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ON SOME PROPERTIES OF ONE-DIMENSIONAL DIFFUSION PROCESSES ON AN INTERVAL

BY

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Abstract. Some equations are obtained for the moments of the first passage time of a one-dimensional time-homogeneous diffusion process, through each of two accessible boundaries α and β, given that the process has started from $x \in (\alpha, \beta)$. Some examples are considered and the results are graphically shown. Moreover, a special class of one-dimensional diffusions, of peculiar importance in biological modeling, is considered; the first passage times, and other properties such as ergodicity and reversibility of the stationary distribution, are investigated for these processes.

1. INTRODUCTION

In this paper we consider a one-dimensional temporally homogeneous diffusion process defined in the interval $[a, b]$ with first and second order infinitesimal moments given by two smooth enough functions $b(x)$ and $\sigma^2(x)$. The associated Itô's SDE is

$$dx(t) = b(x)\,dt + \sigma(x)\,dB_t, \qquad x \in [a, b], \tag{1.1}$$

where B_t is a standard one-dimensional Brownian motion. Notice that the diffusion process is defined in the closed interval $[a, b]$, and not, as usual, in an open set.

Equations of the form (1.1) often arise from stochastic models for biological systems. An example is given by the continuous diffusion approximation of some discrete Markov chains (MCs) with binomial-like transition probabilities. This type of MCs is very interesting and examples are known from population genetics (see e.g. [2]) and from cooperative interactions in protein molecules ([3], [4]). In the diffusion equation which comes from the model of cooperative interactions in proteins ([3], [6]), the drift coefficient $b(x)$ is a polynomial and the diffusion term is given by $\sigma^2(x) = x(1-x) \vee 0$, $x \in [0, 1]$. Equations like (1.1) with these infinitesimal moments are also used in diffusion neural models for synaptic transmission (see e.g. [7], [18]) and, generally, these equations have been often studied in this context.

We emphasize that, especially when a diffusion equation comes from biology, it is prominent to study the qualitative behavior of the solution as a function of the starting point. For instance, it is very interesting to know if the system may or not reach the extreme states a or b, and, if it does, how long it takes. Another important problem is to check if the convergence to stationarity occurs and if the stationary distribution of the process is reversible or not.

The present paper consists of three parts. The first one (Section 2) is devoted to study the first passage problem for the diffusion process X_t which is a solution of the SDE (1.1). Precisely, we study the first exit time of X_t from an interval (α, β) with $a \leqslant \alpha < \beta \leqslant b$ and the exit probabilities through each of the two barriers α and β, under the assumption that $x = \alpha$ and $x = \beta$ can be reached with probability one; of course, in the special case $\alpha = a$ and $\beta = b$, conditions on the behavior at the boundary could be required. Formulas for the moments of n-th order of the first exit time of a diffusion process from an interval (α, β) are known since early 50's (see [8]), in the case when the first exit time is a proper random variable (the existence of the density of this r.v. easily follows by the backward equation). Indeed, let us consider a process X_t which is a solution of the SDE (1.1). If the passage of X_t through the boundaries of (α, β) is a certain event, that is the first exit time $\tau_{\alpha\beta}(x)$ from (α, β) of the solution of (1.1) with starting point $x \in (\alpha, \beta)$ is a proper r.v., then the moments $t^{(n)}(x) = E\left(\tau_{\alpha\beta}(x)^n\right)$ satisfy Darling and Siegert's recursive relations (see [8]):

$$\frac{1}{2}\sigma^2(x)\frac{d^2 t^{(n)}}{dx^2} + b(x)\frac{dt^{(n)}}{dx} = -nt^{(n-1)}, \tag{1.2}$$

$$t^{(0)} = 1, \; t^{(n)}(a) = t^{(n)}(b) = 0, \; n = 1, 2, \ldots$$

These are linear second order ODEs which can be easily solved by quadratures.

In the late 70's, equations analogous to (1.2) were derived for more general Markov processes [28]. Results concerning the moments of first two orders also appeared in [15]. More recently, in [14] a formula for the average conditional exit time of the process through one particular end (α or β) was given.

Many questions about first passage times have been investigated by Ricciardi et al. [19]–[21] in the case when the diffusion process is defined in the open interval (a, b), under the assumption that both the boundary points a and b are inaccessible (in particular, this occurs when the probability current J vanishes at the boundary of (a, b)). Therein, some computationally convenient formulas were obtained for the moments of the first passage time of $X(t)$ through an assigned state $\eta \in (a, b)$. In the case when the steady-state density (i.e. the density of the invariant measure) of the diffusion process is available, such moments can be also calculated by Siegert's [27] recursive method. Moreover, in a more general case (i.e. nonhomogeneous process), Sacerdote [25] has studied various types of recursive equations for the moments of first-passage time through two barriers, $S_1 = S_1(t)$ and $S_2 = S_2(t)$, which are functions of time.

In this paper, we study an essentially different problem where we consider a time-homogeneous diffusion process which never exits from the closed interval $[a, b]$, then taking $(\alpha, \beta) \subset [a, b]$ we suppose that the process starting from an interior point $x \in (\alpha, \beta)$ can reach, with probability one, any of two barriers α and β in a finite time. If $\alpha = a$ and $\beta = b$, then, once the process has reached a boundary point, it may be absorbed (i.e. it remains there for ever) or it may return to the interior of (α, β). This behavior is shown, for instance, by the wide enough class of processes studied in [2], for particular values of the parameters, as well those related to diffusion neural models (see [7] and [18]).

Then we state here some equations for the moments of first and second order of the first exit time through a particular end of the interval (α, β), in the case when the two barriers α and β are both attainable. Of course, we suppose that conditions are satisfied in order that the diffusion process driven by (1.1) never exits from the interval $[a, b]$ (in the case when $\sigma^2(x) = x(1-x) \vee 0$, these conditions on $b(x)$ have been obtained in [2]). Our approach is based on the use of Itô's formula, and some arguments are similar to those used in [15]. Results somehow like these have been obtained by Ewens, by studying the diffusion approximation of conditional Markov chains in population genetics (see [9]). When the two boundaries a and b are absorbing, differential equations for the moments of any order of the first exit (i.e. absorbing) time through any of the boundary can be found.

In terms of a normalized *population*, whose evolution is described by a diffusion process in the interval $[0, 1]$, the first-passage time through the barrier $x = 0$ is nothing but the *extinction time* of the population, while that through $x = 1$ is the time at which the population reaches its maximum size.

Notice that the knowledge of the distribution (or, at least, of its moments) of both these two exit times is particularly interesting in many applications, as is equally important, for instance, while observing interspike intervals in models for neural activity, to know the first time at which the membrane voltage exceeds a voltage threshold.

In the second part of this paper (Section 3), we go back to diffusion processes with second order infinitesimal moment $\sigma^2(x) = x(1-x)$, which are essentially those obtained by the continuous approximation of the MCs considered in [2]; then we study further properties which were not investigated in [2], such as the reversibility of the process and the ergodic property of the transition probability function:

$$P(t, x, E) \to \mu(E) \quad \text{as } t \to \infty,$$

where μ is the invariant measure of the process.

In Section 4, some examples of one-dimensional diffusions on an interval I are considered, and the moments of the first exit time through the ends of I are computed.

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2. FIRST PASSAGE TIMES AND RELATED PROBLEMS

First, we briefly recall some facts from the theory of diffusion processes.

Let $X(t)$ be a temporally homogeneous diffusion process in $[a, b]$ with first and second order moments $b(x)$ and $\sigma^2(x)$, respectively; then the transition probability $P(x, t, E)$ of $X(t)$ is uniquely determined by $b(x)$ and $\sigma^2(x)$. Alternatively, $X(t)$ is a solution of the SDE

(2.1) $$dX(t) = b(X)\,dt + \sigma(X)\,dB_t, \qquad x \in [a, b],$$

where B_t is a standard Brownian motion.

If $f(x)$ is any bounded function defined in $[a, b]$, then

$$u(t, x) = E_x f(X(t)) = \int_a^b f(y) P(x, t, dy), \qquad t \in [0, T],$$

is continuous and bounded together with its derivatives $\partial u/\partial x$, $\partial^2 u/\partial x^2$, $\partial u/\partial t$, and it satisfies Kolmogorov's backward equation:

(2.2) $$\frac{\partial u}{\partial t} = Lu \doteq \left[b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} \right]$$

with terminal condition

$$\lim_{t \to 0} u(t, x) = f(x).$$

In the case when $P(x, t, \cdot)$ has a density $p(x, t, y)$ regular enough, then $p(x, t, y)$ satisfies (2.2) under the condition

$$\lim_{t \to 0} p(x, t, y) = \delta(x - y).$$

Moreover, $p(x, t, y)$ is a fundamental solution of the forward Fokker–Plank equation (FPE):

(2.3) $$\frac{\partial p}{\partial t} = L^* p \doteq -\frac{\partial}{\partial y}[b(y) p(x, t, y)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y) p(x, t, y)],$$

where the operator L^* is the formal adjoint of L. The FPE can be also written as

(2.4) $$\frac{\partial p}{\partial t} + \frac{\partial J}{\partial x} = 0,$$

where J is the *probability current* defined by

$$J(x, t) = b(x)p(x, t) - \frac{1}{2}\frac{d}{dx}(\sigma^2(x)p(x, t)) \tag{2.5}$$

and $p(x, t)$ is the one-time probability

$$p(x, t) = \int p(x_0, t, x)p(x_0, 0)\,dx_0$$

satisfying the initial condition $p(x, t)|_{t=0} = p(x, 0)$.

Formula (2.4) is the FPE written in the form of continuity equation. If the diffusion process X_t has an invariant (or stationary) measure absolutely continuous with respect to the Lebesgue measure, its density $W(x)$ is the solution of $dJ(x)/dx = 0$ with $p(x, t) = W(x)$ or, equivalently, $L^* W = 0$.

