining the density h. Evidently, in each case the sufficient assumption in the implicit form $\sum_{k=-\infty}^{\infty} \|e_n f_k\|^2 < \infty$ is the same but in the explicit form looks different.

Finally, we can prove that weighted square summable processes considered admit normal dilations of Niemi's type. The process of normal type is defined in [3]. The notion of normal process appears also in [8]. Unfortunately, these notions cannot be used in our work. Though the Getoor definition deals with unbounded shift operators, it works only for a process with continuous parameter. On the other hand, the definition from [8] assumes that the shift operators attached to the process are bounded. For our purpose the best choice is to apply the latter by neglecting the boundedness of shift operators and assuming that the considered semigroup is equal to Z or Z^+. Thus the definition of normal process can be restated as follows. The sequence $\{y_n\}_{n\in S}$ in some Hilbert space K is *normal* iff there is a normal (shift) operator N in K (possibly unbounded) such that y_0 is in domain of N^n and $y_n = N^n y_0$ for any $n \in S$. We say that a process $\{x_n\}_{n\in Z}$ in H has *normal dilation* if there exists a normal process $\{y_n\}_{n\in Z}$ in some Hilbert space $K \supset H$ such that $x_n = Py_n$ for $n \in Z$, where P is the orthogonal projection of K onto H. The following theorem gives an explicit construction of this dilation.

THEOREM 2. *Let μ, $\{\beta_n\}_{n\in Z}$, $\{x_n\}_{n\in Z}$ and h be as in Theorem 1. If (2.3) holds and $\beta_0 = 1$, then $\{x_n\}_{n\in Z}$ has a normal dilation. Moreover, the process $\{y_n\}_{n\in Z} \subset L^2(\mu_{\rm rot}, H)$ defined by the formula $y_n = e_n h$, $n \in Z$, is a normal dilation of $\{x_n\}_{n\in Z}$.*

Proof. By Theorem 1 we see that h and $e_n h$ are in $L^2(\mu_{\rm rot}, H)$. Let N be the operator of multiplication by e_1 in $L^2(\mu_{\rm rot}, H)$. Then $y_n = N^n h$. Evidently, the operator N as a multiplication in L^2-space is normal. All we have to show is that $x_n = Py_n$, where P denotes the orthogonal projection of $L^2(\mu_{\rm rot}, H)$ onto the space of constants H. For $x \in H$ we have

$$\langle Py_n, x\rangle = \langle y_n, x\rangle = \int (y_n, x) d\mu_{\rm rot}$$
$$= \int e_n(h, x) d\mu_{\rm rot} = \left(\int e_n h d\mu_{\rm rot}, x\right) = (x_n, x) = \langle x_n, x\rangle.$$

The fifth equality follows from Theorem 1. The proof is complete. ∎

Remark 2. Repeating proofs from [8] one can extend Theorem 1.23 and Corollary 1.24 of [8] to the case of an arbitrary generalized harmonizable sequence (not necessarily with compact support). Thus any generalized harmonizable sequence has a normal dilation. In Theorem 2 we assume more (namely, weighted square summability, which by Theorem 1 is equivalent to generalized harmonizability with representing measure having L^2-density), but our proof does not use any existence type theorems (like the Rosenberg theorem [6] which leads back to the Grothendieck inequality).

3. Sequences harmonizable on compact sets. Our goal in this section is to show that any sequence harmonizable on a compact set admits a representing measure with closed support localized on at most two circles. We also prove the main result of our paper which characterizes sequences harmonizable on compact sets as those whose growth in norm is not greater than exponential. We begin with some notation. For a real number $r > 0$, we denote by Γ_r the circle on the complex plane centered at the origin and with radius r. Given a sequence $x_n \in H$, $n \in \mathbf{Z}$, define $R_+(x)$ and $R_-(x)$ as follows:

