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AN INTERACTING FREE FOCK SPACE AND THE ARCSINE LAW

BY

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Abstract. Motivated by previous investigations of the *interacting* central limit theorem for the quantum Bernoulli process and of the stochastic limit of quantum electrodynamics, we construct some examples of interacting free Fock spaces which realize the first non-Gaussian (neither free, nor Bose nor Fermi or q-deformed) examples of quantum independent increment processes: the mixed momenta are not expressible as products of pair correlations. We give general rules to compute the vacuum expectation of products of creation and annihilation operators. By these rules, any moment of field operator becomes computable. We also obtain the precise expression of the distribution of the field operator. This is *not* the Wigner semi-elliptical law (even if we start from the free Fock space) but in some sense its reciprocal, that is the *arcsine* law.

1. Introduction. *Interacting free Fock* (IFF) space was motivated originally by investigation of the stochastic limit of the quantum electromagnetic field (see [1] and [2]). Later, some example of IFF space appears in consideration of central limit behaviour of the quantum Bernoulli process (see [3]). In [4], IFF space is studied systematically.

It is well known that the *usual free Fock* (let us say UFF) space over a given (pre-)Hilbert space $\mathscr{H}$ is the (pre-)Hilbert space

$$\Gamma(\mathscr{H}) := C \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n}. \tag{1.1}$$

On the n-th space $\mathscr{H}^{\otimes n}$, the usual tensor Hilbert structure is defined. Such a Fock structure comes from the free central limit theorem and is studied by many authors (see [7] and the references therein).

Remark. In the present paper, we shall not distinguish a pre-Hilbert space from its completion, just we say Hilbert space.

Similar to the UFF case, the IFF space, suggested by the central limit theorem of quantum Bernoulli process and the stochastic limit of quantum electromagnetic field, over a given Hilbert space $\mathscr{H}$ is defined as

$$\Gamma_I(\mathscr{H}) := C \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}_n, \tag{1.2}$$

where $\mathscr{H}_1 = \mathscr{H}$, for any $n \geqslant 2$, algebraically $\mathscr{H}_n = \mathscr{H}^{\odot n}$ but on which a different scalar product is defined.

In order to be precise, we shall restrict ourselves in this paper to the following case:

(i) $\mathscr{H} := L^2(M, d\mu)$, where M is a measurable space, and μ is a σ-finite measure on M;

(ii) there exists a sequence of measurable functions $\{\lambda_n(x_1, x_2, \ldots, x_n)\}_{n=1}^{\infty}$ (λ_n: $M^n \to \boldsymbol{R}_+$ is called an *n-th interacting function*) and two sequences of positive numbers $\{b_n, d_n\}_{n=1}^{\infty}$ such that, for any $n \in N$,

$$\frac{\lambda_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})}{\lambda_1(x_1)\cdot\lambda_n(x_2, \ldots, x_n, x_{n+1})} \leqslant d_n, \quad 0 \leqslant \lambda_n \leqslant b_n, \tag{1.3}$$

where (and in the following) we adopt that $0/0 := 1$;

(iii) the n-th Hilbert space $\mathscr{H}_n$ is obtained by introducing on the vector space $\mathscr{H}^{\odot n}$ the scalar product

$$\langle F_n, G_n\rangle := \int \mu(dx_1)\ldots\mu(dx_n)\,\lambda_n(x_1, \ldots, x_n)(\bar{F}_n G_n)(x_1, \ldots, x_n), \tag{1.4}$$

where $F_n, G_n \in \mathscr{H}^{\odot n}$, $\odot$ means the algebraic tensor product, and on $\mathscr{H}_n$ the usual procedure is assumed to be done, i.e. take the quotient space and make completion with respect to the scalar product defined by (1.4).

Of course, if we take $\lambda_n(x_1, x_2, \ldots, x_n)$ as $f(x_1)\ldots f(x_n)$ for some positive measurable function $f \in L^2(M, d\mu)$, the IFF space $\Gamma_I(L^2(M, d\mu))$ becomes a UFF space.

In the present paper, first of all we introduce basic concepts concerning IFF space and discuss some of its general properties. Then, in Section 3, we investigate the IFF space over the Hilbert space $L^2([0, T])$ with interacting function

$$\lambda_n(x_1, x_2, \ldots, x_n) = \chi_{\Delta_n}(x_1, x_2, \ldots, x_n),$$

where

(i) $0 \leqslant T \leqslant +\infty$, and if $T = +\infty$, for any $a \in [0, T)$ the intervals $[a, T]$ and $(a, T]$ are understood as $[a, +\infty)$ and $(a, +\infty)$, respectively;

(ii) by Δ_n we denote the subset of $[0, T]^n$ in which $x_1 < x_2 < \ldots < x_n$.

In this case, we give rules to calculate the vacuum expectation of the function of creation and annihilation operators. As an application of the general rules, we show that, for any $t \in [0, T]$ and any continuous function ξ, the distribution of the field operator with test function $\xi\chi_{[0,t)}$ has the probability density function

$$\frac{1}{\pi\sqrt{2\eta_t - x^2}}\chi_{(-\sqrt{2\eta_t}, \sqrt{2\eta_t})}(x),$$

where $\eta_t := \int_0^t |\xi|^2(s)\,ds$, and for $\eta_t = 0$ the above distribution is considered as the one-point distribution at zero. In particular, for any $t \in (0, T]$, the distribution of the field operator with the test function $\chi_{[0,t)}$ has the probability density function

$$\frac{1}{\pi\sqrt{2t-x^2}}\chi_{(-\sqrt{2t},\sqrt{2t})}(x).$$

This is nothing else but the density function of the arcsine law (or distribution).

A quantum stochastic calculus theory on IFF space is investigated in [5] and some other interacting Fock spaces are discussed in [6].

2. Some discussions on general IFF space. In this section, we give some general properties of IFF space over a certain Hilbert space $L^2(M, \mu)$. Some of them have been studied in [4].

DEFINITION 2.1. The vector

$$\Phi := 1 \oplus 0 \oplus 0 \oplus \ldots$$

is called the *vacuum* of the IFF space $\Gamma_I(\mathscr{H})$. For each $g \in \mathscr{H}$, the linear operator $A^+(g)$, defined by

$$[A^+(g)G_n](x_1, x_2, \ldots, x_{n+1}) := g(x_1)G_n(x_2, \ldots, x_{n+1}) \tag{2.1}$$

for any $n \in N$ and $G_n \in \mathscr{H}_n$,

is called the *creation operator* (with the test function $g \in \mathscr{H}$). In particular, for any $g_1, \ldots, g_n \in \mathscr{H}$,

$$[A^+(g)g_1 \odot g_2 \odot \ldots \odot g_n] = g \odot g_1 \odot \ldots \odot g_n.$$

Remark. By our assumptions on the interacting functions, for any $g \in \mathscr{H}$ the creation operator $A^+(g)$ brings the null element of the $(n+1)$-st space to zero, and therefore the creation operator is well defined.