Remark 2.1. Let $\tau_{\alpha\beta}(x) = \{\inf t \geqslant 0 \mid X_t \notin (\alpha, \beta)\}$ be the first exit time from (α, β) of the solution of (2.1) starting from the interior point $x \in (\alpha, \beta)$. Then, if $\tau = \tau_{\alpha\beta}(x)$ is finite with probability 1, it is well known (see e.g. [12], [13], [26]) that $u(x) = E(\tau_{\alpha\beta}(x))$ is the solution of the Dirichlet problem

$$Lu = \tfrac{1}{2}\sigma^2(x)u'' + b(x)u' = -1, \quad x \in (\alpha, \beta), \qquad u(a) = u(b) = 0. \tag{2.6}$$

Formula (2.6) is called *Dynkin's equation.*

Remark 2.2. The probability of exit through the end $x = \alpha$, say

$$\pi_\alpha(x) = \Pr\big(X(\tau_{\alpha\beta}(x)) = \alpha \mid X(0) = x\big)$$

(under the condition that the boundary is accessible, i.e. $P(\tau_{\alpha\beta}(x) < \infty) = 1$), is the solution of the Dirichlet problem

$$\tfrac{1}{2}\sigma^2(x)u'' + b(x)u' = 0, \quad x \in (\alpha, \beta), \qquad u(\alpha) = 1,\ u(\beta) = 0. \tag{2.7}$$

In an analogous way we can see that $\pi_\beta(x) = \Pr\big(X(\tau_{\alpha\beta}(x)) = \beta \mid X(0) = x\big)$ is the solution of

$$\tfrac{1}{2}\sigma^2(x)u'' + b(x)u' = 0, \quad x \in (\alpha, \beta), \qquad u(\alpha) = 0,\ u(\beta) = 1. \tag{2.7'}$$

A straightforward calculation shows that the solution of (2.6) is

$$u(x) = \theta(\beta)\psi(x)/\psi(\beta) - \theta(x), \tag{2.8}$$

where

$$\begin{gathered} \theta(x) = \int_\alpha^x \xi(s)\,ds, \qquad \phi(x) = \exp\left(-\int_\alpha^x \frac{2b(s)}{\sigma^2(s)}\,ds\right), \\ \psi(x) = \int_\alpha^x \phi(s)\,ds, \qquad \xi(x) = \phi(x)\int_\alpha^x 2[\sigma^2(s)\phi(s)]^{-1}\,ds. \end{gathered} \tag{2.9}$$

The solutions of (2.7) and (2.7′) for $\pi_\alpha(x)$ and $\pi_\beta(x)$ are given by

$$\pi_\beta(x) = \psi(x)/\psi(\beta), \qquad \pi_\alpha(x) = 1-\pi_\beta(x). \tag{2.10}$$

We recall that a boundary point $r \in \{\alpha, \beta\}$ is called *attainable* (or *accessible*) if

$$P_x\big(\lim_{t\to\tau} X(t) = r,\ \tau < \infty\big) > 0,$$

and *unattainable* (or *unaccessible*) otherwise. More generally, following the Feller classification, the boundary point r is called *attractive* if

$$P\big(\lim_{t\to\tau} X(t) = r\big) > 0$$

and *repelling* (or *natural*) otherwise. The classification above can be characterized in terms of integrability conditions of the functions Φ and Ξ which are obtained from ϕ and ξ defined in (2.9), by replacing the lower extremum of integration α by any number $c \in (\alpha, \beta)$. Precisely, we have:

- r is *attractive* if Φ is integrable over a neighborhood of r, and *repelling* otherwise;
- r is *attainable* if Ξ is integrable over a neighborhood of r, and *unattainable* otherwise;
- r is *regular* if the function $[\Phi\sigma^2]^{-1}$ is integrable over a neighborhood of r, *absorbing* otherwise.

If the boundaries a and b are both regular, it is easily seen that the process $X(t)$ has an invariant measure with density $W(x) = \text{const}\cdot[\phi(x)\sigma^2(x)]^{-1}$.

Explicit formulas for the n-th moments of the first exit time are well known (see e.g. equation (3.26) in [25]). Also, by solving directly Darling and Siegert's recursive equations (1.2), we can see that, when $\tau_{\alpha\beta}(x)$ is a proper random variable, the moments $t^{(n)}(x) = E\big(\tau_{\alpha\beta}(x)^n\big)$ are determined by the formula

$$t^{(n)}(x) = n\,[\theta_n(\beta)\,\psi(x)/\psi(\beta) - \theta_n(x)], \tag{2.11}$$

where ϕ, ψ are given by (2.9) and

$$\theta_n(x) = \int_\alpha^x \xi_n(s)\,ds, \qquad \xi_n(x) = \phi(x)\int_\alpha^x 2t^{(n-1)}(s)\,[\sigma^2(s)\,\phi(s)]^{-1}\,ds.$$

Formula (2.11) reduces to equation (3.26) of [25] by considering a suitable transformation which brings the process $X(t)$ into another process $Y(t)$ having infinitesimal variance equal to 2 (such a new process is defined by $Y(t) = \sqrt{2}\int_0^{X(t)} (1/\sigma(s))\,ds$; see also [10]).

In the case when $\alpha = a$, $\beta = b$ and the invariant density $W(x) = C\,[\phi(x)\sigma^2(x)]^{-1}$ exists, then

$$\xi_n(x) = \phi(x)\int_a^x 2t^{(n-1)}(s)\,W(s)\cdot C^{-1}\,ds = 2C^{-1}\,\phi(x)\,\mu\big(t^{(n-1)}\,1_{(a,x)}\big),$$

where $d\mu(s) = W(s)\,ds$ is the invariant measure.

Now, we will consider the main results of the paper, i.e. some equations for computing the moments of first and second order of the exit time of the diffusion process driven by (2.1) through a given end (α or β) of the interval (α, β).

The first result concerns a way to compute the average first exit time through the end $x = \alpha$. Indeed, this result has already appeared in [14] and was obtained following a different approach based on the continuity equation for the probability current. We derive this result following ideas similar to those used in [15] to obtain the expectation of first exit time from an interval.

THEOREM 2.3. *Let $X(t)$ be the diffusion process which is the solution of*

$$dX(t) = b\big(X(t)\big)dt + \sigma\big(X(t)\big)dB_t. \tag{2.12}$$

Let us suppose that both the ends of (α, β) are reachable (accessible); let $\pi_\alpha(x)$ be the probability that the process starting at $x \in (\alpha, \beta)$ reaches the boundary of (α, β) for the first time at the end $x = \alpha$, and let $\tau_\alpha(x)$ be the first arrival time at the boundary with the condition that the exit takes place at $x = \alpha$. Let $T_\alpha(x)$ be the solution of the problem

$$(Lz)(x) = \tfrac{1}{2}\sigma^2(x)z''(x) + b(x)z'(x) = -\pi_\alpha(x), \qquad x \in (\alpha, \beta), \qquad z(\alpha) = z(\beta) = 0. \tag{2.13}$$

Then $T_\alpha(x) = E\big(\pi_\alpha(x)\tau_\alpha(x)\big) = \pi_\alpha(x)E\big(\tau_\alpha(x)\big)$.

Analogously, if $\pi_\beta(x)$ is the probability to exit at the right of (α, β), $\tau_b(x)$ is the first exit time through $x = \beta$, and $T_\beta(x) = \pi_\beta(x)E\big(\tau_\beta(x)\big)$, then $T_\beta(x)$ satisfies the problem

$$LT_\beta(x) = -\pi_\beta(x), \; x \in (\alpha, \beta), \qquad T_\beta(\alpha) = T_\beta(\beta) = 0. \tag{2.13'}$$

Proof. Fix $x \in (a, b)$ and let $X(t)$ be the solution of (2.12) starting from x. Since Itô's formula holds for functions of the solution process on bounded Markov time intervals, if $\tau(t) = \min(t, \tau_\alpha)$, we get

$$T_\alpha\big(X(\tau(t))\big) = T_\alpha(x) + \int_0^{\tau(t)} LT_\alpha\big(X(s)\big)ds + \int_0^{\tau(t)} T'_\alpha\big(X(s)\big)\sigma\big(X(s)\big)dB_s.$$

Then, as $t \to \infty$, we obtain

$$T_\alpha\big(X(\tau_\alpha(x))\big) = T_\alpha(x) + \int_0^{\tau_\alpha(x)} \big(-\pi_\alpha\big(X(s)\big)\big)ds + \int_0^{\tau_\alpha(x)} T'_\alpha\big(X(s)\big)\sigma\big(X(s)\big)dB_s.$$

From this, by using the boundary condition $T_\alpha\big(X(\tau_\alpha(x))\big) = T_\alpha(\alpha) = 0$, and taking the expectation of both sides, we have

$$T_\alpha(x) = E\Big(\int_0^{\tau_\alpha} \pi_\alpha\big(X(s)\big)ds\Big). \tag{2.14}$$

But the last quantity is equal to $E(\tau_\alpha(x)\pi_\alpha(x))$. Indeed, by Itô's formula we obtain

$$\pi_\alpha(X(s)) = \pi_\alpha(x) + \int_0^s L\pi_\alpha(X(t))\,dt + \int_0^s \pi'_\alpha(X(t))\,\sigma(X(t))\,dB_t.$$

Then, by (2.14) and (2.7),

$$T_\alpha(x) = E(\tau_\alpha(x)\pi_\alpha(x)) + E\left[\int_0^{\tau_\alpha} ds \int_0^s \pi'_\alpha(X(t))\,\sigma(X(t))\,dB_t\right]$$

and the last expectation is zero, since, changing the order of integration, the integral is equal to

$$\int_0^{\tau_\alpha} \pi'_\alpha(X(t))\,\sigma(X(t))\,dB_t \int_t^{\tau_\alpha} ds = \int_0^{\tau_\alpha} (\tau_\alpha - t)\,\pi'_\alpha(X(t))\,\sigma(X(t))\,dB_t.$$

The proof is completed.

By solving equation (2.13) by quadratures, we obtain the formula

$$E(\tau_\alpha(x)) = \frac{1}{\pi_\alpha(x)}[\tilde{\theta}_\alpha(\beta)\,\psi(x)/\psi(\beta) - \tilde{\theta}_\alpha(x)].$$

Analogously,

$$E(\tau_\beta(x)) = \frac{1}{\pi_\beta(x)}[\tilde{\theta}_\beta(\beta)\,\psi(x)/\psi(\beta) - \tilde{\theta}_\beta(x)].$$

Here

$$\tilde{\xi}_A(x) = \phi(x)\int_\alpha^x 2\pi_A(s)\,[\sigma^2(s)\,\phi(s)]^{-1}\,ds,$$

$$\tilde{\theta}_A(x) = \int_\alpha^x \tilde{\xi}_A(t)\,dt, \quad A = \alpha, \beta, \qquad \psi(x) = \int_\alpha^x \phi(t)\,dt.$$

Remark 2.4. Notice that, in order to find a Dirichlet problem whose solution is $\tau_\alpha(x)$, a nontrivial difficulty is by the fact that the boundary condition at the right, $\tau_\alpha(\beta)$, is not known. An instructive calculation shows that, if one conditions the process to exit through the end $x = \alpha$, one obtains for the process the SDE

$$dX(t) = \tilde{b}(x) + \sigma(x)\,dB_t,$$

where the drift term is modified in

$$\tilde{b}(x) = b(x) + \pi'_\alpha(x)\,\sigma^2(x)/\pi_\alpha(x).$$

Indeed, let us write $\tau_\alpha = \tau + (\tau_\alpha - \tau) = \tau + r$ and rewrite the equation $L(E(\pi_\alpha \tau_\alpha)) = -\pi_\alpha$ as

$$L(E(\pi_\alpha \tau)) + L(E(\pi_\alpha r)) = -\pi_\alpha.$$

Now, for all $x \in (\alpha, \beta)$ we have

$$L\big(E(\pi_\alpha \tau + \pi_\alpha r)\big)$$
$$= (r+\tau)L\big(E(\pi_\alpha)\big) + \pi_\alpha\big(L(E(\tau)) + L(E(r))\big) + \big((E(\tau))' + (E(r))'\big)\sigma^2(x)\pi'_\alpha$$
$$= -\pi_\alpha + \pi_\alpha L\big(E(r)\big) + \sigma^2(x)\pi'_\alpha\big((E(\tau))' + (E(r))'\big)$$

because $L\pi_\alpha = 0$, $L\big(E(\tau)\big) = -1$, $x \in (\alpha, \beta)$. Then the equation $L\big(E(\pi_\alpha \tau_\alpha)\big) = -\pi_\alpha$ can be written in the form

$$\sigma^2(x)\pi'_\alpha\big(E(\tau_\alpha)\big)' + \pi_\alpha b(x)\,[(E\tau_\alpha)' - (E\tau)'] + \tfrac{1}{2}\sigma^2(x)\,[(E\tau_\alpha)'' - (E\tau)''] = 0,$$

that is

$$\sigma^2(x)\pi'_\alpha(E\tau_\alpha)' - \pi_\alpha L(E\tau) + \pi_\alpha L(E\tau_\alpha) = 0$$

and, since $L(E\tau) = -1$, after some manipulation we finally obtain

$$\tfrac{1}{2}(E\tau_\alpha)''\sigma^2(x) + [b(x) + \pi'_\alpha\sigma^2(x)/\pi_\alpha](E\tau_\alpha)' = -1. \tag{2.15}$$

Thus, $E\tau_\alpha$ solves the same equation of $E\tau$, with the drift term modified to be $\tilde{b}(x) = b(x) + \pi'_\alpha\sigma^2(x)/\pi_\alpha$. (This kind of result can be found for instance in [9] in the case when α and β are absorbing.) Obviously, $E\big(\tau_\alpha(\alpha)\big) = 0$, but the problem remains how to assingn the boundary condition at $x = \beta$.