$$R_+(x) = \limsup_{n\to\infty} \|x_n\|^{1/n}$$

and

$$R_-(x) = (\limsup_{n\to\infty} \|x_{-n}\|^{1/n})^{-1}.$$

We admit for $R_+(x)$ and $R_-(x)$ to be equal to infinity. We shall say that a sequence $\{x_n\}_{n\in\mathbf{Z}}$ has an *exponential growth* if there are $a, b \geqslant 0$ such that $\|x_n\| \leqslant ab^{|n|}$ for any integer n. For $z \in \mathbf{C}$, denote by δ_z the point mass probability measure concentrated at the point $\{z\}$.

Now we can prove the main result of this section.

THEOREM 3. *If $x = \{x_n\}_{n\in\mathbf{Z}} \subset H$, then the following conditions are equivalent:*

(3.1) *x is harmonizable on a compact set,*

(3.2) *x has an exponential growth,*

(3.3) *$R_+(x) < \infty$ and $R_-(x) > 0$.*

If one of the above conditions holds, then

(3.4) *for any $r, R \in (0, \infty)$ such that $r < R_-(x)$ and $R > R_+(x)$ there is $h \in L^2(\mu_{\mathrm{rot}}, H)$, $\mu = \frac{1}{2}(\delta_r + \delta_R)$, such that*

(a) *$Ph = h$, where P denotes the orthogonal projection of $L^2(\mu_{\mathrm{rot}}, H)$ onto the closed span of $\bigcup_{n\in\mathbf{Z}} \bar{e}_n H$,*

(b) *$x_n = \int_{\Gamma_r \cup \Gamma_R} e_n h d\mu_{\mathrm{rot}}$.*

Moreover, for fixed r and R the function h having the properties (a) *and* (b) *is unique and takes the form*

(3.5) $$h = \sum_{n=-\infty}^{\infty} 2(r^{2n} + R^{2n})^{-1} \bar{e}_n x_n.$$

Proof. By the definition of generalized harmonizable process we see that there is a measure ξ supported on a compact subset K of C^* such that $x_n = \int_K e_n d\xi$. Since K is compact in C^*, there is $b > 0$ such that $|e_n(z)| \leqslant b^{|n|}$ for any $z \in K$ and any integer n. Consequently, $\|x_n\| \leqslant |\xi|(K) b^{|n|}$, which completes the proof of the implication (3.1)$\Rightarrow$(3.2).

The implication (3.2)$\Rightarrow$(3.3) is obvious. To show that (3.3) implies (3.4) observe first that $\beta_n = \frac{1}{2}(r^{2n} + R^{2n})$ is a nondegenerate moment sequence with representing measure μ. Since $R > R_+(x)$ and $r < R_-(x)$, one can show that the series $\sum_{k=-\infty}^{\infty} \beta_k^{-1} \|x_k\|^2$ converges. Hence the observation that $\mu_{\rm rot}$ is supported on $\Gamma_r \cup \Gamma_R$, together with Theorem 1, shows (3.4), the uniqueness of h and (3.5).

Since the implication (3.4)$\Rightarrow$(3.1) is obvious and we have just proved the implication (3.3)$\Rightarrow$(3.4), we infer that (3.3)$\Rightarrow$(3.1), which completes the proof. ■

Applying Theorem 3 and the Kolmogoroff factorization theorem we can state necessary and sufficient conditions for a matrix to be a covariance matrix of a sequence harmonizable on a compact set.

COROLLARY. *A matrix* $\{C(n, m)\}_{n,m\in Z}$ *is a covariance matrix of some sequence harmonizable on a compact set iff the following conditions hold*:

(a) $\{C(n, m)\}_{n,m\in Z}$ *is positive definite, i.e.*,

$$\sum_{n,m=-N}^{N} C(n, m)\lambda_n \bar{\lambda}_m \geqslant 0,$$

for any choice of $N \geqslant 0$ *and* $\lambda_{-N}, \ldots, \lambda_N \in C$;

(b) $\limsup C(n, n)^{1/n} < \infty$ *as* $n \to \infty$ *and* $\limsup C(-n, -n)^{1/n} < \infty$ *as* $n \to \infty$.