We have proved in [4] the following result:

LEMMA 2.2. *For any $g \in \mathscr{H}$, the creation operator $A^+(g)$ maps $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$ and is bounded on each $\mathscr{H}_n$:*

$$\|A^+(g)\| \leqslant \sqrt{d_n} \cdot \|g\|. \tag{2.2}$$

Given a test function $g \in \mathscr{H}$, the creation operator $A^+(g)$ is not necessarily bounded on the IFF space $\Gamma_I(\mathscr{H})$ (of course, it is bounded on each $\mathscr{H}_n$) and the boundedness depends on the sequence of the interacting functions $\{\lambda_n\}_{n=1}^{\infty}$. But the discussion in [4] makes sure that $A^+(g)$ is an operator densely defined on the IFF space $\Gamma_I(\mathscr{H})$, and the set

$$\Gamma_0 := \{ \sum_{n=0}^{N} c_n G_n \colon N \in \boldsymbol{N},\ G_n \in \mathscr{H}_n,\ c_n \in \boldsymbol{C},\ n = 0, 1, 2, \ldots\} \tag{2.3}$$

is included in $\mathscr{D}(A^+(g))$, the domain of $A^+(g)$.

Since $A^+(g)$ is an operator densely defined, its essential adjoint exists; it will be denoted by $A(g)$ and called the *annihilation operator* (with the test function $g \in \mathscr{H}$). For any $g \in \mathscr{H}$, the annihilation operator $A(g)$ has the following properties:

$$A(g)\mathscr{H}_0 = 0, \quad A(g)\colon \mathscr{H}_n \to \mathscr{H}_{n-1}, \tag{2.4a}$$

$$\Gamma_0 \subset \mathscr{D}(A(g)), \tag{2.4b}$$

and moreover, for any $n = 1, 2, \ldots, m \in \boldsymbol{N}$, $F_{m-1} \in \mathscr{H}_{m-1}$, $G_n \in \mathscr{H}_n$,

$$\langle F_{m-1}, A(g) G_n \rangle = \delta_{n,m} \langle g \odot F_{m-1}, G_n \rangle. \tag{2.4c}$$

On the IFF space, although the creation operator is defined as that on UFF space, the annihilation operator takes a different form and it depends strongly on the interacting functions. In fact, we have

LEMMA 2.3. *For any $n \in \boldsymbol{N}$, the annihilation operator $A(f)$ on the $(n+1)$-st space $\mathscr{H}_{n+1}$ is bounded and for the bound $\sqrt{d_{n+1}} \cdot \|f\|$ and for any $G_{n+1} \in \mathscr{H}_{n+1}$*

$$\begin{aligned}(A(f) G_{n+1})(x_1, \ldots, x_n) \\ = \int \mu(dx) \frac{\lambda_{n+1}(x, x_1, \ldots, x_n)}{\lambda_n(x_1, \ldots, x_n)} \bar{f}(x) G_{n+1}(x, x_1, \ldots, x_n),\end{aligned} \tag{2.5}$$

where (and in the following) the 0-th interacting function λ_0 is defined to be 1.

Remark. The proof is essentially the same as that of the lemmata (2.4) and (2.5) of [4].

In the following, we consider creation and annihilation operators only on Γ_0 (i.e. we do not distinguish the operators themselves from their restrictions on Γ_0), and therefore they are the adjoints of each other.

Now we try to understand the vacuum expectation of a product of many creation and annihilation operators:

$$\langle \Phi, A^{\varepsilon(1)}(f_1) \ldots A^{\varepsilon(m)}(f_m) \Phi \rangle, \tag{2.6}$$

where $m \in \boldsymbol{N}$, $\varepsilon \in \{0, 1\}^m$, $f_1, \ldots, f_m \in \mathscr{H}$ and, for any $\varepsilon' \in \{0, 1\}$,

$$A^{\varepsilon'} := \begin{cases} A & \text{if } \varepsilon' = 0, \\ A^+ & \text{if } \varepsilon' = 1. \end{cases} \tag{2.7}$$

As a consequence of the following facts: the creation operator sends $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$, and the annihilation operator sends $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$ and brings $\mathscr{H}_0$ to zero (in the following we shall call them the *basic facts*) we know that (2.6) is equal to zero if m is odd. Thus we need only to study

$$(2.8)\qquad \langle \Phi, A^{\varepsilon(1)}(f_1)\ldots A^{\varepsilon(2n)}(f_{2n})\Phi\rangle .$$

Again due to the basic facts we know that the quantity (2.8) differs from zero only if $\varepsilon\in\{0, 1\}_+^{2n}$, where $\{0, 1\}_+^{2n}$ is the subset of $\{0, 1\}^{2n}$ determined by the following properties:

(i) $\sum_{k=1}^{2n}\varepsilon(k) = n$;

(ii) for any $k\in\{1, 2, \ldots, 2n\}$,

$$|\{j\geqslant k\colon \varepsilon(j) = 1\}| \geqslant |\{j\geqslant k\colon \varepsilon(j) = 0\}|.$$

Remark. The condition (i) means that in (2.8) there are n creation operators and n annihilation operators; the condition (ii) means that, for any $k\in\{1, 2, \ldots, 2n\}$, among the operators $A^{\varepsilon(k)}(f_k)$, $A^{\varepsilon(k+1)}(f_{k+1}), \ldots, A^{\varepsilon(2n)}(f_{2n})$ the number of annihilation operators is not greater than the number of creation operators.

In [4] (see also [2]) we have proved that each $\varepsilon\in\{0, 1\}_+^{2n}$ determines exactly *one* non-crossing pair partition $\{(l_h, r_h)\}_{h=1}^{n}$ on the set $\{1, 2, \ldots, 2n\}$ (which is described by the usual non-crossing principle: for each $k<h$, if $l_h<r_k$, then $r_h<r_k$) such that

$$\{l_h\}_{h=1}^{n} = \{j\in\{1, 2, \ldots, 2n\}\colon \varepsilon(j) = 0\}.$$

Moreover, one of the two sets $\{l_h\}_{h=1}^{n}$, $\{r_h\}_{h=1}^{n}$ can be assumed to be ordered and we shall assume that $l_1<l_2<\ldots<l_n$; $\{l_h\}_{h=1}^{n}$ (respectively, $\{r_h\}_{h=1}^{n}$) will be called the *left* (respectively, *right*) *index-set* of the non-crossing pair partition.

Theorem (2.12) of [4] states that for any $\varepsilon\in\{0, 1\}_+^{2n}$ there exists a function of n variables

$$K(\{l_h, r_h\}_{h=1}^{n};\ y_1, \ldots, y_n)$$

(where $\{l_h, r_h\}_{h=1}^{n}$ is the left-right index-set of the $\varepsilon\in\{0, 1\}_+^{2n}$) such that the quantity (2.8) is equal to

$$(2.9)\qquad \int \mu(dy_1)\ldots\mu(dy_n)\prod_{h=1}^{n}\big(\bar{f}_{l_h}(y_h) f_{r_h}(y_h)\big)\cdot K(\{l_h, r_h\}_{h=1}^{n};\ y_1, \ldots, y_n).$$

Moreover, the function K is determined uniquely by the $\varepsilon\in\{0, 1\}_+^{2n}$ (or, equivalenty, by $\{l_h, r_h\}_{h=1}^{n}$) and the first n interacting functions $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Now we shall give some principles to compute the vacuum expectation of the product of creation and annihilation. This is equivalent to determining the function K in (2.9).