Remark 2.5. By summing the equations in (2.13) and (2.13′), since $\pi_\alpha(x) + \pi_\beta(x) = 1$, the function $T_\alpha(x) + T_\beta(x)$ satisfies (2.6), and for the uniqueness of the solution of the Dirichlet problem we finally get $T_\alpha + T_\beta = E(\tau)$, that is

$$E\big(\tau_{\alpha\beta}(x)\big) = E\big(\tau_\alpha(x)\big)\pi_\alpha(x) + E\big(\tau_\beta(x)\big)\pi_\beta(x). \tag{2.16}$$

Remark 2.6. By Remark 2.4 it follows that $z = E\big(\tau_\alpha(x)\big)$ is the solution of the Dirichlet problem:

$$\frac{1}{2}z''\sigma^2(x) + \left(b(x) + \frac{\pi'_\alpha(x)\sigma^2(x)}{\pi_\alpha(x)}\right)z' = -1, \qquad z(\alpha) = 0,\ z(\beta) = z_\beta, \tag{2.17}$$

where the boundary condition z_β is unknown (cf. e.g. [9]).

On the other hand, by Theorem 2.3, $z(x)\pi_\alpha(x)$ is also the solution of (2.13); then $z'(\alpha)$ can be easily calculated and it turns out to be equal to $\tilde{\theta}_\alpha(\beta)/\psi(\beta)$. Thus $E(\tau_\alpha)$ solves the Cauchy problem:

$$\frac{1}{2}z''\sigma^2(x) + \left(b(x) + \frac{\pi'_\alpha(x)\sigma^2(x)}{\pi_\alpha(x)}\right)z' = -1, \tag{2.18}$$
$$z(\alpha) = 0,\ z'(\alpha) = \tilde{\theta}_\alpha(\beta)/\psi(\beta).$$

In order to obtain the second order moment of the first exit time through a particular end of the interval (α, β), we shall make use of a drift transformation. Indeed, by means of the Girsanov formula, we shall get the desired moment in a different suitable probability space.

Let us consider, for instance, the end α, and choose $\bar{T}>0$ such that $\tau_\alpha(x) \leqslant \bar{T}$ for all $x \in (\alpha, \beta)$; then, let us consider the process $\tilde{X}(t)$ which is the solution of the SDE:

$$d\tilde{X}(t) = \tilde{b}(\tilde{X}(t))\,dt + \sigma(\tilde{X}(t))\,d\tilde{B}_t, \quad \tilde{X}(0) = x \in (\alpha, \beta), \tag{2.19}$$

where

$$\tilde{b}(x) = b(x) + \frac{\pi'_\alpha(x)\,\sigma^2(x)}{\pi_\alpha(x)}$$

and $\tilde{B}_t$ is a Brownian motion defined in a new probability space $(\Omega, \mathscr{F}, \tilde{P})$ to be chosen in a suitable way.

We make the following assumption:

(A) There exists $\delta = \delta(x) > 0$ such that, with probability one, $\tilde{X}(t) \in [\alpha, \beta-\delta)$ for all $t \in [0, \bar{T}]$.

For instance, (A) is fulfilled if the boundary $x = \beta$ is repelling for the process $\tilde{X}(t)$. Notice that, if $b(x)$ and $\sigma(x)$ are sufficiently regular in order to assure the uniqueness of the solution of (1.1), then the solution of (2.19) is defined up to the (random) explosion time at which $\tilde{X}(t)$ reaches the end $x = \beta$ (in fact, $\tilde{b}(x)$ becomes unbounded at $x = \beta$). However, if the assumption (A) is satisfied, this explosion time is infinite with probability one.

Now, define

$$\eta(x) = \frac{b(x)-\tilde{b}(x)}{\sigma(x)} = -\frac{\pi'_\alpha(x)\,\sigma(x)}{\pi_\alpha(x)}, \quad \gamma(t) = \eta(\tilde{X}(t)),$$
$$\zeta_0^t(\gamma) = \int_0^t \gamma(u)\,d\tilde{B}_u - \tfrac{1}{2}\int_0^t |\gamma(u)|^2\,du. \tag{2.20}$$

Notice that, because of assumption (A), the following condition holds:

$$E_{\tilde{P}}\left(\exp(\varrho\,|\gamma(t)|^2)\right) \leqslant c, \quad t \in [0, \bar{T}], \ \varrho, c > 0, \tag{2.21}$$

where $E_{\tilde{P}}$ denotes expectation with respect to the probability measure $\tilde{P}$. Then all the conditions are satisfied to apply the Girsanov theorem (see e.g. [11]). Thus, by means of the Girsanov formula, we infer that, for $0 \leqslant t \leqslant \bar{T}$, the process $\tilde{X}(t)$ is also the solution of the SDE:

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dB_t, \quad X(0) = x, \tag{2.22}$$

where the Brownian motion $\tilde{B}_t$ is chosen in a way such that

$$B_t = \tilde{B}_t - \int_0^t \gamma(s)\,ds. \tag{2.23}$$

Moreover, the probability measure $\tilde{P}$ defined in $(\Omega, \mathcal{F})$ is absolutely continuous with respect to P and it is given by the formula

$$d\tilde{P}(\omega) = \exp\left(-\zeta_0^{\bar{T}}(\gamma)\right) dP(\omega), \quad \omega \in \Omega. \tag{2.24}$$

In other words, if the assumption (A) is satisfied, then $X(t) = \tilde{X}(t)$, $t \in [0, \tau_\alpha(x)]$; thus the process $\tilde{X}(t)$ is nothing but (in a different probability space) the process $X(t)$ conditioned to exit for the first time from (α, β) through the left end α. Then, by using the argument above, we are able to obtain the following:

THEOREM 2.7. *Let us suppose that for some* $\varepsilon \geqslant 0$

$$\pi_\alpha(x) = O(\beta - x)^{1/2+\varepsilon} \quad \text{as } x \to \beta \tag{2.25}$$

and set $v(x) \doteq E_{\tilde{P}}(\tau_\alpha^2(x))$; *then* $v(x)$ *is the solution of the problem*

$$\tfrac{1}{2}\sigma^2(x)v''(x) + \tilde{b}(x)v'(x) = -2E(\tau_\alpha(x)), \quad x \in (\alpha, \beta), \tag{2.26}$$

$$v(\alpha) = 0, \quad v(\beta) \text{ finite},$$

where

$$\tilde{b}(x) = b(x) + \frac{\pi_\alpha'(x)\sigma^2(x)}{\pi_\alpha(x)}$$

and $E(\tau_\alpha)$ *is given by Theorem* 2.3.

Proof. Let $\tilde{X}(t)$ be the solution of (2.19); first, we observe that because of (2.25) the end β is repelling for the process $\tilde{X}(t)$. Indeed, since the boundary β is attainable for the original process $X(t)$, in order that it might be repelling for $\tilde{X}(t)$ it is sufficient to assume that the further factor of the function Φ in (2.9), i.e.

$$\exp\left(-\int_\alpha^x (2\pi_\alpha'(s)/\pi_\alpha(s))\,ds\right) = 1/(\pi_\alpha(x))^2$$

is not integrable near $x = \beta$. This condition is exactly (2.25).

Now, let $z(t)$ be the solution of (2.18); then, by Itô's formula applied to the process $\tilde{X}(t)$,

$$z(\tilde{X}(\tau_t)) - z(x) = \int_0^{\tau_t} z'(\tilde{X}(s))\sigma(\tilde{X}(s))\,d\tilde{B}_s + \int_0^{\tau_t}(-1)\,ds, \tag{2.27}$$

where $\tau_t = \min(\tau_\alpha, t)$. Then, for $t \to \infty$, using the fact that $\tilde{X}(\tau_\alpha(x)) = X(\tau_\alpha(x)) = \alpha$ and the boundary condition $z(\alpha) = 0$ we obtain

$$\tau_\alpha(x) = z(x) + \int_0^{\tau_\alpha} z'(\tilde{X}(s))\sigma(\tilde{X}(s))\,d\tilde{B}_s. \tag{2.28}$$

Taking the $\tilde{P}$-expectation we have (because of Remark 2.6)

$$E_{\tilde{P}}(\tau_\alpha(x)) = z(x) = E_P(\tau_\alpha(x)). \tag{2.29}$$

Moreover, taking the $\tilde{P}$-expectation of the square of both members in (2.28), we obtain

$$E_{\tilde{P}}(\tau_\alpha^2) = E_{\tilde{P}} \int_0^{\tau_\alpha} [z'(\tilde{X}(s))\sigma(\tilde{X}(s))]^2 ds + z^2(x). \tag{2.30}$$

Now, let $w(t)$ be the solution of the problem

$$\tfrac{1}{2}\sigma^2(x)w'' + \tilde{b}(x)w' = -[z'(x)\sigma(x)]^2, \qquad w(\alpha) = 0, \ w(\beta) = w_\beta \text{ finite}. \tag{2.31}$$

Once again by Itô's formula and the boundary condition on $w(x)$ we obtain

$$w(x) = E_{\tilde{P}} \int_0^{\tau_\alpha} [z'(\tilde{X}(s))\sigma(\tilde{X}(s))]^2 ds. \tag{2.32}$$

Then, from (2.30) it follows that

$$v(x) = E_{\tilde{P}}(\tau_\alpha^2(x)) = z^2(x) + w(x). \tag{2.33}$$

Moreover,

$$\tfrac{1}{2}\sigma^2(x)v'' + \tilde{b}(x)v' = [\tfrac{1}{2}\sigma^2(x)w'' + \tilde{b}(x)w' + (z'\sigma)^2] + 2z[\tfrac{1}{2}\sigma^2(x)z'' + \tilde{b}(x)z'] = -2z \tag{2.34}$$

since the expression in the first brackets is zero by (2.31), and the second one is equal to -1 by (2.18). Finally, the result follows by (2.34) and the fact that $v(\alpha) = z^2(\alpha) + w(\alpha) = 0$.

