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SOME REMARKS ON MEASURES WITH n-DIMENSIONAL VERSIONS

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Abstract. A nondegenerate probability measure ν on $\mathbf{R}^n$ is an n-dimensional version of a symmetric measure μ on $\mathbf{R}$ if there exists $c\colon \mathbf{R}^n \to [0, \infty)$ such that $\hat{\nu}(ta) = \hat{\mu}(|t| c(a))$, $t \in \mathbf{R}$, $a \in \mathbf{R}^n$. If the function c is an L_p-norm on $\mathbf{R}^n$, we call the measure ν *p-elliptically contoured*. The main result of this paper is that if μ has an ε-order for $\varepsilon > 0$, then every its n-dimensional version is p-elliptically contoured for some $p \in (0, 2]$. We show also that $\operatorname{supp}(\mu) = \mathbf{R}$ if only μ has an n-dimensional version which is not 2-elliptically contoured.

Distributions on $\mathbf{R}^n$ having all one-dimensional projections the same up to a scale parameter play a particular role in statistics and probability theory. For example, symmetric Gaussian measures and symmetric stable measures have this property. The investigation of this class of measures was started by Eaton [4] in 1981 and continued by Cambanis et al. [2] in 1983. It is still unknown however how large this class is, and this paper is devoted to the investigation of some its properties.

By a *nondegenerate distribution* on $\mathbf{R}^n$ we will understand a distribution for which the linear support is equal to $\mathbf{R}^n$. By $\mathscr{L}(X)$ we denote the distribution of a random vector X.

DEFINITION 1. The nondegenerate distribution ν of a symmetric random vector $(X_1, \ldots, X_n) \in \mathbf{R}^n$ is said to be an *n-dimensional version of a symmetric distribution μ of a random variable $X \in \mathbf{R}$* if for every $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$ there exists $c(a) \geqslant 0$ such that

$$\mathscr{L}\left(\sum a_i X_i\right) = \mathscr{L}\left(c(a) X\right)$$

or, equivalently,

$$\hat{\nu}(ta) = \hat{\mu}\left(c(a)t\right), \quad a \in \mathbf{R}^n,\ t \in \mathbf{R},$$

where $\hat{\nu}$ and $\hat{\mu}$ are the corresponding characteristic functions.

In general, we know very little about the function c. It is known (see [4]) that $c(ta) = |t| c(a)$ for every $t \in \mathbf{R}$ and $a \in \mathbf{R}^n$. It is almost evident also that c is a continuous function on $\mathbf{R}^n$, so it is equivalent to any norm on $\mathbf{R}^n$.

There are very close connections between measures μ having n-dimensional versions and symmetric stable measures on $\mathbf{R}$, as the above definition is almost the same as the definition of stable distribution. The only difference is that we do not assume here the independence of X_i's. It is trivial then that if at least two of X_i's are independent, then all X_i's and X are symmetric and stable on R^n.

If ν is a symmetric p-stable measure on R^n, then (see, e.g., [6]) its characteristic function is of the form

$$\hat{\mu}(a) = \exp\{-c(a)^p\}, \quad a \in R^n,$$

where

$$(*) \qquad c(a)^p = \int_{S^{n-1}} \ldots \int |\langle a, x\rangle|^p \lambda(dx), \quad a \in R^n,$$

for some finite measure λ on the unit sphere $S^{n-1} \in R^n$. This means that every symmetric p-stable measure ν on R^n is an n-dimensional version of the symmetric p-stable measure γ_p on R with the characteristic function $\exp\{-|t|^p\}$. Moreover, every n-dimensional version of the measure γ_p is symmetric and p-stable as stable is a distribution having all one-dimensional projections symmetric and p-stable.

As we can see it will not be surprising if it turns out that every function c: $R^n \to [0, \infty)$, appearing in Definition 1, is given by the formula $(*)$ for some $p > 0$ and a finite measure λ on $S^{n-1} \subseteq R^n$. In fact, as far as we know, there exists no example of a measure ν on R^n being an n-dimensional version of some measure μ on R with the function c which cannot be written in the form $(*)$ for any $p \in (0, 2]$ and any finite measure λ. That is why we introduce the following

Definition 2. A symmetric measure ν on R^n is called *p-elliptically contoured*, $p > 0$, if its characteristic function is of the form

$$\hat{\nu}(ta) = f\big(tc(a)\big), \quad a \in R^n, \ t \in R,$$

where f: $[0, \infty) \to R$ is a continuous function and $c(a)$ is given by the formula $(*)$ for some finite measure λ on S^{n-1}.