It is well known that creation and annihilation operators on the usual (uninteracting) Boson, Fermion and free Fock spaces (over a certain Hilbert space $\mathscr{H}$) have an important property: for any $m\in N$, $g_1, \ldots, g_m\in\mathscr{H}$ and

$\varepsilon \in \{0, 1\}^m$, the vacuum expectation of the product

$$A^{\varepsilon(1)}(g_1)\dots A^{\varepsilon(m)}(g_m) \tag{2.10}$$

vanishes certainly in the following two cases:

(i) m is odd;

(ii) $m = 2n$ for some $n \in N$ but $\varepsilon \in \{0, 1\}^{2n} \backslash \{0, 1\}^{2n}_+$.

In the case of $\varepsilon \in \{0, 1\}^{2n}_+$ with the left index-set $\{l_h\}_{h=1}^n$, the vacuum expectation of (2.10) takes the form

$$\sum_{\{l_h, r_h\}_{h=1}^n \in \{\text{p.p.}2n\}} f(\{l_h, r_h\}_{h=1}^n) \prod_{h=1}^n \langle \Phi, A(g_{l_h}) A^+(g_{r_h}) \Phi \rangle, \tag{2.11}$$

where (here and in the sequel) {p.p.$2n$} means the totality of all pair partitions on the set $\{1, 2, \dots, 2n\}$. The factor $f(\{l_h, r_h\}_{h=1}^n)$ depends on the Fock structure:

(i) $f(\{l_h, r_h\}_{h=1}^n) = 1$ if we consider the Boson case;

(ii) $f(\{l_h, r_h\}_{h=1}^n) = (-1)^{\|\{l_h, r_h\}_{h=1}^n\|}$ if we consider the Fermion case, where $\|\{l_h, r_h\}_{h=1}^n\|$ is the index of the permutation

$$\{1, 2, 3, 4, \dots, 2n-1, 2n\} \to \{l_1, r_1, l_2, r_2, \dots, l_n, r_n\};$$

(iii) in the UFF case,

$$f(\{l_h, r_h\}_{h=1}^n) = \begin{cases} 1 & \text{if } \{l_h, r_h\}_{h=1}^n \in \{\text{n-c.p.p.}2n\}, \\ 0 & \text{otherwise}, \end{cases}$$

where (and in the following) by {n-c.p.p.$2n$} we denote the totality of all non-crossing pair partitions on the set $\{1, 2, \dots, 2n\}$.

Roughly speaking, in all the above three cases, any $(2n-1)$-point function is equal to zero and any $2n$-point function equal a finite sum of products of n two-point functions. In other words, under the vacuum state, creation and annihilation operators have the (mean-zero) Gaussianity property.

Now let us consider the IFF space. It is clear that, under the vacuum state, creation and annihilation operators do not, in general, have the above property. For example, in the case of $n = 2$, the expression

$$\langle \Phi, A(g_1) A(g_2) A^+(g_3) A^+(g_4) \Phi \rangle \tag{2.12}$$

is equal to

$$\int \mu(dx) \mu(dy) (\bar{g}_2 \cdot g_3)(x) (\bar{g}_1 \cdot g_4)(y) \lambda_2(x, y),$$

and it is in general impossible to represent it in a form like

$$\sum_{\sigma \in \mathscr{S}_4} c(g_1, g_2, g_3, g_4; \sigma) \int \mu(dx) \mu(dy) (\bar{g}_{\sigma(2)} \cdot g_{\sigma(3)})(x) (\bar{g}_{\sigma(1)} \cdot g_{\sigma(4)})(y)$$

(since the second interacting function λ_2 is not necessarily a function with separable variables), where by $\mathscr{S}_n$ we denote the n-permutation group. Therefore, under the vacuum state, the creation and annihilation operators do not, in general, have the Gaussianity property.

But, on IFF space, under the vacuum state, the creation and annihilation operators have some properties similar to the Gaussianity. In order to see this let us introduce some notation.

DEFINITION 2.4. For each $n \in \boldsymbol{Z}$, $m, k \in \boldsymbol{N}$, and a function f defined on the set $\{n+1, n+2, \ldots, n+m\}$, the *$\pm k$-shift of f* is a function defined on the set $\{n+1 \pm k, n+2 \pm k, \ldots, n+m \pm k\}$:

$$(u_{\pm}^{k} f)(h \pm k) := f(h) \quad \text{for all } h \in \{n+1, n+2, \ldots, n+m\}.$$

With this notation we can state a simple result:

LEMMA 2.5. *For each $n \in \boldsymbol{N}$, $p < n$ and $\varepsilon \in \{0, 1\}_{+}^{2n}$ (with the left-right index-set $\{l_h, r_h\}_{h=1}^{n}$), if we define a map ε', which takes values in $\{0, 1\}$, by*

$$(2.13) \quad \varepsilon'|_{\{1,2,\ldots,l_p-1\}} := \varepsilon|_{\{1,2,\ldots,l_p-1\}}, \quad \varepsilon'(l_p) := \varepsilon(r_p+1),$$
$$\varepsilon'(l_p+1) := \varepsilon(r_p+2), \quad \ldots, \quad \varepsilon'(2n+l_p-r_p-1) := \varepsilon(2n),$$

then

$$\varepsilon' \in \{0, 1\}_{+}^{2\frac{2n+l_p-r_p-1}{2}}.$$

Proof. First of all, we have to show that $2n+l_p-r_p-1$ is even. In fact, by the non-crossing principle, if $l_d \in (l_p, r_p)$, then it is certainly true that $r_d \in (l_p, r_p)$. Therefore, r_p-l_p+1 $(\leqslant 2n)$ is always even, and so is $2n+l_p-r_p-1 = 2n-(r_p-l_p+1)$.

Again by the non-crossing principle, the restriction of the given $\varepsilon \in \{0, 1\}_{+}^{2n}$ on the set $\{1, 2, \ldots, 2n\} \backslash \{l_p, l_p+1, \ldots, r_p\}$ permits a unique non-crossing pair partition

$$\{(l_h, r_h)\}_{h \in \{1,2,\ldots,n\} \backslash \{d: l_p \leqslant l_d \leqslant r_p\}}.$$

So the assertion is proved.

Let us examine now the quantity (2.8) for $\varepsilon \in \{0, 1\}_{+}^{2n}$. It is obvious that $l_1 = 1$, r_1 is even and

$$\varepsilon|_{\{1,2,\ldots,r_1\}} \in \{0, 1\}_{+}^{2\frac{r_1}{2}}.$$

By the non-crossing principle, r_1+1 must be a left index, say l_{d_1}. Moreover, we know that $r_{d_1}-l_{d_1}+1 = r_{d_1}-r_1$ is even, $r_{d_1}+1$ is a left index, and

$$(u_{-}^{r_1} \varepsilon)|_{\{1,2,\ldots,r_{d_1}-l_{d_1}+1\}} \in \{0, 1\}_{+}^{2\frac{r_{d_1}-l_{d_1}+1}{2}}.$$