Remark 2.8. If we define the differential operator $\tilde{L}$ by

$$\tilde{L}(u) = \tfrac{1}{2}\sigma^2(x)u'' + \tilde{b}(x)u', \tag{2.35}$$

it is easily seen that

$$\pi_\alpha \tilde{L}u = \pi_\alpha Lu + \pi'_\alpha \sigma^2 u' = L(\pi_\alpha u). \tag{2.36}$$

Consequently,

$$\tilde{L}(u) = \frac{1}{\pi_\alpha} L(\pi_\alpha u).$$

Then, if $v(x) = E_{\tilde{P}}(\tau_\alpha^2(x))$ is the solution of (2.26), then $\pi_\alpha(x)E_{\tilde{P}}(\tau_\alpha^2(x))$ is the solution of the following problem:

$$L(z) = -2\pi_\alpha E(\tau_\alpha), \qquad z(\alpha) = z(\beta) = 0. \tag{2.37}$$

Finally, defining $T_\alpha^{(n)}(x) = E_{\tilde P}(\pi_\alpha(x)\tau_\alpha^n(x))$, $n = 1, 2$, it is easily seen that $T_\alpha^{(n)}(x)$, $n = 1, 2$, is the solution of the equation

$$LT_\alpha^{(n)}(x) = -nT_\alpha^{(n-1)}(x) \tag{2.38}$$

with boundary conditions

$$T_\alpha^{(n)}(x)(\alpha) = T_\alpha^{(n)}(\beta) = 0. \tag{2.39}$$

A differential equation like (2.26) was stated by Ewens [9] for a diffusion process arising by the approximation of a Markov chain from population genetics. Here, we obtain a more formal verification of the boundary or initial conditions for the equation.

The equation (2.38) with boundary conditions (2.39) coincides with Darling and Siegert's result for the $\tilde P$-moments of the unconditional exit time $\tau_{\alpha\beta}$ from (α, β), with $T_\alpha^{(n)}$ in place of $t^{(n)}$ for $n = 1, 2$ (see equation (1.2)). We wonder if (2.38) and (2.39) also hold for $n > 2$. At the moment, we are not able to show this without any further assumption on the nature of the boundaries α and β, not even by means of the techniques used by Darling and Siegert in the case of the exit time from (α, β) (such as the Laplace transform, etc.).

Remark 2.9. An equation analogous to (2.37) holds for $E_{\hat P}(\tau_\beta^2(x))$ under the condition that $\pi_\beta(x) = O(x-\alpha)^{1/2+\varepsilon}$ as $x \to \alpha$. Here $\hat P$ is another suitable probability measure absolutely continuous with respect to P, obtained by the same construction which precedes Theorem 2.7. Summing up we conclude that $E_{\tilde P}(\tau_\alpha^2(x))$ is the solution of the problem

$$LE_{\tilde P}(\pi_\alpha\tau_\alpha^2) = -2E_{\tilde P}(\pi_\alpha\tau_\alpha), \qquad E_{\tilde P}(\pi_\alpha\tau_\alpha^2)|_{x=\alpha,\beta} = 0, \tag{2.40}$$

while $E_{\hat P}(\tau_\beta^2(x))$ is the solution of the problem

$$LE_{\hat P}(\pi_\beta\tau_\beta^2) = -2E_{\hat P}(\pi_\beta\tau_\beta), \qquad E_{\hat P}(\pi_\beta\tau_\beta^2)|_{x=\alpha,\beta} = 0. \tag{2.41}$$

Moreover, Darling and Siegert's equation for $n = 2$ gives

$$LE_P(\tau_{\alpha\beta}^2) = -2E_P(\tau_{\alpha\beta}), \qquad E_P(\tau_{\alpha\beta}^2)|_{x=\alpha,\beta} = 0. \tag{2.42}$$

By summing (2.40) and (2.41), we obtain

$$LE_{\tilde P}(\pi_\alpha\tau_\alpha^2)+LE_{\hat P}(\pi_\beta\tau_\beta^2) = -2\big(E_{\tilde P}(\pi_\alpha\tau_\alpha)+E_{\hat P}(\pi_\beta\tau_\beta)\big)$$
$$= -2\big(E_P(\pi_\alpha\tau_\alpha)+E_P(\pi_\beta\tau_\beta)\big) = -2E_P(\tau_{\alpha\beta}) = LE_P(\tau_{\alpha\beta}^2).$$

Then, for the uniqueness of the solution of the Dirichlet problem, we finally get

$$E_P(\tau_{\alpha\beta}^2(x)) = \pi_\alpha(x)E_{\tilde P}(\tau_\alpha^2(x))+\pi_\beta(x)E_{\hat P}(\tau_\beta^2(x)). \tag{2.43}$$

The explicit formulas for the second order moments of $\tau_\alpha(x)$ and $\tau_\beta(x)$ are easily obtained by quadratures of the corresponding differential equations.

Precisely, we have

$$E_A(\tau_A^2(x)) = \frac{2}{\pi_A(x)}[\hat{\theta}_A(\beta)\psi(x)/\psi(\beta)-\hat{\theta}_A(x)], \tag{2.44}$$

$$A = \alpha,\ \beta,\ E_\alpha = E_{\tilde{p}},\ E_\beta = E_{\tilde{p}},$$

where

$$\hat{\theta}_A(x) = \int_\alpha^x \xi_A(s)\,ds$$

and

$$\xi_A(x) = \phi(x)\int_\alpha^x 2\pi_A(s)E_A(\tau_A(s))(\sigma^2(s)\phi(s))^{-1}\,ds.$$

Let us suppose now that both the boundaries a and b are absorbing. In this case, we can consider the diffusion process conditioned to exit from (a, b) through one of the ends (of course, once the process has reached this boundary, it remains there for ever); the first exit time through an end (a or b) coincides with the absorbing time at this boundary. Such a conditioned process was considered by Ewens [9] for a diffusion arising by the approximation of a Markov chain from population genetics.

Let us assume for simplicity that the transition probability of the diffusion process carried by (2.1) has a density $p(x, t, y)$ (this is the case, for instance, if there exists an invariant measure absolutely continuous with respect to the Lebesgue measure), and denote by $\tilde{p}_A(x, t, y)$ ($A = a, b$) the corresponding transition probability density for the process conditioned to be absorbed at the end $A = a, b$. As easily seen, we have

$$\tilde{p}_A(x, t, y) = p(x, t, y)\pi_A(y)/\pi_A(x), \tag{2.45}$$

where $\pi_A(x)$ is given by (2.10).

Now, we search for the infinitesimal moments of the (conditional) diffusion process whose transition probability density is $\tilde{p}_A(x, t, y)$. For the drift coefficient $\tilde{b}_A(x)$ we must have

$$\tilde{b}_A(x) = \lim_{t\to 0}\frac{1}{t}\int (y-x)\tilde{p}_A(x, t, y)\,dy \tag{2.46}$$

$$= \lim_{t\to 0}\frac{1}{t}\int (y-x)p(x, t, y)\pi_A(y)/\pi_A(x)\,dy.$$

By the Taylor expansion of $\pi_A(x)$ we obtain

$$\pi_A(y) = \pi_A(x)+(y-x)\pi'_A(x)+O(y-x)^2.$$

Then, substituting in (2.46), we get

$$\tilde{b}_A(x) = \lim_{t\to 0}\frac{1}{t}\int (y-x)p(x, t, y)[1+(y-x)\pi'_A(x)/\pi_A(x)+O(y-x)^2]\,dy \tag{2.47}$$

$$= b(x)+\frac{\pi'_A(x)}{\pi_A(x)}\lim_{t\to 0}\frac{1}{t}\int (y-x)^2 p(x, t, y)\,dy = b(x)+\frac{\pi'_A(x)}{\pi_A(x)}\sigma^2(x).$$

Here the integration is over the set $|y-x| \leqslant \varepsilon$ and we have used the fact that the integral of $(y-x)\cdot O(y-x)^2 p(x, t, y)$ is zero because it gives the infinitesimal moment of third order (cf. [22] and [23]).

An analogous calculation shows that for the diffusion coefficient the equality

$$\tilde{\sigma}_A^2(x) = \sigma^2(x) \tag{2.48}$$

holds since now the additional term is an integral giving the infinitesimal moment of higher order.

Therefore, in the case when both a and b are absorbing barriers, the diffusion process conditioned to exit through the end $A = a, b$ has the infinitesimal operator given by

$$\tilde{L}_A(f) = \tilde{b}_A(x) f'(x) + \tfrac{1}{2}\sigma^2(x) f''(x), \tag{2.49}$$

where

$$\tilde{b}_A(x) = b(x) + \frac{\pi_A'(x)}{\pi_A(x)}\sigma^2(x).$$

For the conditional diffusion process the boundary $B = \{a, b\}\backslash A$ is obviously repelling; then the case of a and b absorbing reduces to that of Darling and Siegert for two barriers, since the exit time $\tau_{\alpha\beta}(x)$ necessarily must coincide with $\tau_A(x)$. Thus, by applying a slight variant of Darling and Siegert's method, we finally obtain

PROPOSITION 2.10. *If two barriers a and b are both absorbing, then the moments $\tilde{t}_A^{(n)}(x) = E(\tau_A^{(n)}(x))$ of any order n of the absorbing time at $A = a, b$ are the solutions of the following equations with initial conditions:*

$$\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}\tilde{t}_A^{(n)} + \tilde{b}(x)\frac{d}{dx}\tilde{t}_A^{(n)} = -n\tilde{t}_A^{(n-1)}, \tag{2.50}$$

$$\tilde{t}_A^{(0)} = 1,\ \tilde{t}_A^{(n)}(A) = 0,\ \tilde{t}_A^{(n)\prime}(A) \text{ finite},\ n = 1, 2, \ldots$$

Notice that the initial value of the derivative $\tilde{t}_A^{(n)\prime}(A)$ is unknown; (2.50) can be transformed into equations with boundary conditions by considering the barriers A and $B-\varepsilon$. Then the solutions of (2.50) are obtained as limits of the solutions of Darling and Siegert's equations for the last two-barrier problem with zero boundary conditions when $\varepsilon \to 0$ (cf. [19] for an analogous case).

Alternatively, if we set

$$\tilde{T}_A^{(n)}(x) = E(\pi_A(x)\tau_A^{(n)}(x)) = \pi_A(x)\tilde{t}_A^{(n)}(x),$$

it is easily seen that $\tilde{T}_A^{(n)}(x)$, $n = 1, 2, \ldots$, is the solution of the Dirichlet problem

$$L\tilde{T}_A^{(n)}(x) = -nT_A^{(n-1)}(x), \qquad \tilde{T}_A^{(n)}(A) = \tilde{T}_A^{(n)}(B) = 0. \tag{2.51}$$

Remark 2.11. The above procedure cannot be followed as it stands if a and b are not absorbing. Indeed, the use of the operator $\tilde{L}$ permits in Theorem 2.7 to find (in a certain probability space) the second order moment of the first exit time from A if the assumption (2.25) (or its analogue regarding the behavior of $\pi_A(x)$ near the ends of (α, β) is satisfied. However, it must be underlined that without the absorbing assumption, and any further condition, the operator $\tilde{L}_A$ does not represent the differential operator of the diffusion process conditioned to exit from A.

Nevertheless, if $x = \beta$ is repelling for the diffusion corresponding to $\tilde{L}_\alpha$ (condition (2.25)), then $\tilde{X}(t)$ can exit in a finite time only through the end α, i.e. $\tau_\alpha(x) = \tau_{\alpha\beta}(x)$. Once again, applying Darling and Siegert's method we see that the equations (2.50) hold for the $\tilde{P}$-moments of any order of the first exit time through the end α. Of course, analogue equations hold for the $\hat{P}$-moments relative to the end β if the analogue of (2.25) is satisfied.