There is a full characterization of p-elliptically contoured measures in finite and infinite dimensional spaces for $p = 2$ and $p = 1$ (see [2], [10]–[12]). In [1] one can find a full characterization of measures on R having n-dimensional p-elliptically contoured version for every $n \in N$. But we know very little about p-elliptically contoured measures on R^n if $p \notin \{1, 2\}$ and $n \in N$ is fixed.

Now let c: $R^n \to [0, \infty)$. We define $M(c, n)$ as the set of all probability measures μ on R having an n-dimensional version ν on R^n with a given function c, i.e., such that

$$\hat{\nu}(ta) = \hat{\mu}\big(tc(a)\big), \quad a \in R^n, \ t \in R.$$

It is easy to see that the set $M(c, n)$ is convex, weakly sequentially closed and closed with respect to convolution. The following theorem asserts that every n-dimensional version of a measure having p-th order, $p \in (0, 2]$, is p-elliptically contoured.

THEOREM 1. *Assume that there exists* $\varepsilon > 0$ *and* $\mu_0 \in M(c, n)$, $\mu_0 \neq \delta_0$, *such that* $\int |x|^\varepsilon \mu_0(dx) < \infty$. *Then there exists* $p \in (0, 2]$ *such that* $c(a)$ *can be given by the formula* (*) *for some finite measure* λ *on* S^{n-1}. *Moreover, if* p_0 *is the greatest such* p, *then* $\gamma_q \in M(c, n)$ *for every* $q \leqslant p_0$.

Proof. Without loss of generality we can assume that $\varepsilon \leqslant 2$ and that $\int |x|^\varepsilon \mu_0(dx) = 1$. If the measure ν on $\mathbf{R}^n$ is the n-dimensional version of the measure μ_0, then for every $a \in \mathbf{R}^n$ we have

$$c(a)^p = \int |c(a)x|^p \mu_0(dx) = \int \ldots \int |\langle a, x\rangle|^p \nu(dx).$$

Now, in the usual way (see, e.g., [6]) we construct an infinitely divisible probability measure $\exp\{m\}$ on $\mathbf{R}^n$ as the weak limit of measures $\exp\{m_\delta\}$ when $\delta \searrow 0$, where

$$m_\delta(A) = \int_\delta^\infty \nu(A/s) s^{-\varepsilon-1} ds, \qquad A \in \mathscr{B}(\mathbf{R}^n).$$

We obtain

$$\begin{aligned}[\exp\{m\}]^\wedge(ta) &= \exp\{-\int \ldots \int_{\mathbf{R}^n} \int_0^\infty (1-\cos\langle ta, sx\rangle) s^{-\varepsilon-1} ds\, \nu(dx)\} \\ &= \exp\{-\int \ldots \int_{\mathbf{R}^n} |t\langle a, x\rangle|^\varepsilon \nu(dx)\} \\ &= \exp\{-|t|^\varepsilon c(a)^\varepsilon\}.\end{aligned}$$

We see then that $\exp\{-c(a)^\varepsilon\}$ is a positive definite function on $\mathbf{R}^n$ (as the characteristic function of the measure $\exp\{m\}$), so the function $c(a)^\varepsilon$ is negative definite on $\mathbf{R}^n$. We define

$$p = \sup\{\varepsilon \in (0, 2]: c(a)^\varepsilon \text{ is negative definite on } \mathbf{R}^n\}.$$

As the limit of negative definite functions is also negative definite, it follows that $c(a)^p = \lim c(a)^\varepsilon$ as $\varepsilon \to p$ is negative definite on $\mathbf{R}^n$, and $\exp\{-c(a)^p\}$ is the characteristic function of some probability measure ν_p on $\mathbf{R}^n$.

Observe that all one-dimensional projections of ν_p are symmetric, p-stable and belong to $M(c, n)$. Hence (see [8] and [9]) ν_p is p-stable, so there exists a finite measure λ on S^{n-1} such that

$$c(a)^p = \int \ldots \int_{S^{n-1}} |\langle a, x\rangle|^p \lambda(dx), \qquad a \in \mathbf{R}^n.$$

Now $\gamma_p \in M(c, n)$. To see that $\gamma_q \in M(c, n)$ for every $0 < q \leqslant p$ notice that the following measure is an n-dimensional version of the measure γ_q with the same function c as for ν_p:

$$\nu_p \circ \gamma_{q/p}^{+}(A) := \int \nu_p(As^{-1/p})\mu_{q/p}(ds), \qquad A \in \mathscr{B}(\boldsymbol{R}^n),$$

where $\gamma_{q/p}^{+}$ is the (q/p)-stable measure on $(0, \infty)$ with the Laplace transform $\exp\{-t^{q/p}\}$. Indeed, we have

$$(\nu_p \circ \mu_{q/p})^{\wedge}(a) = \int_0^{\infty} \nu_p(s^{1/p}a)\gamma_{q/p}^{+}(ds)$$

$$= \int_0^{\infty} \exp\{-sc(a)^p\}\gamma_{q/p}^{+}(ds) = \exp\{-c(a)^q\}. \blacksquare$$