Although the main part of the paper concerns sequences harmonizable on a compact set, the more general setting developed in Section 2 is also important. Namely, we can give an example of a generalized harmonizable sequence which does not admit any representing measure on a compact set.

EXAMPLE 1. Let $\{x_n\}_{n\in Z}$ be an arbitrary sequence of vectors in H such that $\|x_n\|^4 = \gamma(|n|)$, $n \in Z$, where $\gamma(n) = \int_1^\infty t^n \varrho(t)dt$, $n \in Z$, and $\varrho\colon [0, \infty) \to R$ is defined by

$$\varrho(t) = \begin{cases} e^{-t} & \text{for } t \geqslant 1, \\ t^{-2} e^{-1/t} & \text{for } 0 \leqslant t < 1. \end{cases}$$

Then one can prove that the sequence $\{x_n\}_{n\in Z}$ satisfies the condition (2.3) with the nondegenerate moment sequence $\{\beta_n\}_{n\in Z}$ defined by $\beta(n) = \int_0^\infty t^n \varrho(t)dt$, $n \in Z$. By Theorem 1, the sequence $\{x_n\}_{n\in Z}$ is generalized harmonizable. Suppose for a moment that $\{x_n\}_{n\in Z}$ is harmonizable on a compact subset of C^*. Then (see Theorem 3) there exist positive real numbers a and b such that $\|x_n\|^4 \leqslant ab^{|n|}$, $n \in Z$. Thus $e^{-1} n! \leqslant \gamma(|n|) \leqslant ab^n$ for $n \geqslant 0$, which is impossible.

Remark 3. Any sequence $x\colon \boldsymbol{Z} \to H$ harmonizable on a compact set has a representing measure supported on two circles centered at the origin. If $R_+(x) < R_-(x)$, then x has a representing measure supported on one circle. If $R_+(x) < 1 < R_-(x)$, then x is harmonizable in the Rozanov sense.

Remark 4. Theorem 3 gives sufficient conditions for a sum of two circles $\Gamma_r \cup \Gamma_R$ to be a support of some representing measure of the process $\{x_n\}_{n\in \boldsymbol{Z}}$. The necessary conditions are a little weaker. Namely, if ξ is the representing measure of a generalized harmonizable sequence $x\colon \boldsymbol{Z} \to H$ and the closed support of ξ is contained in the sum $\Gamma_r \cup \Gamma_R$ with $0 < r < R$, then $0 < r \leqslant R_-(x)$ and $R_+(x) \leqslant R < \infty$. If the closed support of ξ is contained in Γ_r with some $r > 0$, then $R_+(x) \leqslant r \leqslant R_-(x)$.

Remark 5. Evidently, for a process x which is harmonizable on a compact set, a choice of the measure $\mu = \frac{1}{2}(\delta_r + \delta_R)$ (with r and R as in (3.4)) allows us to define the Stieltjes moment sequence $\{\beta_n\}_{n\in \boldsymbol{Z}}$ having the property (2.3). Thus Theorem 2 gives the direct proof of the existence of normal dilations for sequences harmonizable on compact sets. The result is exactly the same as Theorem 1.23 of [8].

Now we consider the question when the representing measure of a sequence harmonizable on a compact set may be located on its extremal circles. The problem of characterization of such sequences having an absolutely continuous representing measure is still open. Denote by m_r, $r > 0$, the normalized Lebesgue measure supported on Γ_r, i.e., $m_r = (\delta_r)_{\mathrm{rot}}$. The following equivalence is a simple consequence of Theorem 1 (ii).