Finally, we observe that, in the case when the boundaries a and b are both repelling for the original diffusion process in $[a, b]$, and the density of the invariant measure of the process is known, a recursive Siegert's method [27] is available to calculate the moments of the first exit time through any of two ends of an interval (α, β), $a < \alpha < \beta < b$.

3. DIFFUSIONS ARISING BY APPROXIMATION OF MARKOV CHAINS WITH BINOMIAL-LIKE TRANSITION PROBABILITIES

In this section we consider the diffusion processes which arise by the continuous approximation of discrete Markov chains with binomial-like transition probabilities. Precisely, let us consider the homogeneous Markov chain X_k having transition probabilities

$$(3.1)\qquad P(X_{k+1} = m \mid X_k = n) = \binom{N}{m} \theta\left(\frac{n}{N}\right)^m \left(1-\theta\left(\frac{n}{N}\right)\right)^{N-m},$$

where X_k represents a *population* at time k which can take values $n \in \{0, 1, \ldots, N\}$, N being a fixed positive integer (that is the maximum allowed size of the population), and the parameter $\theta = \theta(n/N)$ is a function of the fraction n/N of *individuals* at time k over the maximum size of the population. This type of MCs has been treated at length in [2], where it was assumed that θ is a polynomial function of its argument, i.e.

$$(3.2)\qquad \theta(x) = a_0 + a_1 x + \ldots + a_r x^r, \qquad a_i = \text{const},\ i = 0, 1, \ldots, r.$$

For a survey of examples of such MCs from biology, see [2]. As stated there, the continuous approximation of the normalized MC (i.e. X_k/N) leads to considering a diffusion process X_t in $[0, 1]$ such that the second order infinitesimal moment is

$$\sigma(x) = \sqrt{x(1-x) \vee 0}$$

and the drift term $b(x)$ is a polynomial function of the same degree r of $\theta(x)$. For such a diffusion, the study of the corresponding SDE (1.1) is more complicated, because $\sigma(x)$ vanishes at the ends of the interval (the condition $\sigma^2(x) > 0$ is usually assumed for diffusion processes in order to get peculiar properties for certain related parabolic PDEs); furthermore, it is not Lipschitz continuous, thus the results concerning the uniqueness of the solution starting from a given initial point $x \in [0, 1]$ as well as the continuity of the associated transition probabilities (Feller property) do not hold immediately (see [2] for a discussion). These properties as well as the exit problem for the diffusion process $X(t) \in [0, 1]$ and attainability of the boundary have been investigated in [2]. Indeed, under certain conditions on $b(x)$, the process X_t does not exit from [0, 1] for any time; moreover, if the function $[x(1-x)\Phi(x)]^{-1}$ is integrable in [0, 1] (for the definition of Φ, see (2.9) and the text to follow), there exists a unique absolutely continuous invariant measure μ whose density is a beta function if $r = 1$, while for $r > 1$ it is a beta function multiplied by a factor $e^{-q(x)}$, where $q(x)$ is a polynomial of degree $r-1$. Furthermore, in the case $r = 1$, the ergodic property holds for the transition probability function $P(t, x, E)$, that is $P(t, x, E) \to \mu(E)$ as $t \to \infty$ for any μ-measurable set $E \subset [0, 1]$ (see [2]). Also, for $r = 1$ the process X_t is reversible (see [6]). Here, we want to remove the restriction on the drift term $b(x)$ to be a polynomial by allowing it to be any bounded continuous function on [0, 1], and we will show that many of the properties above remain valid. Then, let us consider the Itô SDE

$$dX_t = b(X_t) + \sqrt{x(1-x) \vee 0}, \qquad x \in [0, 1], \tag{3.3}$$

where $b(x)$ is a bounded Lipschitz-continuous function defined in [0, 1]; the first result that immediately follows is:

THEOREM 3.1. *The SDE* (3.3) *has a unique strong solution* $X(t)$ *for any initial condition* $X(0) \in [0, 1]$. *Moreover, the continuity with respect to the initial condition holds for the transition probability function* (*Feller property*).

For the proof see [5].

Note that some care has to be taken in the proof, since the diffusion coefficient is not a Lipschitz-continuous function.

By exactly the same proof of Theorem 2.3 in [2], we obtain

THEOREM 3.2. *If* $b(x)$ *in* (3.3) *satisfies* $b(0) \geqslant 0$, $b(1) \leqslant 0$, *then the solution of the SDE* (3.3) *with initial condition* $X(0) \in [0, 1]$ *remains in the interval* [0, 1] *for all the time* $t \geqslant 0$.

In order to obtain the ergodic property for the transition probability function we make use of an argument already utilized in [5], which consists in comparing the diffusion process X_t with two extreme processes. In fact, we obtain

THEOREM 3.3. *Under the assumptions of Theorem* 3.2, *if*

$$T_t f(x) = \int_0^1 P(t, x, dy) f(y),$$

then

$$T_t f(x) \to \mu(f) \quad \text{as } t \to \infty,$$

where $P(t, x, E)$ is the transition probability function of the process X_t described by the SDE (3.3), *and μ is any invariant measure (i.e. $\mu(T_t f) = \mu(f)$). This implies that the invariant measure is unique and the ergodic property holds: $P(t, x, E) \to \mu(E)$.*

Proof. By assumptions, there exist two polynomials of degree one, $p^-(x)$ and $p^+(x)$, such that

$$p^-(x) \leqslant b(x) \leqslant p^+(x), \quad x \in [0, 1], \tag{3.4}$$

and

$$p^\pm(0) > 0, \quad p^\pm(1) < 0. \tag{3.5}$$

Now, let $X^-(t)$ and $X^+(t)$ be, respectively, the solutions of the SDEs on $[0, 1]$:

$$dY = p^-(Y)dt + \sqrt{Y(1-Y) \vee 0}, \quad Y(0) = x, \tag{3.6}$$

and

$$dY = p^+(Y)dt + \sqrt{Y(1-Y) \vee 0}, \quad Y(0) = x. \tag{3.7}$$

Then, by a standard comparison theorem (see, e.g., Theorem 1.1 of [17], p. 352), the process X_t, being the solution of (3.3), satisfies with probability one:

$$X^-(t) \leqslant X(t) \leqslant X^+(t) \quad \text{for all } t. \tag{3.8}$$

The processes X^- and X^+, having polynomials of degree one as drift terms, admit invariant densities of beta-type. Then, exactly in the same way as in [5] to prove Lemmas 3.3, 3.4 and 3.5, respectively, we are able to get the analogous results, i.e.:

(i) $P(t, x, (0, 1)) = 1$;

(ii) for all $t > 0$ and any open set $B \subset (0, 1)$ there exists $\delta > 0$ such that

$$\inf_{x \in [0,1]} P(t, x, B) \geqslant \delta;$$

(iii) let $f \in C[0, 1]$ and let $t_0 > 0$ be fixed; then for every compact subset K of $(0, 1)$ the family $\{v_t\}_{t \geqslant t_0} \subset C(K)$ defined as v_t: $K \to R$, $x \to v_t(x) = T_t f(x)$, is a relatively compact subset of $C(K)$.

Finally, following the proof of Theorem 3.6 in [5] and using (i), (ii) and (iii), we obtain the desired result. The uniqueness of the invariant measure follows by the bounded convergence theorem and by the definition of invariant measure.

Remark 3.4. Reasoning as in [5] one can show that the invariant measure μ is absolutely continuous with respect to the Lebesgue measure on $[0, 1]$. Consequently, the transition probability function $P(t, x, E)$ has a density, since the absolute continuity of the invariant measure implies the existence of the density of the transition probability function.

Remark 3.5. We are not able to compute explicitly the transition probability function $P(t, x, E)$ of the Markov process related to the SDE (3.3) for any bounded continuous drift $b(x)$. However, in the special case when $b(x)$ is a polynomial of degree one, an explicit formula for the transition probability density function $p(x, t, y)$ can be given. Indeed (see, e.g., [16], [18]), $p(x, t, y)$ can be written as an infinite series of hypergeometric functions as follows:

$$(3.9)\qquad p(x, t, y) = y^{A-1}(1-y)^{B-A-1} \times \sum_{i=0}^{\infty} \frac{(B+2i-1)\Gamma(B+i-1)}{\Gamma(B-A)\Gamma(A+i)} P_i^{\gamma,\delta}(2y-1)\tilde{F}_i(1-x)\exp(-\lambda_i t),$$

where, if $b(x) = b_0 + b_1 x$,

$$(3.10)\qquad \begin{gathered} A = 2b_0, \quad B = -2b_1, \quad \delta = A-1, \quad \gamma = B-A-1, \\ \lambda_i = i\big(-2b_1+(i-1)\big)/2, \quad \tilde{F}_i(y) = F(B+i-1,\ -i,\ B-A;\ y), \end{gathered}$$

F is a hypergeometric function, and $P_i^{\gamma,\delta}(y)$ is a Jacobi polynomial (see the Appendix). Alternatively, $p(x, t, y)$ can be written as (see [5]):

$$(3.11)\qquad p(x, t, y) = \sum_{n=0}^{\infty} \exp(l_n t)\,\phi_n(x)\,\phi_n(y)\,u_{b_0,b_1}(y),$$

where $u_{b_0,b_1}(y)$ is the beta-density of parameters $2b_0$ and $-2b_1-2b_0$, i.e.

$$(3.12)\qquad u_{b_0,b_1}(y) = y^{2b_0-1}(1-y)^{-2(b_0+b_1)-1}\frac{\Gamma(-2b_1)}{\Gamma(2b_0)\Gamma\big(-2(b_0+b_1)\big)}, \qquad y\in[0, 1],$$

and $\{\phi_n\}_n$ is a set of eigenfunctions of the generator of the diffusion process described by the SDE (3.3) with $b(x) = b_0 + b_1 x$, and l_n are the corresponding eigenvalues such that $l_0 = 0$ ($\phi_0 = 1$) and $l_n < 0$ for all $n \geq 1$. Here $\phi_n(x)$ is the normalized Jacobi polynomial on $[0, 1]$ with weight $u_{b_0,b_1}(x)$ (see [5]). From

(3.9) it follows that as $t \to \infty$:

$$\lim_{t\to\infty} p(x,t,y) \tag{3.13}$$

$$= y^{A-1}(1-y)^{B-A-1}\frac{(B-1)\Gamma(B-1)\Gamma(B-A)}{\Gamma^2(B-A)\Gamma(A)}\tilde{F}_0(1-y)\tilde{F}_0(1-x)$$

$$= y^{2b_0-1}(1-y)^{-2(b_0+b_1)-1}\frac{\Gamma(2b_1)}{\Gamma(2b_0)\Gamma(-2(b_0+b_1))} = u_{b_0,b_1}(y).$$

This limit can be also achieved easily by taking the limit as t goes to inifinity in (3.11) and by using the fact that $l_0 = 0$, $l_n < 0$ for all $n \geqslant 1$.

While the convergence in (3.13) occurs at an exponential rate (i.e. in the case of a linear drift), for a general continuous drift $b(x)$ we are not able to obtain the rate of convergence of $P(t, x, E)$ to $\mu(E)$ or of $p(x, t, y)$ to the density μ.