By repeating the argument, having obtained the pieces

$$1, \ldots, r_1, l_{d_1}, \ldots, r_{d_1}, l_{d_2}, \ldots, r_{d_2}, \ldots, l_{d_i}, \ldots, r_{d_i},$$

where $r_1+1 = l_{d_1}$, $r_{d_1}+1 = l_{d_2}$, $\ldots$, $r_{d_{i-1}}+1 = l_{d_i}$ since the $r_{d_i}+1$ must be a left index, say $l_{d_{i+1}}$, we have the next piece

$$l_{d_{i+1}}, l_{d_{i+1}}+1, \ldots, r_{d_{i+1}}.$$

Thus, in fact, we have proved

LEMMA 2.6. *Each* $\varepsilon\in\{0,1\}_+^{2n}$ *determines uniquely an* $m\leqslant n$, $1\leqslant n_1,\ldots,n_m\leqslant n$, *and* $\varepsilon_i\in\{0,1\}_+^{2n_i}$ $(i=1,2,\ldots,m)$ *such that*

$$\sum_{i=1} n_i = n \quad \text{and} \quad (u_-^{2n_j}\varepsilon)|_{\{1,2,\ldots,2n_{j+1}-2n_j\}} = \varepsilon_i.$$

Moreover, for any $i=1,2,\ldots,m$, *if we denote by* $\{l_h^{(i)}, r_h^{(i)}\}_{h=1}^{n_{i+1}-n_i}$ *the left-right index-set associated with* $\varepsilon_i\in\{0,1\}_+^{2n_{i+1}-2n_i}$, *then* $r_1^{(i)}=2n_{i+1}-2n_i$.

This result shows that the restriction of $\varepsilon\in\{0,1\}_+^{2n}$ on the set $\{2n_j+1, 2n_j+2,\ldots,2n_{j+1}\}$ has the same properties as the shift of the restriction (this shift is an element of $\{0,1\}_+^{2n_{j+1}-2n_j}$). Therefore, we shall not distinguish between the restriction itself and its shift. Taking into account these arguments, we prefer to rewrite (2.8) in the form

$$\langle\Phi, \prod_{h=1}^{r_1} A^{\varepsilon(h)}(g_h) \prod_{h=l_{d_1}}^{r_{d_1}} A^{\varepsilon(h)}(g_h)\ldots \prod_{h=l_{d_m}}^{r_{d_m}} A^{\varepsilon(h)}(g_h)\Phi\rangle, \tag{2.14}$$

where

(i) $1\leqslant m\leqslant n$, $d_j\leqslant n$ $(j=1,2,\ldots,m)$, $d_m=n$, and they are determined uniquely by the given ε,

(ii) $l_{d_1}=r_1+1$, $l_{d_2}=r_{d_1}+1,\ldots, l_{d_m}=r_{d_{m-1}}+1$.

Remark. In the language of diagrams, each $\varepsilon\in\{0,1\}_+^{2n}$ permits to draw a unique non-crossing diagram with vertices $\{1,2,\ldots,2n\}$. Any edge in the diagram connects two vertices l_h, r_h $(h=1,\ldots,n)$. The decomposition (2.14) means that the diagram determined by $\varepsilon\in\{0,1\}_+^{2n}$ can be decomposed into m pieces with the following properties:

(i) there is no edge connecting two vertices from two different pieces;

(ii) in any piece, the first vertex is paired with the last vertex;

(iii) the restriction of the full diagram (determined by the given $\varepsilon\in\{0,1\}_+^{2n}$) on each piece gives a non-crossing diagram on the corresponding vertices.

In the following, we shall call an $\varepsilon\in\{0,1\}_+^{2n}$ *totally connected* if $r_1=2n$, and call the decomposition (stated in the above) the *totally connected decomposition* of the given $\varepsilon\in\{0,1\}_+^{2n}$.

Now we are ready to state what we call the *factorization principle.*

LEMMA 2.7. *For each* $\varepsilon\in\{0,1\}_+^{2n}$, *the expression* (2.14) *is equal to*

$$\langle\Phi, \prod_{h=1}^{r_1} A^{\varepsilon(h)}(g_h)\Phi\rangle\cdot\langle\Phi, \prod_{h=l_{d_1}}^{r_{d_1}} A^{\varepsilon(h)}(g_h)\Phi\rangle\ldots\langle\Phi, \prod_{h=l_{d_m}}^{r_{d_m}} A^{\varepsilon(h)}(g_h)\Phi\rangle. \tag{2.15}$$

Proof. By the definition of creation and annihilation operators, for any vector G in a certain $\mathscr{H}_n$, any $f, f_1,\ldots,f_n\in\mathscr{H}$, and any $\varepsilon\in\{0,1\}^n$, the action of the operator $\prod_{h=1}^n A^{\varepsilon(h)}(f_h)A(f)$ on the vector G is equal to that of the operator $\prod_{h=1}^n A^{\varepsilon(h)}(f_h)$ on the vector $A(f)G$, i.e.

$$[\sum_{h=1}^n A^{\varepsilon(h)}(f_h)A(f)]\,G = \prod_{h=1}^n A^{\varepsilon(h)}(f_h)[A(f)G].$$

Hence, the quantity (2.14) is equal to

$$(2.16)\quad \langle \Phi, \prod_{h=1}^{r_1} A^{\varepsilon(h)}(g_h)\cdot \prod_{h=l_{d_1}}^{r_{d_1}} A^{\varepsilon(h)}(g_h)\dots \prod_{h=l_{d_{m-1}}}^{r_{d_{m-1}}} A^{\varepsilon(h)}(g_h)\Big[\prod_{h=l_{d_m}}^{r_{d_m}} A^{\varepsilon(h)}(g_h)\Phi\Big]\rangle .$$

In (2.16), since $\{l_{d_m}, l_{d_m}+1, \dots, r_{d_m}\}$ is a totally connected piece determined by the ε, the vector $\prod_{h=l_{d_m}}^{r_{d_m}} A^{\varepsilon(h)}(g_h)\Phi$ must be a *number*, i.e. it belongs to $\mathscr{H}_0 = C$. This argument guarantees that (2.14) is equal to

(2.17)

$$\langle \Phi, \prod_{h=1}^{r_1} A^{\varepsilon(h)}(g_h)\cdot \prod_{h=l_{d_1}}^{r_{d_1}} A^{\varepsilon(h)}(g_h)\dots \prod_{h=l_{d_{m-1}}}^{r_{d_{m-1}}} A^{\varepsilon(h)}(g_h)\Phi\rangle\cdot\langle \Phi, \prod_{h=l_{d_m}}^{r_{d_m}} A^{\varepsilon(h)}(g_h)\Phi\rangle .$$

By repeating the above argument to the first scalar product in (2.17) and applying the induction, we complete the proof.

As we have seen in the IFF case, the $2n$-point function in general cannot be reduced to a product of n 2-point functions. The factorization principle shows that the $2n$-point function can be reduced to a product of some scalar products and each scalar product is the vacuum expectation of a certain product of creation and annihilation operators. Moreover, each product of creation and annihilation operators has the totally connected property. Thus, in order to understand the precise form of the $2n$-point function, one needs to have a rule to compute the vacuum expectation of totally connected type products of creation and annihilation operators. In the next section, we shall give a general method to calculate the vacuum expectation of totally connected type products of creation and annihilation operators in some special case and, as application, the distribution of the field operator will be investigated.