The maximal p we have found in Theorem 1 is a characterizing constant of the set $M(c, n)$ or of the function c on $\boldsymbol{R}^n$. Therefore, let us define

$$p(c) = \sup\{p \in (0, 2]: \exists \mu \in M(c, n), \mu \neq \delta_0, \int |x|^p \mu(dx) < \infty\}$$

or, equivalently,

$$p(c) = \sup\{p \in (0, 2]: c(a)^p \text{ is negative definite on } \boldsymbol{R}^n\},$$

where $\sup \emptyset = 0$. Now, if $p(c) > 0$, then every n-dimensional version of any measure from $M(c, n)$ has to be $p(c)$-elliptically contoured. So only in the case $p(c) = 0$ maybe we would be able to find c which is not any L^p-norm for any $p \in (0, 2]$. In 1985 Kuritsyn and Schestiakov [7] showed that the function $\exp\{-(|x|^p + |y|^p)^{1/p}\}$ is a characteristic function for every $p > 2$. They expressed in this way the fact that every two-dimensional normed space embeds isometrically into some L^1-space or, equivalently, that every norm on $\boldsymbol{R}^2$ is negative definite. The two-dimensional measures obtained in [7] are special cases of 1-elliptically contoured measures. So the problem whether or not there exists an n-dimensional version of a symmetric measure on $\boldsymbol{R}$ other than p-elliptically contoured, $p \in (0, 2]$, remains open.

The following result gives us some more information about measures having an n-dimensional version.

THEOREM 2. *Let $\mu \in M(c, n)$, $\mu \neq \delta_0$, $n \geqslant 2$, and let ν be an n-dimensional version of μ. Then either* $\mathrm{supp}(\mu)$ *is a compact set (and then ν is 2-elliptically contoured) or* $\mathrm{supp}(\mu) = \boldsymbol{R}$.

Proof. It is easy to see that if $\mu \in M(c, n)$, $n \geqslant 2$, then $\mu \in M(c', 2)$, where $c'(a) = c((a_1, a_2, 0, \ldots, 0))$ for $a = (a_1, a_2) \in \boldsymbol{R}^2$. Assume then, without loss of generality, that $\mu \in M(c, 2)$, $c(1, 0) = 1$, and ν is a two-dimensional version of μ. Since

$$\iint \exp\{i\langle ta, x\rangle\}\nu(dx) = \int \exp\{ic(ta)x\}\mu(dx) = \hat{\mu}(tc(a)),$$

for every $t, s \in \boldsymbol{R}$, $t < s$, we have

$$\nu\left\{t < \frac{\langle a, x\rangle}{c(a)} < s\right\} = \mu\{t < x < s\}.$$

Suppose now that $\operatorname{supp}(\mu) \neq \boldsymbol{R}$; then there exist $t, s \in \boldsymbol{R}$, $t < s$, such that $\mu\{t < x < s\} = 0$ (by symmetry of μ we can assume that $t > 0$). The sets

$$A(a) = \left\{t < \frac{\langle a, x\rangle}{c(a)} < s\right\}, \quad a \in \boldsymbol{R}^2,$$

are open cylinders in $\boldsymbol{R}^2$ and it is easy to see that

$$\{x \in \boldsymbol{R}^2 \colon \|x\| > Mt\} \subseteq \bigcup_a A(a),$$

where $M = \sup\{c(a)\colon \|a\| = 1, a \in \boldsymbol{R}^2\}$ and $\|\cdot\|$ is the Euclidean norm on $\boldsymbol{R}^2$. Now let $K \subseteq \{x \in \boldsymbol{R}^2\colon \|x\| > Mt\}$ be a compact set. There exists a finite set $a_1, \ldots, a_k \in \boldsymbol{R}^2$ such that $K \subseteq \bigcup A(a_i)$ and we obtain

$$\nu(K) \leqslant \sum \nu(A(a_i)) = 0.$$

This means that μ as well as ν have compact supports, so they in particular have the second moment, and then

$$\int\int |\langle a, x\rangle|^2 \nu(dx) = \int |c(a)x|^2 \mu(dx) = c(a)^2 \int |x|^2 \mu(dx) < \infty.$$

Consequently, the function $c(a)$ is given by an L^2-norm on $\boldsymbol{R}^2$, i.e., the measure ν is 2-elliptically contoured. ■

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