Reversibility of the process. Now, we turn to study the reversibility of the diffusion process described by the SDE (3.3). We claim that, also by restricting our consideration to the case when $b(x)$ is a polynomial, the process occurs to be reversible only when $b(x)$ is a polynomial of degree one (the reversibility of the process in this case has been already proved in [6]). Indeed, the diffusion process is reversible if the generator L is self-adjoint, i.e.

$$\langle Lf, h\rangle_\mu = \langle f, Lh\rangle_\mu \tag{3.14}$$

for any functions $f, h \in L^2_\mu(0,1)$, where $\langle\cdot\rangle_\mu$ is the scalar product in L^2_μ, and $\mu(dx) = u(x)\,dx$ is the invariant measure of the process ($u(x)$ is the product of a beta-function of the form $\text{const}\cdot x^{C-1}(1-x)^{D-1}$ times a factor $e^{-q(x)}$; see the beginning of Section 3). A straightforward calculation shows that

$$\langle Lf, h\rangle_\mu \tag{3.15}$$

$$= \int_0^1 (Lfu)h\,dx = \int_0^1 \left(Lf(x)e^{-q(x)}\cdot\text{const}\cdot x^{C-1}(1-x)^{D-1}\right)h\,dx$$

$$= \text{const}\cdot\int_0^1\left(\frac{1}{2}x^C(1-x)^D e^{-q(x)}f'(x)\right)' h(x)\,dx - \int_0^1 u(x)R(x)f'(x)h(x)\,dx,$$

where $R(x)$ is a polynomial of degree $r-1$, if degree of $b(x)$ is r. Notice by the same calculation that, when $r = 1$, $R(x)$ is identically zero. An easy computation shows that the third integral in (3.15) is equal to $\langle f, Lh\rangle_\mu$, so we obtain

$$\langle Lf, h\rangle_\mu = \langle f, Lh\rangle_\mu - \int_0^1 u(x)R(x)f'(x)h(x)\,dx. \tag{3.16}$$

Then the process is reversible only in the case when $R(x)$ is identically zero. Thus, we have obtained

THEOREM 3.6. *The diffusion process described by the SDE* (3.3) *with invariant density* μ *is reversible if and only if the drift* $b(x)$ *is a polynomial of degree one.*

This fact could explain why the SDE (3.3) with a linear drift $b(x) = b_0 + b_1 x$ has been widely applied in literature to describe successfully a lot of physical and biological models. We underline that, when the drift term $b(x)$ is linear, the density of the invariant measure is a beta-function; moreover, the transition probability density can be calculated explicitly and all the calculations involved are easier. This is just the case of the moments of the first-passage times, for instance, those which can be computed more or less explicitly when $b(x)$ is linear. Really, in this case, Siegert's equation for the Laplace transform $g_S(\lambda | x)$ of the density of the first-passage time of the solution starting at x through a barrier S can be identified with the so-called Gaussian equation for hypergeometric functions (see e.g. [1]). Thus, $g_S(\lambda | x)$ can be written as a product of a constant multiplied by a hypergeometric function or also as a sum of exponential functions (see [18] and [24]).

4. EXAMPLES

In this section we will consider some examples of SDEs like (2.1) and, by using the formulas of Section 2, we shall compute the moments of the first-passage time through each of two accessible barriers α and β. For some examples the computation is carried on in a theoretical way; for others, the integrals involved in the formulas cannot be theoretically computed, thus they are obtained numerically.

4.1. The Brownian motion. We take $b(x) = 0$ and $\sigma(x) = 1$ in (2.1), $\alpha = 0$ and $\beta = 1$ and we consider the SDE:

(4.1) $$dX(t) = dB_t, \quad x \in [0, 1].$$

By formulas (2.9), we have

(4.2) $$\phi(x) = 1, \quad \xi(x) = \int_0^x 2ds = 2x,$$
$$\psi(x) = \int_0^x dt = x, \quad \theta(x) = \int_0^x 2tdt = x^2.$$

Then, by (2.10),

(4.3) $$\pi_1(x) = \psi(x)/\psi(1) = x, \quad \pi_0(x) = 1 - \pi_1(x) = 1 - x.$$

Moreover,

$$\tilde{\xi}_0(x) = \phi(x)\int_0^x 2\pi_0(s)\,ds = 2(x-x^2/2),$$

$$\tilde{\theta}_0(x) = \int_0^x \tilde{\xi}_0(t)\,dt = 2\int_0^x (t-t^2/2)\,dt = x^2(1-x/3),$$

$$\tilde{\xi}_1(x) = \phi(x)\int_0^x 2\pi_1(s)\,ds = x^2,$$

$$\tilde{\theta}_1(x) = \int_0^x \tilde{\xi}_1(t)\,dt = \int_0^x t^2\,dt = x^3/3.$$

Then

$$E(\tau_0(x)) = \frac{1}{\pi_0(x)}\left[\frac{\tilde{\theta}_0(1)}{\psi(1)}\psi(x) - \tilde{\theta}_0(x)\right] = \frac{x(2-x)}{3}, \tag{4.4}$$

$$E(\tau_1(x)) = \frac{1}{\pi_1(x)}\left[\frac{\tilde{\theta}_1(1)}{\psi(1)}\psi(x) - \tilde{\theta}_1(x)\right] = \frac{1-x^2}{3}. \tag{4.5}$$

From (4.4), (4.5) and the fact that

$$E(\tau_{01}(x)) = E(\tau_0(x))\pi_0(x) + E(\tau_1(x))\pi_1(x)$$

the well-known formula follows for the mean exit time of the Brownian motion from the interval [0, 1]:

$$E(\tau_{01}(x)) = x(1-x). \tag{4.6}$$

Now, we will consider the second order moments of the first-passage times through the barriers $\alpha = 0$ and $\beta = 1$. Since the assumption (2.25) of Theorem 2.7 is fulfilled, $E_{\tilde{P}}(\tau_0^2(x))$ is the solution of the problem

$$\frac{1}{2}v''(x) - \frac{v'(x)}{1-x} = -\frac{2}{3}(2x-x^2), \quad v(0) = 0,\ v(1) \text{ finite} \tag{4.7}$$

or, analogously, $\pi_0(x)E_{\tilde{P}}(\tau_0^2(x))$ is the solution of the corresponding equation (2.38) with conditions (2.39). By solving (4.7) and by imposing $v(1)$ to be finite, after some straightforward calculation we obtain

$$E_{\tilde{P}}(\tau_0^2(x)) = \frac{1}{45}(3x^4 - 12x^3 + 8x^2 + 8x). \tag{4.8}$$

For the second order moment of the first passage time through $x = 1$, we can solve, for instance, the problem (2.38), (2.39), that is

$$\tfrac{1}{2}T'' = -\tfrac{2}{3}x(1-x^2), \qquad T(0) = T(1) = 0. \tag{4.9}$$

As easily seen, the solution of (4.9) is $T(x) = \frac{1}{45}x(3x^4-10x^2+7)$, and then we obtain

$$E_{\hat{P}}(\tau_1^2(x)) = \frac{T(x)}{x} = \frac{1}{45}(3x^4-10x^2+7). \tag{4.10}$$

From (2.43), (4.8) and (4.10) we get

$$\begin{aligned} E_P(\tau_{01}^2(x)) &= (1-x)\cdot\frac{1}{45}(3x^4-12x^3+8x^2+8x)+x\cdot\frac{1}{45}(3x^4-10x^2+7) \\ &= \frac{1}{3}(x^4-2x^3+x). \end{aligned} \tag{4.11}$$

4.2. Diffusion arising by approximation of Markov chains with binomial-like transition probabilities. Let $b(x)$ be a bounded Lipschitz-continuous function, $\sigma(x) = \sqrt{x(1-x)\vee 0}$, and $a = \alpha = 0$, $\beta = b = 1$; then we have the SDE already considered in (3.3):

$$dX(t) = b(x)\,dt + \sqrt{x(1-x)\vee 0}\,dB_t, \qquad x \in [0, 1]. \tag{4.12}$$

(i) *The case of zero drift* ($b(x) = 0$). By formulas (2.9), we have

$$\begin{aligned} &\phi(x) = 1, \qquad \xi(x) = \int_0^x 2s(1-s)\,ds = x^2 - \frac{2}{3}x^3, \\ &\psi(x) = \int_0^x dt = x, \qquad \theta(x) = \int_0^x \left(t^2 - \frac{2}{3}t^3\right) dt = \frac{x^3}{3} - \frac{x^4}{6}. \end{aligned} \tag{4.13}$$

Then, by (2.10),

$$\pi_0(x) = 1-x, \qquad \pi_1(x) = x. \tag{4.14}$$

Thus, the assumption (2.25) and its analogue for the right end are satisfied. By (2.13), $\pi_0(x)E(\tau_0(x))$ is the solution of the following problem:

$$\tfrac{1}{2}x(1-x)T''(x) = -(1-x), \qquad T(0) = T(1) = 0. \tag{4.15}$$

By solving (4.15), we easily obtain

$$E(\tau_0(x)) = -\frac{2x\ln x}{1-x}. \tag{4.16}$$

Analogously, $\pi_1(x)E(\tau_1(x))$ is the solution of the problem

$$\tfrac{1}{2}x(1-x)T''(x) = -x, \qquad T(0) = T(1) = 0 \tag{4.17}$$

and then we obtain

$$(4.18)\qquad E(\tau_1(x)) = -\frac{2(1-x)\ln(1-x)}{x}.$$

Moreover, by (2.16) we get

$$(4.19)\qquad E(\tau_{01}(x)) = -2(x\ln x + (1-x)\ln(1-x)).$$

For what concerns the second order moments of the first-passage times through the ends $\alpha = 0$ and $\beta = 1$, by Theorem 2.7 we know that $\pi_0(x)E_{\tilde{P}}(\tau_0^2(x))$ is the solution of the problem

$$(4.20)\qquad \tfrac{1}{2}\sigma^2(x)T''(x) = -2\pi_0(x)E(\tau_0(x)), \qquad T(0) = T(1) = 0.$$

Then, first, we have to solve the equation

$$(4.21)\qquad T''(x) = 8\frac{\ln x}{1-x}$$

whose solution cannot be explicitly obtained. However, we can expand $1/(1-x)$ in a geometric series and integrate the series; thus, after an integration by parts we obtain

$$(4.22)\qquad T'(x) = 8\sum_{k=0}^{\infty}\frac{1}{k+1}x^{k+1}\left(\ln x - \frac{1}{k+1}\right) + C.$$

By a further integration

$$T(x) = 8\sum_{k=0}^{\infty}\frac{x^{k+2}}{(k+1)(k+2)}\left(\ln x - \frac{1}{k+2} - \frac{1}{k+1}\right) + Cx + D.$$

Finally, by imposing the conditions $T(0) = T(1) = 0$, the constants C and D are found and we obtain

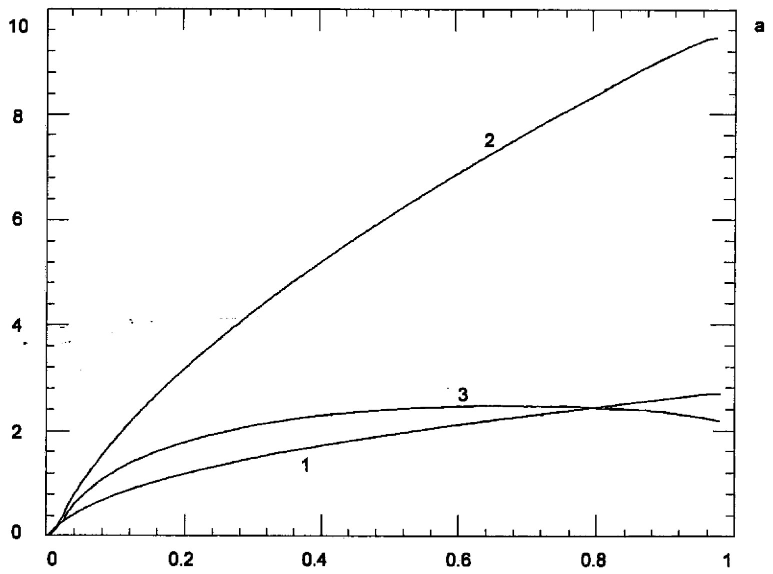
$$(4.23)\qquad E_{\tilde{P}}(\tau_0^2(x)) = \frac{8}{1-x}\left[\sum_{k=0}^{\infty}\frac{(x-x^{k+2})(2k+3)}{(k+1)^2(k+2)^2} + \sum_{k=0}^{\infty}\frac{x^{k+2}\ln x}{(k+1)(k+2)}\right].$$

For instance, by using the fact that the sum of the harmonic series is $\pi^2/6$, by (4.23), after some calculations, we get

$$E_{\tilde{P}}(\tau_0^2(1)) = 8((\pi^2/6) - 1).$$

Analogous calculations can be carried on for $E_{\tilde{P}}(\tau_1^2(x))$, and then for $E_P(\tau_{01}(x))$, but once again we cannot obtain explicit formulas, but only series expansions.