3. The IFF space $\Gamma_I(L^2([0, T]))$ with $\lambda_n = \chi_{\Delta_n}$. In this section, we investigate the IFF space $\Gamma_I(L^2([0, T]))$ with the interacting functions $\{\chi_{\Delta_n}\}_{n=1}^{\infty}$. In order to distinguish our special case from the general case, we shall use a^+ and a, instead of A^+ and A, respectively, to denote creation and annihilation operators.

By definition, for any $n \in N$, F_n, $G_n \in \mathscr{H}_n$,

$$(3.1)\qquad \langle F_n, G_n\rangle := \int_{[0,T]^n} (\bar{F}_n G_n)(x_1, \dots, x_n)\chi_{\Delta_n}(x_1, \dots, x_n)\,dx_1\dots dx_n .$$

For any test function $f \in L^2([0, T])$, the creation operator $a^+(f)$ brings, by definition, the n-th space $\mathscr{H}_n$ to the $(n+1)$-st space $\mathscr{H}_{n+1}$. For any $F_n \in \mathscr{H}_n$, F_n is a function of $n+1$ variables and, more precisely,

$$(3.2)\qquad (a^+(f)F_n)(x_1, \dots, x_{n+1}) = f(x_1)\cdot F_n(x_2, \dots, x_{n+1}).$$

It is easy to see that the creation operator $a^+(f)$ is a bounded operator and its norm is less than or equal to the L^2-norm of f. So the annihilation operator $a(f) := (a^+(f))^*$ is bounded.

LEMMA 3.1. *For any* $f \in \mathscr{H}$, $n \in N$ *and* $G_{n+1} \in \mathscr{H}_{n+1}$,

$$(3.3)\qquad \big(a(f)G_{n+1}\big)(x_1, \ldots, x_n) = \int_0^T \bar{f}(x)\, G_{n+1}(x, x_1, \ldots, x_n)\, \chi_{[0,x_1)}(x)\, dx;$$

in particular, for any $f_1, \ldots, f_n \in \mathscr{H}$,

$$(3.4)\qquad a(f)a^+(f_1)\ldots a^+(f_n)\Phi = \int_0^T (\bar{f}f_1)(x)\, a^+(f_2\chi_{(x,T]})\, a^+(f_3)\ldots a^+(f_n)\Phi\, dx$$

$$= \int_0^T (\bar{f}f_1)(x)\, a^+(f_2\chi_{(x,T]})\, a^+(f_3\chi_{(x,T]})\ldots a^+(f_n\chi_{(x,T]})\Phi\, dx.$$

Proof. Once (3.3) is proved, by noticing that, for any $y \in [0, T]$,

$$\int_0^T F(x, y)\, \chi_{[0,y)}(x)\, dx = \int_0^T F(x, y)\, \chi_{(x,T]}(y)\, dx,$$

we get immediately the first equality of (3.4). Then, starting from this and noticing that in our case, for any element F_n in the n-th Hilbert space $\mathscr{H}_n$, $F_n = F_n \cdot \chi_{\Delta_n}$ is true, we find the second equality of (3.4). Therefore, we need only to prove (3.3) in order to complete the proof of the lemma.

For any $F_n \in \mathscr{H}_n$,

$$(3.5)\qquad \langle a(f)G_{n+1}, F_n\rangle = \langle G_{n+1}, a^+(f)F_n\rangle$$

$$= \int_{[0,T]^{n+1}} \bar{G}_{n+1}(x, x_1, \ldots, x_n)\, f(x)\, F_n(x_1, \ldots, x_n)\, \chi_{\Delta_{n+1}}(x, x_1, \ldots, x_n)\, dx dx_1 \ldots dx_n$$

$$= \int_{[0,T]^n} \Big[\int_0^T \bar{f}(x)\, G_{n+1}(x, x_1, \ldots, x_n)\, \chi_{[0,x_1)}(x)\, dx\Big]^* \times F_n(x_1, \ldots, x_n)\, \chi_{\Delta_n}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

So the assertion is obtained.

By the discussion in Section 2, to calculate the vacuum expectation of a product of creation and annihilation operators, we need only to give a rule to compute

$$(3.6)\qquad \langle \Phi, a^{\varepsilon(1)}(f_1)\ldots a^{\varepsilon(2n)}(f_{2n})\Phi\rangle$$

for $\varepsilon \in \{0, 1\}_+^{2n}$ totally connected.

LEMMA 3.2. *For any* $n \in N, f_1, \ldots, f_{2n} \in \mathscr{H}$, *and* $\varepsilon \in \{0, 1\}_+^{2n}$ *totally connected,* (3.6) *is equal to*

$$(3.7)\qquad \int_0^T (\bar{f}_1 f_{2n})(x)\, \langle \Phi, a^{\varepsilon(2)}(f_2\chi_{[0,x)})\ldots a^{\varepsilon(2n-1)}(f_{2n-1}\chi_{[0,x)})\Phi\rangle\, dx.$$

Proof. Denote by $\{l_h, r_h\}_{h=1}^{n}$ the non-crossing pair partition on $\{1, 2, \ldots, 2n\}$. Since we consider the totally connected case, r_1 must be equal to $2n$, and this implies that $l_n < 2n-1$ for any $n \geqslant 2$.

For $n = 1$ the lemma is obvious. For $n = 2$ and $\varepsilon \in \{0, 1\}_{+}^{2n}$ totally connected the expression (3.6) becomes

$$\langle \Phi, a(f_1) a(f_2) a^+ (f_3) a^+ (f_4) \Phi \rangle = \langle a^+ (f_2) a^+ (f_1) \Phi, a^+ (f_3) a^+ (f_4) \Phi \rangle,$$

which, by definition, is equal to

$$\int_{[0,T]^2} (\bar{f}_2 f_3)(x_1)(\bar{f}_1 f_4)(x_2) \chi_{\Delta_2}(x_1, x_2)\, dx_1\, dx_2 \tag{3.8}$$

$$= \int_0^T (\bar{f}_1 f_4)(x_2)\, dx_2 \int_0^{x_2} (\bar{f}_2 f_3)(x_1)\, dx_1 = \int_0^T (\bar{f}_1 f_4)(x_2)\, dx_2 \int_0^1 (\bar{f}_2 f_3 \chi_{[0,x_2)})(x_1)\, dx_1,$$

and this is clearly equal to

$$\int_0^T (\bar{f}_1 f_4)(x_2) \langle \Phi, a(f_2 \chi_{[0,x_2)}) a^+ (f_3 \chi_{[0,x_2)}) \Phi \rangle\, dx_2.$$

Suppose that the lemma is satisfied for all integers less than or equal to $n-1$. We shall consider two cases: $l_n = 2n-2$ (i.e. $r_n = 2n-1$) and $l_n < 2n-2$.