If $x\colon \boldsymbol{Z} \to H$ and $0 < r \leqslant R < \infty$, then the following conditions are equivalent:

(3.6) There are $f \in L^2(m_r, H)$ and $g \in L^2(m_R, H)$ such that

$$x_n = \int_{\Gamma_r} e_n f \, dm_r + \int_{\Gamma_R} e_n g \, dm_R.$$

(3.7) $$\sum_{n=0}^{\infty} R^{-2n} \|x_n\|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} r^{2n} \|x_{-n}\|^2 < \infty.$$

It is easy to see that if we assume that in (3.7) the first series converges and the second one diverges, then we get square summable density on one circle and nonexistence of square summable density on the other. Evidently, any $R > R_+(x)$ and $0 < r < R_-(x)$ have the property (3.7). Thus the above is more interesting in the case $R = R_+(x)$ and $r = R_-(x)$. The square summable sequences considered in [8] are exactly those which have the property (3.7) with $r = R$ and the square summable sequences considered by Niemi in [5] are exactly those for which $r = R = 1$.

Now we prove some necessary conditions of the absolute continuity of representing measures on extremal circles. For a given measure ξ on $\Gamma_r \cup \Gamma_R$

($r < R$) we will write $\xi = \xi_r + \xi_R$, where ξ_r (ξ_R, respectively) is supported on Γ_r (Γ_R, respectively). If ξ is the representing measure of a generalized harmonizable sequence x, then

$$x_n = \int_{\Gamma_r} e_n d\xi_r + \int_{\Gamma_R} e_n d\xi_R.$$

PROPOSITION. *Suppose ξ is a representing measure of the sequence $\{x_n\}_{n\in Z}$ which is supported on $\Gamma_r \cup \Gamma_R$ $(0 < r \leqslant R < \infty)$. Then*

(i) *the sequences $\{\|x_n\|/R^n\}_{n=0}^{\infty}$ and $\{\|x_{-n}\| r^n\}_{n=0}^{\infty}$ are bounded;*

(ii) *if $\xi_R \ll m_R$ $(\xi_r \ll m_r)$, then*

$$\operatorname*{w\text{-}lim}_{n\to\infty} x_n R^{-n} = 0 \qquad (\operatorname*{w\text{-}lim}_{n\to\infty} x_{-n} r^n = 0).$$

Proof. Suppose that $r < R$. For a given measure η on Γ_R denote by $\tilde{\eta}$ the measure on Γ such that $\tilde{\eta}(E) = \eta(RE)$ for any Borel subset E of Γ. Then for $x \in H$ and $n \in Z$ we have

$$(x_n, x) = r^n \int_{\Gamma} z^n d(\tilde{\xi}_r(z), x) + R^n \int_{\Gamma} z^n d(\tilde{\xi}_R(z), x).$$

Consequently, $x_n = r^n(\tilde{\xi}_r)^\wedge(-n) + R^n(\tilde{\xi}_R)^\wedge(-n)$ for $n \in Z$, where $^\wedge$ denotes the Fourier transform of measure. Thus for positive n we have

$$\|x_n/R^n\| \leqslant (r/R)^n \|(\tilde{\xi}_r)^\wedge(-n)\| + \|(\tilde{\xi}_R)^\wedge(-n)\| \qquad \text{for } n \geqslant 0.$$

Since $\{\|(\tilde{\xi}_r)^\wedge(-n)\|\}_{n\in Z^+}$ and $\{\|(\tilde{\xi}_R)^\wedge(-n)\|\}_{n\in Z^+}$ are commonly bounded as the sequences of Fourier coefficients of bounded measures (see [2]), we get the boundedness of $\{\|x_n\|/R^n\}_{n=0}^{\infty}$. The proof of the other part of the condition (i) is similar.