(ii) *The case of nonzero drift.* We outline the calculation in the case when $b(x)$ is a polynomial of degree one, the general case is only some more com-



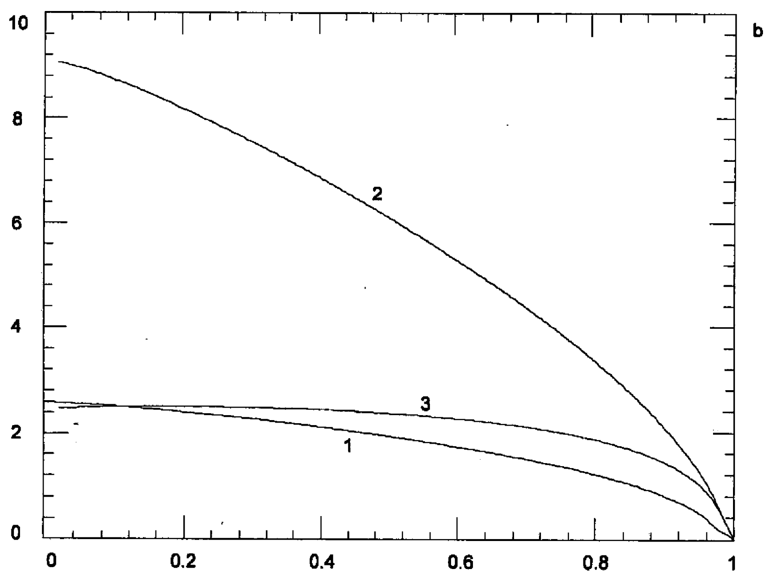


Fig. 1. Exit probabilities and moments of the first exit time for the SDE (4.12) in a symmetric case $\lambda = \mu = 0.25$

a — plot of the mean exit time at the left $E(\tau_0(x))$ (curve *1*), of the second order moment $E_{\tilde{P}}(\tau_0^2(x))$ (curve *2*), and of the variance $\mathrm{Var}_{\tilde{P}}(\tau_0(x)) = E_{\tilde{P}}(\tau_0^2(x)) - (E(\tau_0(x)))^2$ (curve *3*), as a function of the starting point $x \in [0, 1]$

b — the same as in Fig. 1a for the exit time at the right

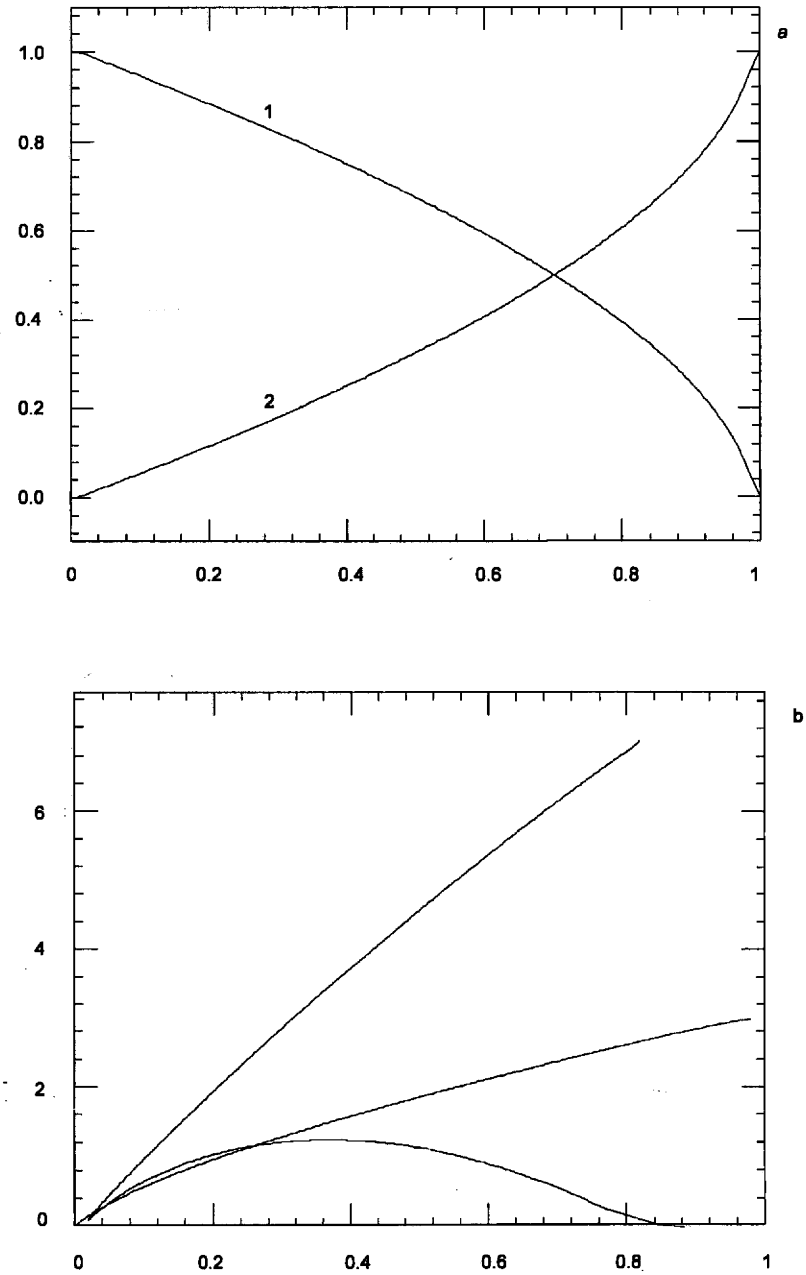


Fig. 2. Exit probabilities and moments of the first exit time for the SDE (4.12) in a nonsymmetric case $\lambda = 0.01$, $\mu = 0.125$

a — plot of the exit probability at the left (curve *1*) and at the right (curve *2*) as a function of the starting point $x \in [0, 1]$

b — as in Fig. 1a, i.e. for the exit time at the left

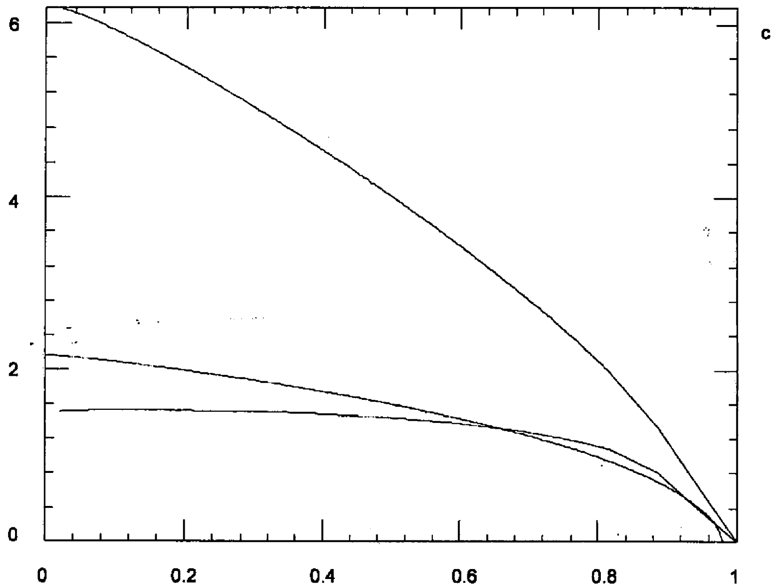


Fig. 2 (continued)
c – as in Fig. 1b, i.e. for the exit time at the right

putationally complicated. Indeed, we suppose that $b(x) = \lambda - 2\mu x$, where λ and μ are nonnegative constants such that $\lambda < 2\mu$, $\lambda < 1/2$ and $4\mu - 2\lambda < 1$. Then $a = 0$ and $b = 1$ are both attainable boundaries (see e.g. [2]).

In the present case, it is convenient to consider barriers α and β such that $0 < \alpha < \beta < 1$, and then to take the limit as $\alpha \to 0$ and $\beta \to 1$. By a straightforward calculation we get

$$\phi(x) = \frac{\alpha^{2\lambda}}{(1-\alpha)^{2(\lambda-2\mu)}} \frac{(1-x)^{2(\lambda-2\mu)}}{x^{2\lambda}}. \tag{4.24}$$

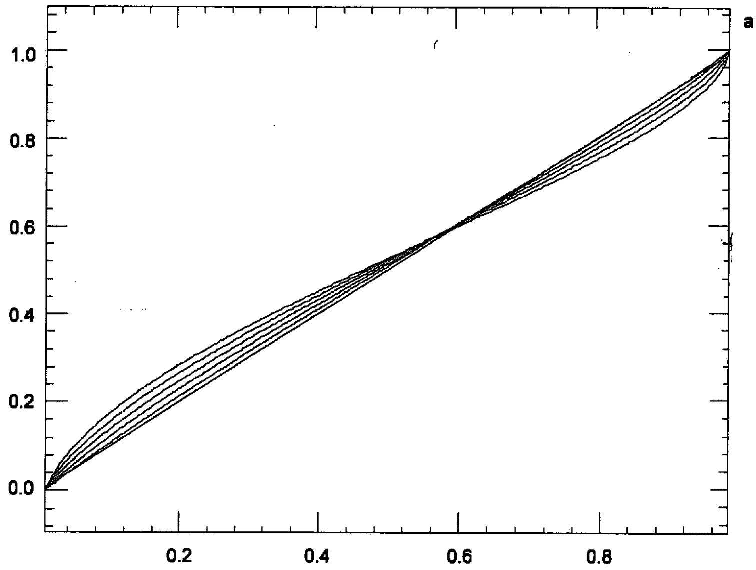
From (4.24) and (2.10) we obtain

$$\pi_\beta(x) = 1 - \pi_\alpha(x) = \frac{\int_\alpha^x (1-t)^{2(\lambda-2\mu)} t^{-2\lambda} dt}{\int_\alpha^\beta (1-t)^{2(\lambda-2\mu)} t^{-2\lambda} dt}. \tag{4.25}$$

Thus, passing to the limit as $\alpha \to 0$ and $\beta \to 1$, $\pi_1(x)$ turns out to be the distribution function of a random variable with beta-density of parameters $1-2\lambda$ and $2\lambda - 4\mu + 1$. As is easily seen

$$\pi_1(x) \sim \text{const} \cdot x^{1-2\lambda} \quad \text{as } x \to 0.$$

Then, in order that the condition analogous to (2.25) holds, it is sufficient to take $\lambda \leqslant 1/4$.