In the case of $l_n = 2n-2$, by Lemma 3.1, the expression (3.6) is equal to

$$\int_0^T (\bar{f}_{2n-2} f_{2n-1})(y) \langle \Phi, a^{\varepsilon(1)}(f_1) \ldots a^{\varepsilon(2n-3)}(f_{2n-3}) a^{\varepsilon(2n)}(f_{2n} \chi_{(y,T]}) \Phi \rangle\, dy. \tag{3.9}$$

Notice that $\{l_h, r_h\}_{h=1}^{n-1}$ is the non-crossing pair partition on the set $\{1, 2, \ldots, 2n-3, 2n\}$ which is still totally connected. Therefore, by the induction assumption, (3.9) becomes

$$\int_0^T (\bar{f}_{2n-2} f_{2n-1})(y)\, dy \int_0^T (\bar{f}_1 f_{2n} \chi_{(y,T]})(x) \times \langle \Phi, a^{\varepsilon(2)}(f_2 \chi_{[0,x)}) \ldots a^{\varepsilon(2n-3)}(f_{2n-3} \chi_{[0,x)}) \Phi \rangle\, dx. \tag{3.10}$$

By changing the order of integrals, (3.10) takes the form

$$\int_0^T (\bar{f}_1 f_{2n})(x)\, dx \int_0^x (\bar{f}_{2n-2} f_{2n-1})(y) \langle \Phi, a^{\varepsilon(2)}(f_2 \chi_{[0,x)}) \ldots a^{\varepsilon(2n-3)}(f_{2n-3} \chi_{[0,x)}) \Phi \rangle\, dy$$

$$= \int_0^T (\bar{f}_1 f_{2n})(x) \langle \Phi, a^{\varepsilon(2)}(f_2 \chi_{[0,x)}) \ldots a^{\varepsilon(2n-3)}(f_{2n-3} \chi_{[0,x)}) \times a^{\varepsilon(2n-2)}(f_{2n-2} \chi_{[0,x)}) a^{\varepsilon(2n-1)}(f_{2n-1} \chi_{[0,x)}) \Phi \rangle\, dx$$

and this is exactly what we want to prove.

In the case of $l_n < 2n-2$, by the non-crossing principle, we know that $r_n = l_n + 1$. Therefore, (3.6) is equal to

$$(3.11)\qquad \int_0^T (\bar{f}_{l_n} f_{r_n})(y) \langle \Phi,\, a^{\varepsilon(1)}(f_1) \dots a^{\varepsilon(l_n-1)}(f_{l_n-1})\, a^{\varepsilon(r_n+1)}(f_{r_n+1}\, \chi_{(y,T]}) \times a^{\varepsilon(r_n+2)}(f_{r_n+2}) \dots a^{\varepsilon(2n)}(f_{2n})\, \Phi \rangle \, dy.$$

Now, applying the induction assumption to the vacuum expectation in (3.11), we infer that it is equal to

$$(3.12)\qquad \int_0^T (\bar{f}_{l_n} f_{r_n})(y)\, dy \int_0^T (\bar{f}_1 f_{2n})(x) \langle \Phi,\, a^{\varepsilon(1)}(f_1\, \chi_{[0,x)}) \dots a^{\varepsilon(l_n-1)}(f_{l_n-1}\, \chi_{[0,x)}) \times a^{\varepsilon(r_n+1)}(f_{r_n+1}\, \chi_{[0,x)}\, \chi_{(y,T]})\, a^{\varepsilon(r_n+2)}(f_{r_n+2}\, \chi_{[0,x)}) \dots a^{\varepsilon(2n-1)}(f_{2n-1}\, \chi_{[0,x)})\, \Phi \rangle \, dx.$$

Since the (r_n+1)-st test function is $f_{r_n+1}\, \chi_{[0,x)}\, \chi_{(y,T]}$, we can rewrite (3.12) in the form

$$\int_0^T (\bar{f}_{l_n} f_{r_n})(y)\, dy \int_y^T (\bar{f}_1 f_{2n})(x) \langle \Phi,\, a^{\varepsilon(1)}(f_1\, \chi_{[0,x)}) \dots a^{\varepsilon(l_n-1)}(f_{l_n-1}\, \chi_{[0,x)}) \times a^{\varepsilon(r_n+1)}(f_{r_n+1}\, \chi_{[0,x)}\, \chi_{(y,T]})\, a^{\varepsilon(r_n+2)}(f_{r_n+2}\, \chi_{[0,x)}) \dots a^{\varepsilon(2n-1)}(f_{2n-1}\, \chi_{[0,x)})\, \Phi \rangle \, dx,$$

which is equal to

$$(3.13)\qquad \int_0^T (\bar{f}_1 f_{2n})(x)\, dx \int_0^T (\bar{f}_{l_n} f_{r_n}\, \chi_{[0,x)})(y) \langle \Phi,\, a^{\varepsilon(2)}(f_2\, \chi_{[0,x)}) \dots a^{\varepsilon(l_n-1)}(f_{l_n-1}\, \chi_{[0,x)}) \times a^{\varepsilon(r_n+1)}(f_{r_n+1}\, \chi_{[0,x)}\, \chi_{(y,T]})\, a^{\varepsilon(r_n+2)}(f_{r_n+2}\, \chi_{[0,x)}) \dots a^{\varepsilon(2n-1)}(f_{2n-1}\, \chi_{[0,x)})\, \Phi \rangle \, dy.$$

It is obvious, by Lemma 3.1, that for any $x \in [0, T]$

$$(3.14)\qquad \int_0^T (\bar{f}_{l_n} f_{r_n}\, \chi_{[0,x)})(y) \langle \Phi,\, a^{\varepsilon(1)}(f_1\, \chi_{[0,x)}) \dots a^{\varepsilon(l_n-1)}(f_{l_n-1}\, \chi_{[0,x)}) \times a^{\varepsilon(r_n+1)}(f_{r_n+1}\, \chi_{[0,x)}\, \chi_{(y,T]})\, a^{\varepsilon(r_n+2)}(f_{r_n+2}\, \chi_{[0,x)}) \dots a^{\varepsilon(2n-1)}(f_{2n-1}\, \chi_{[0,x)})\, \Phi \rangle \, dy = \langle \Phi,\, a^{\varepsilon(2)}(f_2\, \chi_{[0,x)}) \dots a^{\varepsilon(2n-1)}(f_{2n-1}\, \chi_{[0,x)})\, \Phi \rangle.$$

Thus, by induction, the proof is completed.

Remark. In the sequel, we shall call the method presented in Lemma 3.2 the *totally connected principle*. By combining the totally connected principle with the factorization principle, the vacuum expectation of any polynomial of creation and annihilation operators can be calculated. Moreover, since on the IFF space both creation and annihilation operators are bounded, we know that any continuous function of these operators can be calculated, and so, equivalently, can be calculated any element in the C^*-algebra generated by creation and annihilation operators.

As an application of the totally connected and the factorization principles, the distribution of the field operator can be computed. The second part of this section is devoted to this problem.

By the general result given in the preceding section, we know that for any test function $\xi \in L^2([0, T])$, for any $n \in N$, the vacuum expectation of $(a(\xi)+a^+(\xi))^{2n-1}$ is zero, i.e. with respect to the vacuum state, all the odd powers of the field operator have expectation value zero.

For any $t \in [0, T]$ and any function ξ defined on $[0, T]$ such that $\xi\chi_{[0,t)} \in L^2([0, T])$, let us define

$$(3.15) \qquad u_{\xi,n}(t) := \langle \Phi, \big(a(\xi\chi_{[0,t)})+a^+(\xi\chi_{[0,t)})\big)^{2n}\Phi\rangle.$$

It is clear that $u_{\xi,0}(t) = 1$ and $u_{\xi,n}(0) = 0$ for all $n \in N$. In the following, we shall assume that $t \in (0, T]$.