To prove (ii) observe that if $(\xi_R, x) \ll m_R$, then $(\tilde{\xi}_R, x)$ is absolutely continuous with respect to the normalized Lebesgue measure m on Γ. Thus $d(\tilde{\xi}_R, x) = h_x dm$ with some $h_x \in L^1$. This implies that

$$(x_n, x) = r^n \int_{\Gamma} z^n d(\tilde{\xi}_r(z), x) + R^n \hat{h}_x(-n),$$

where $\hat{h}_x(n)$ denotes the n-th Fourier coefficient of h_x. Thus

$$|(R^{-n}x_n, x)| \leqslant (r/R)^n \left|\int_{\Gamma} z^n d(\tilde{\xi}_r(z), x)\right| + |\hat{h}_x(-n)|.$$

Since $r < R$, the integrals $\int_{\Gamma} z^n d(\tilde{\xi}_r(z), x)$ are commonly bounded and $\hat{h}_x(-n)$ tends to 0 for $n \to \infty$ as a sequence of Fourier coefficients of some L^1-function (see [2]), we get

$$\operatorname*{w\text{-}lim}_{n\to\infty} x_n R^{-n} = 0.$$

The proof of the other part of (ii) is analogous. Similar arguments can be used to prove the properties (i) and (ii) in the case $r = R$. ∎

Finally, we give two examples which explain what may happen with representing measure on the extremal circles. For this we prove lemma which shows some method of constructing harmonizable sequences on their extremal circles.

LEMMA. *Let $0 < r < R < \infty$. Suppose we are given two sequences $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ such that*

$$y_n = \int_{\Gamma_r} z^{-n} d\eta_r \ \text{for } n \geqslant 0, \qquad z_n = \int_{\Gamma_R} z^n d\eta_R \quad \text{for } n > 0,$$

where η_r and η_R are arbitrary measures supported on Γ_r and Γ_R, respectively. Then the sequence $\{x_n\}_{n\in\mathbf{Z}}$ defined by

$$x_n = \begin{cases} z_n, & n > 0, \\ y_{-n}, & n \leqslant 0, \end{cases}$$

is generalized harmonizable with the representing measure ξ on $\Gamma_r \cup \Gamma_R$ given by the formulas $\xi = \xi_r + \xi_R$, $\xi_R = \eta_R + h_R dm_R$, $\xi_r = \eta_r + h_r dm_r$, where $h_R \in L^2(m_R, H)$ and $h_r \in L^2(m_r, H)$ take the form

$$h_R(z) = -\sum_{m=1}^{\infty} a_m z^{-m}, \qquad z \in \Gamma_R,\ a_m = \int_{\Gamma_r} e_m d\eta_r,\ m \geqslant 1, \tag{3.8}$$

$$h_r(z) = -\sum_{m=0}^{\infty} b_m z^m, \qquad z \in \Gamma_r,\ b_m = \int_{\Gamma_R} e_{-m} d\eta_R,\ m \geqslant 0. \tag{3.9}$$

Proof. First we prove that the series in (3.8) converges in $L^2(m_R, H)$. Since the summands in the series (3.8) are orthogonal and

$$\|a_m z^{-m}\|^2_{L^2(m_R,H)} \leqslant (r/R)^{2m} |\eta_r|(\Gamma_r)^2,$$

the series (3.8) converges. Similarly we show the convergence of the series in (3.9). Now for $n > 0$ we have

$$\begin{aligned} &\int_{\Gamma_r \cup \Gamma_R} z^n d\xi(z) \\ &= \int_{\Gamma_R} z^n d\eta_R(z) + \int_{\Gamma_R} z^n h_R(z) dm_R(z) + \int_{\Gamma_r} z^n d\eta_r(z) + \int_{\Gamma_r} z^n h_r(z) dm_r(z) \\ &= x_n - \sum_{m=1}^{\infty} a_m \int_{\Gamma_R} z^{n-m} dm_R(z) + \int_{\Gamma_r} z^n d\eta_r(z) - \sum_{m=0}^{\infty} b_m \int_{\Gamma_r} z^{n+m} dm_r(z) \\ &= x_n - a_n + a_n - 0 = x_n. \end{aligned}$$

Analogously we prove that

$$x_n = \int_{\Gamma_r \cup \Gamma_R} z^n d\xi(z) \qquad \text{for } n \leqslant 0. \ \blacksquare$$

The following example is an application of the Lemma.