Let us put

$$\eta_t := \int_0^t |\xi|^2(s)\,ds,$$

and it is obvious that if $\eta_t = 0$, then $u_{\xi,0}(t) = 1$ and $u_{\xi,n}(t) = 0$ for all $n \in N$ and all $t \in [0, T]$.

It is easy to see that

$$u_{\xi,0}(t) = 1, \qquad u_{\xi,1}(t) = \eta_t \qquad \text{for all } t \in (0, T].$$

For any $n \geqslant 2$, we expand $\big(a(\xi\chi_{[0,t)})+a^+(\xi\chi_{[0,t)})\big)^{2n}$ in a sum of products of $2n$ creation and annihilation operators:

$$(3.16) \qquad \big(a(\xi\chi_{[0,t)})+a^+(\xi\chi_{[0,t)})\big)^{2n} = \sum_{\varepsilon \in \{0,1\}^{2n}} a^{\varepsilon(1)}(\xi\chi_{[0,t)})\ldots a^{\varepsilon(2n)}(\xi\chi_{[0,t)}).$$

From the discussions in Section 2 we know that many terms on the right-hand side of (3.16) have vacuum expectation of value zero. More precisely,

$$(3.17) \qquad u_{\xi,n}(t) = \sum_{\varepsilon \in \{0,1\}_+^{2n}} \langle \Phi, a^{\varepsilon(1)}(\xi\chi_{[0,t)})\ldots a^{\varepsilon(2n)}(\xi\chi_{[0,t)})\Phi\rangle.$$

LEMMA 3.3. *For any $n \in N$ and $t \in (0, T]$, $\{u_{\xi,n}(t)\}_{n=0}^{\infty}$ satisfies the system of difference-differential equations*

$$(3.18) \quad u_{\xi,n+1}(t) = \sum_{k=0}^{n} \int_0^t |\xi|^2(s)\,u_{\xi,k}(s)\cdot u_{\xi,n-k}(t)\,ds, \qquad u_{\xi,0}(t) = 1, \qquad u_{\xi,1}(t) = \eta_t.$$

Proof. As we have seen, each $\varepsilon \in \{0, 1\}_+^{2n}$ determines exactly one non-crossing pair partition $\{l_h, r_h\}_{h=1}^{n}$ on the set $\{1, 2, 3, \ldots, 2n\}$. By the non-crossing principle, $r_1 \in \{2, 4, \ldots, 2n\}$. Now we split the set $\{0, 1\}_+^{2n}$ into n parts according to all possible values of the first right index r_1:

$$(3.19) \qquad \{0, 1\}_+^{2n} = \bigcup_{k=1}^{n} \{0, 1\}_{+,k}^{2n} =: \bigcup_{k=1}^{n} \{\varepsilon \in \{0, 1\}_+^{2n}\colon r_1 = 2k\}.$$

Thus we find that

$$(3.20)\qquad u_{\xi,n+1}(t) = \sum_{k=1}^{n+1} \sum_{\varepsilon\in\{0,1\}_{+,k}^{2(n+1)}} \langle \Phi, a^{\varepsilon(1)}(\xi\chi_{[0,t)})\dots a^{\varepsilon(2n+2)}(\xi\chi_{[0,t)})\Phi\rangle$$

and, by the factorization principle, (3.20) is equal to

$$(3.21)\qquad \sum_{k=1}^{n+1} \sum_{\varepsilon\in\{0,1\}_{+,k}^{2(n+1)}} \langle \Phi, a(\xi\chi_{[0,t)})a^{\varepsilon(2)}(\xi\chi_{[0,t)})\dots a^{\varepsilon(2k-1)}(\xi\chi_{[0,t)})a^{+}(\xi\chi_{[0,t)})\Phi\rangle$$
$$\times\langle \Phi, a^{\varepsilon(2k+1)}(\xi\chi_{[0,t)})\dots a^{\varepsilon(2n+2)}(\xi\chi_{[0,t)})\Phi\rangle.$$

By the totally connected principle, the first scalar product in (3.21) is equal to

$$\int_0^T |\xi|^2(s)\chi_{[0,t)}(s)\langle \Phi, a^{\varepsilon(2)}(\xi\chi_{[0,t)}\chi_{[0,s)})\dots a^{\varepsilon(2k-1)}(\xi\chi_{[0,t)}\chi_{[0,s)})\Phi\rangle\, ds$$
$$= \int_0^t |\xi|^2(s)\langle \Phi, a^{\varepsilon(2)}(\xi\chi_{[0,s)})\dots a^{\varepsilon(2k-1)}(\xi\chi_{[0,s)})\Phi\rangle\, ds.$$

For each $k = 1, 2, \dots, n+1$, as ε runs over $\{0, 1\}_{+,k}^{2n+2}$, its restriction on the set $\{2k+1, 2k+2, \dots, 2n+2\}$ runs over all non-crossing pair partitions on the set $\{2k+1, 2k+2, \dots, 2n+2\}$; its restriction on the set $\{2, 3, \dots, 2k-1\}$ runs over all non-crossing pair partitions on the set $\{2, 3, \dots, 2k-1\}$. Thus, we find that

$$u_{\xi,n+1}(t) = \sum_{k=1}^{n+1} \Big(\int_0^t |\xi|^2(s)u_{\xi,k-1}(s)\,ds\Big)u_{\xi,n+1-k}(t)$$
$$= \sum_{k=0}^{n} \Big(\int_0^t |\xi|^2(s)u_{\xi,k}(s)\,ds\Big)u_{\xi,n-k}(t).$$

Therefore, the assertion is proved since $u_{\xi,0}(t) = 1$ and $u_{\xi,1}(t) = \eta_t$ are obvious.

Now we investigate the generating function of $\{u_{\xi,n}(t)\}_{n=0}^{\infty}$. In order to obtain a non-trivial generating function, we need the following comparison theorem:

LEMMA 3.4. *For any $t\in(0, T]$ and $n\in N$,*

$$(3.22)\qquad u_{\xi,n}(t) \leqslant \Big(\int_0^T |\xi|^2(s)\,ds\Big)^n c_n,$$

where c_n is the n-th Catalan number.

Proof. It is clear that for any $t\in(0, T]$, $n\in N$, $0 \leqslant u_{\xi,n}(t) \leqslant u_{\xi,n}(T) =: u_n$, we have

$$u_{\xi,0}(t) = 1 = c_0,\qquad u_{\xi,1}(t) = \int_0^t |\xi|^2(s)\,ds \leqslant u_{\xi,1}(T) = \int_0^T |\xi|^2(s)\,ds = \Big(\int_0^T |\xi|^2(s)\,ds\Big)c_1.$$

By induction,

$$0 \leqslant u_{\xi,n+1}(t) = \sum_{k=0}^{n} \Big(\int_0^t |\xi|^2(s)\, u_{\xi,k}(s)\, ds\Big) u_{\xi,n-k}(t)$$

$$\leqslant \sum_{k=0}^{n} \int_0^T |\xi|^2(s)\, ds \Big(\int_0^T |\xi|^2(s) ds\Big)^k c_k \cdot \Big(\int_0^T |\xi|^2(s)\, ds\Big)^{n-k} c_{n-k}$$

$$= \Big(\int_0^T |\xi|^2(s)\, ds\Big)^{n+1} \sum_{k=0}^{n} c_k \cdot c_{n-k}.$$

By the property of the Catalan numbers, we know that $\sum_{k=0}^{n} c_k \cdot c_{n-k}$ is nothing but c_{n+1}, and therefore the assertion of the lemma is obtained.

Notice that the series $\sum_{n=0}^{\infty} x^n c_n$ has a positive radius of convergence, and so does the series $\sum_{n=0}^{\infty} x^n u_{\xi,n}(t)$.

In the following, we take ξ as a continuous function (in fact, the most interesting case is $\xi = 1$). For any $t \in (0, T]$, let us define

$$S_t(x) := S(t, x) := \sum_{n=0}^{\infty} x^n u_{\xi,n}(t). \tag{3.23}$$

Lemma 3.5. *For any $t \in (0, T]$ the generating function $S(t, x)$ is equal to*

$$\frac{1}{\sqrt{1-2\eta_t x}}$$

in its interval of convergence.

Proof. It is clear that

$$S(t, x) = 1 + \eta_t x + \sum_{n=1}^{\infty} x^{n+1} u_{\xi,n+1}(t).$$

By Lemma 3.4, we have

(3.24)

$$S(t, x) = 1 + \eta_t x + \sum_{n=1}^{\infty} x^{n+1} \sum_{k=0}^{n} \Big(\int_0^t |\xi|^2(s)\, u_{\xi,k}(s)\, ds\Big) u_{\xi,n-k}(t)$$

$$= 1 + \eta_t x + \eta_t \sum_{n=1}^{\infty} x^{n+1} u_{\xi,n}(t) + \sum_{n=1}^{\infty} x^{n+1} \sum_{k=1}^{n} \Big(\int_0^t |\xi|^2(s)\, u_{\xi,k}(s)\, ds\Big) u_{\xi,n-k}(t)$$

$$= 1 + \eta_t x + \eta_t x\, (S(t, x) - 1) + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} x^{n+1} \Big(\int_0^t |\xi|^2(s)\, u_{\xi,k}(s)\, ds\Big) u_{\xi,n-k}(t)$$

$$= 1 + xS(t, x) \int_0^t |\xi|^2(s)\, S(s, x)\, ds.$$

Define

$$R(t, x) := \int_0^t |\xi|^2 (s) S(s, x) ds;$$

then from (3.24) we obtain a differential equation

(3.25) $$\frac{\partial R}{\partial t} = |\xi|^2 (t) + x \frac{\partial R}{\partial t} \cdot R, \qquad R(0, x) = 0.$$

Integrating the two sides of equation (3.25) we obtain

(3.26) $$R = \eta_t + x\left(R^2 - \int R \frac{\partial R}{\partial t} dt\right) + c,$$

where c is a constant. Moreover, for any $x \neq 0$, from (3.25) we get

$$\frac{\partial R}{\partial t} \cdot R = \frac{1}{x}\left(\frac{\partial R}{\partial t} - |\xi|^2 (t)\right).$$

Applying this formula to (3.26) we have

(3.27) $$R = \eta_t + x\left(R^2 - \frac{1}{x} R + \frac{\eta_t}{x}\right) + c = 2\eta_t + xR^2 - R + c.$$

By the condition $R(0, x) = 0$, it follows that $c = 0$. Therefore, any solution of (3.25) must satisfy the algebraic equation

(3.28) $$xR^2 - 2R + 2\eta_t = 0.$$

The solution of equation (3.28) takes clearly the form

(3.29a) $$R = \frac{1}{x}(1 \pm \sqrt{1 - 2\eta_t x}).$$

Again, by the condition $R(0, x) = 0$, we have

(3.29b) $$R = \frac{1}{x}(1 - \sqrt{1 - 2\eta_t x}).$$

Thus, the generating function $S_t(x)$ can be easily computed by the formula

(3.30) $$S_t(x) = \frac{1}{|\xi|^2 (t)} \frac{\partial R}{\partial t} = \frac{1}{\sqrt{1 - 2\eta_t x}}.$$

Notice that $S_t(0) = 1$, so (3.30) is valid also for $x = 0$.

THEOREM 3.6. *For any* $t \in (0, T]$, *the random variable*

$$c_{\xi,t} := a(\xi\chi_{[0,t)}) + a^+ (\xi\chi_{[0,t)})$$

is absolutely continuous and its density function is given by

(3.31a) $$L_{\xi,t}(x) := \frac{1}{\pi\sqrt{2\eta_t - x^2}} \chi_{(-\sqrt{2\eta_t}, \sqrt{2\eta_t})}(x).$$

In particular, for $\xi = 1$, the random variable $c_t := a(\chi_{[0,t)}) + a^+(\chi_{[0,t)})$ is absolutely continuous and its density function is given by

(3.31b) $$L_t(x) := \frac{1}{\pi\sqrt{2t - x^2}} \chi_{(-\sqrt{2t}, \sqrt{2t})}(x).$$

Proof. Since the random variable $c_{\xi,t}$ is bounded, its distribution is determined by its moments, i.e. by its generating function. Therefore, to prove the theorem it is sufficient to show that the generating function determined by the density function $L_{\xi,t}$ is the same as S_t at least on a small interval around the origin.

It is obvious that for any $n \in N$

$$\int_{-\infty}^{\infty} x^{2n-1} L_{\xi,t}(x)\, dx = 0.$$

Let us put

(3.32) $$v_n(t) := \int_{-\infty}^{\infty} x^{2n} L_{\xi,t}(x)\, dx.$$

We have to prove the following formula:

(3.33) $$S(t, y) = \frac{1}{\sqrt{1 - 2\eta_t y}} = \sum_{n=0}^{\infty} y^n v_n(t).$$

In fact,

(3.34) $$\sum_{n=0}^{\infty} y^n v_n(t) = \sum_{n=0}^{\infty} y^n \int_{-\infty}^{\infty} x^{2n} L_{\xi,t}(x)\, dx = 1 + \sum_{n=1}^{\infty} y^n \int_{-\sqrt{2\eta_t}}^{\sqrt{2\eta_t}} x^{2n} \frac{1}{\pi\sqrt{2\eta_t - x^2}}\, dx.$$

By changing the variables

$$u := \frac{x}{\sqrt{2\eta_t}},$$

the integral on the right-hand side of (3.34) takes the form

(3.35) $$(2\eta_t)^n \int_{-1}^{1} \frac{u^{2n}}{\pi\sqrt{1-u^2}}\, du.$$

Moreover, by using the shorthand notation

$$(2n-1)!! := (2n-1)(2n-3) \cdot \ldots \cdot 3 \cdot 1$$

and doing a simple calculation, one can rewrite (3.35) as

$$\frac{2^{n+1}\eta_t^n}{\pi}\int_0^{\pi/2}\sin^{2n}\theta\,d\theta = \frac{\eta_t^n(2n-1)!!}{n!}. \tag{3.36}$$

Therefore, the generating function of $L_{\xi,t}$ is equal to

$$1+\sum_{n=1}^{\infty}\frac{y^n\eta_t^n\cdot(2n-1)!!}{n!} = \frac{1}{\sqrt{1-2\eta_t y}}, \tag{3.37}$$

where the equality in (3.37) is nothing but the Taylor formula.

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