EXAMPLE 2. Let $H = C$, $0 < r < R < \infty$ and $\eta_R = \delta_R$, $\eta_r = \delta_r$. Define

$$x_n = \begin{cases} R^n & \text{for } n > 0, \\ r^n & \text{for } n \leqslant 0. \end{cases}$$

Then $R_+(\{x_n\}) = R$ and $R_-(\{x_n\}) = r$. It follows from the Lemma that

$$x_n = \int_{\Gamma_r \cup \Gamma_R} z^n d\xi(z).$$

Thus the sequence $\{x_n\}_{n\in Z}$ has a representing measure on its extremal circles. Since

$$\lim_{n\to\infty} x_n R^{-n} = 1 \quad \text{and} \quad \lim_{n\to\infty} x_{-n} r^n = 1,$$

we can use the Proposition (ii) to see that $\{x_n\}_{n\in Z}$ has no representing measure absolutely continuous with respect to the Lebesgue measure on $\Gamma_r \cup \Gamma_R$.

The next example shows that there are sequences which have no representing measure on their extremal circles.

EXAMPLE 3. Let $H = C$, $0 < r < R < \infty$ and

$$x_n = \begin{cases} nR^n & \text{for } n > 0, \\ r^n & \text{for } n \leqslant 0. \end{cases}$$

Then $R_+(\{x_n\}) = R$ and $R_-(\{x_n\}) = r$. Since the sequence $\|x_n\|/R^n = n$ for $n > 0$ is unbounded, we can infer from the Proposition (i) that the process $\{x_n\}_{n\in Z}$ has no representing measure on $\Gamma_r \cup \Gamma_R$. It follows from Theorem 3 applied to the sequence

$$\tilde{y}_n = \begin{cases} x_n & \text{for } n \geqslant 0, \\ 0 & \text{for } n < 0 \end{cases}$$

that for $R_1 > R$ and $n > 0$ we have $x_n = \int_{\Gamma_{R_1}} e_n d\eta_{R_1}$ with some η_{R_1} defined on Γ_{R_1}. Evidently, for $n \leqslant 0$ we have $x_n = \int_{\Gamma_r} z^n d\delta_r(z)$. Thus, by the Lemma, $\{x_n\}_{n\in Z}$ has a representing measure on $\Gamma_r \cup \Gamma_{R_1}$.

It is easy to see that using the method as in Example 3 we can also show that the sequences defined by the formulas

$$x_n = \begin{cases} R^n & \text{for } n > 0, \\ nr^n & \text{for } n \leqslant 0 \end{cases}$$

and

$$x_n = \begin{cases} nR^n & \text{for } n > 0, \\ nr^n & \text{for } n \leqslant 0 \end{cases}$$

have no representing measures on their extremal circles. For the last example even taking one circle not being extremal does not help.

Remark 6. Most results of our paper (like Theorem 3) can be reformulated for generalized harmonizable one-sided sequences $x\colon \boldsymbol{Z}^+ \to H$ by replacing $\boldsymbol{Z}$ by $\boldsymbol{Z}^+$ and two circles by one (using only R_+). In particular, any one-sided sequence harmonizable on a compact set has a representing measure supported on one circle centered at the origin. Moreover, a sequence $x\colon \boldsymbol{Z} \to H$ is harmonizable on a compact set iff the sequences $y\colon \boldsymbol{Z}^+ \to H$ and $z\colon \boldsymbol{Z}^+ \to H$ defined by $y_n = x_n$ and $z_n = x_{-n}$ are harmonizable on compact sets.

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