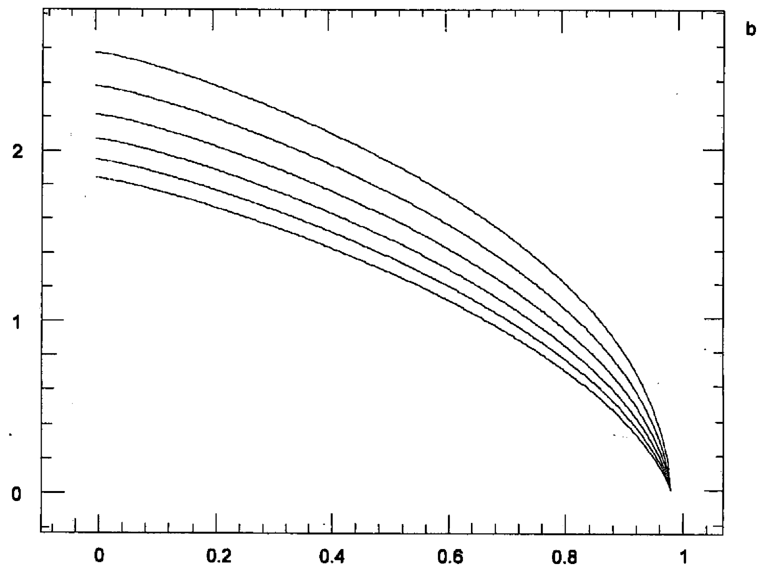
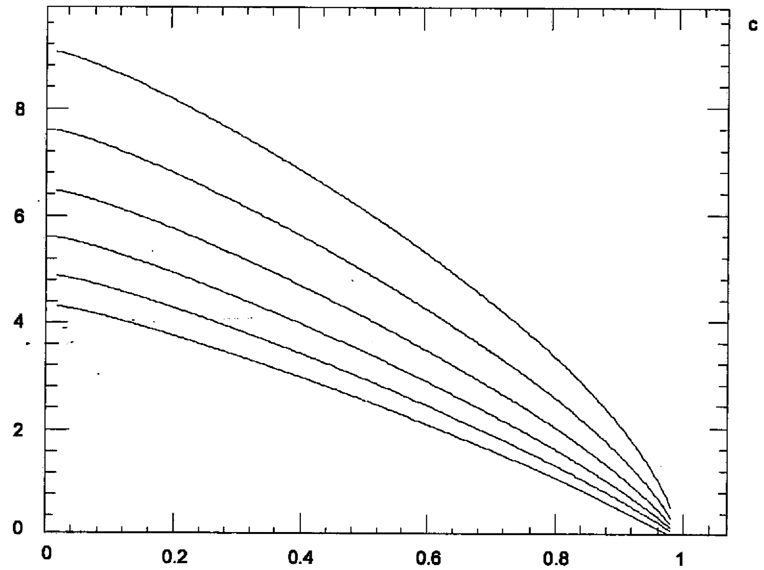


Fig. 3. Exit probabilities and moments of the first exit time at the right for the SDE (4.12) in the symmetric case for a set of decreasing values of $\lambda = \mu$ from 0.25 to 0 (step 0.05). The exit times decrease as λ decreases

a — plots of the exit probabilities at the right, $\pi_1(x)$, as a function of the starting point $x \in [0, 1]$, for the above values of λ

b — plots of the mean exit times at the right, $E(\tau_1(x))$, as a function of the starting point $x \in [0, 1]$, for the above values of λ



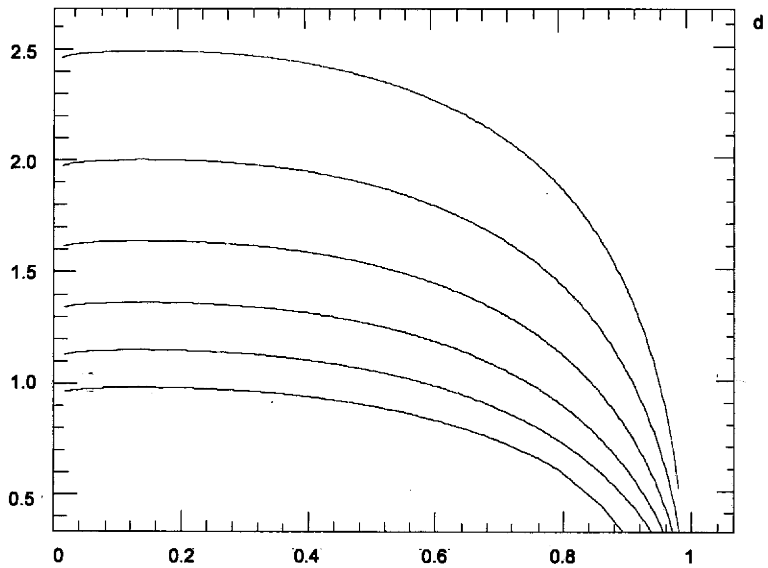


Fig. 3 (continued)

c — plots of the second order moments of the first exit times at the right, $E_{\hat{P}}(\tau_1^2(x))$, as a function of the starting point $x \in [0, 1]$, for the above values of λ

d — plots of the variance of the first exit times at the right, $\mathrm{Var}_{\hat{P}}(\tau_1(x)) = E_{\hat{P}}(\tau_1^2(x)) - (E(\tau_1(x)))^2$, as a function of the starting point $x \in [0, 1]$, for the above values of λ

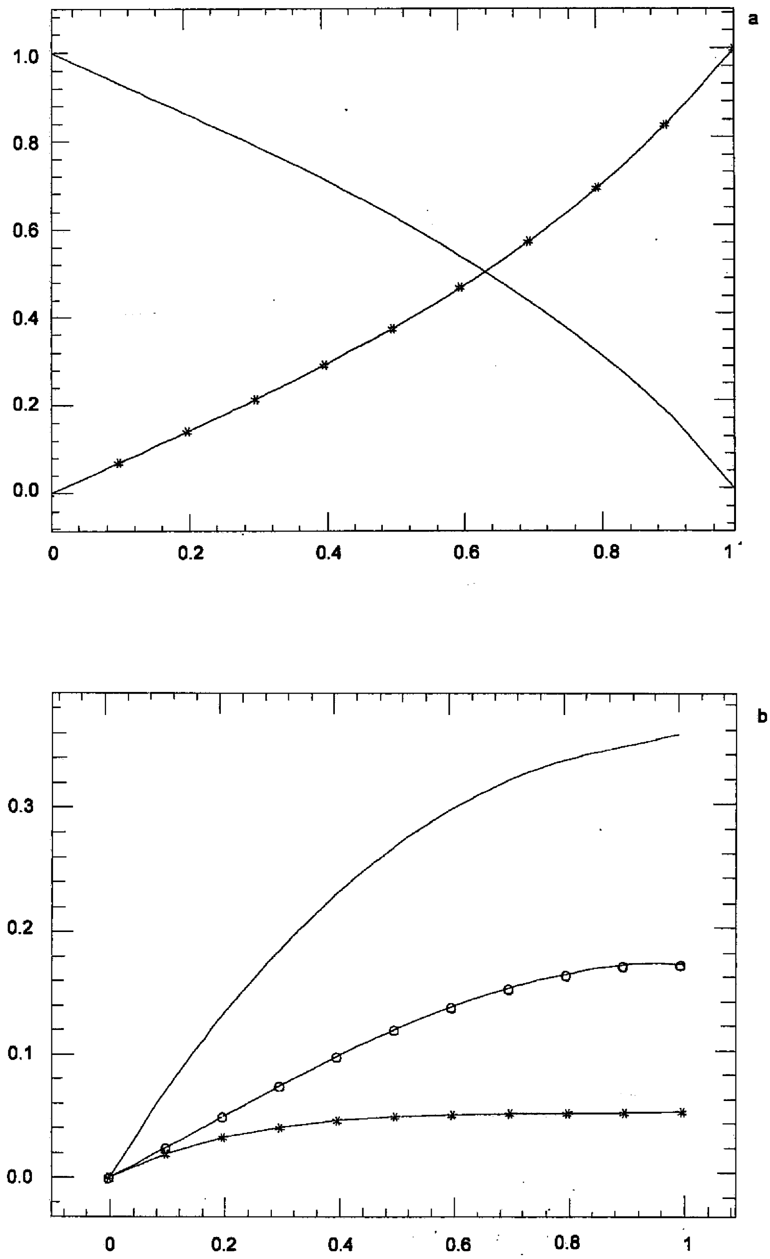
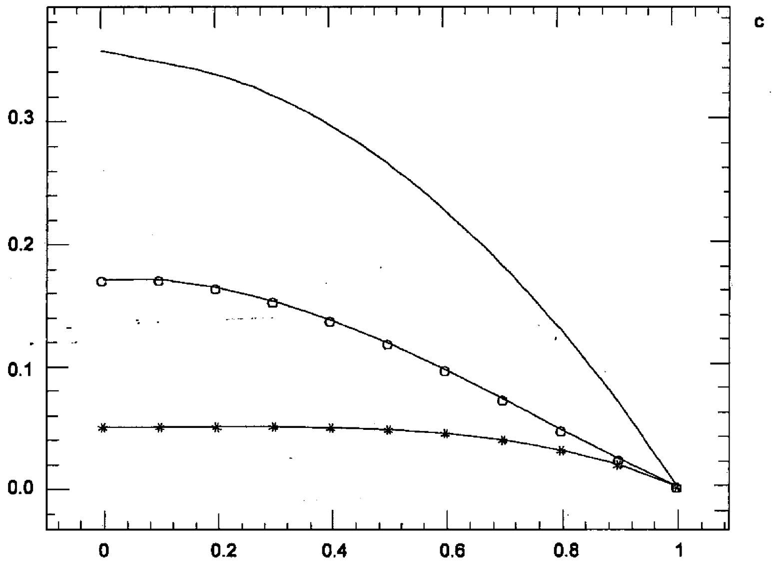


Fig. 4. Exit probabilities and moments of the first-passage time for the Ornstein-Uhlenbeck process with $b = 1$ and $\sigma = 1$ (see (4.26))

a – plot of the exit probability at the left of the interval $[0, 1]$, $\pi_0(x)$, $(-)$, and at the right, $\pi_1(x)$, $(*)$, as a function of the starting point $x \in [0, 1]$

b – plot of the mean exit time at the left, $E(\tau_0(x))$, $(-)$, of the second order moment, $E_{\tilde{P}}(\tau^2(x))$, (o), and of the variance $\mathrm{Var}_{\tilde{P}}(\tau_0(x))$, $(*)$, as a function of the starting point $x \in [0, 1]$



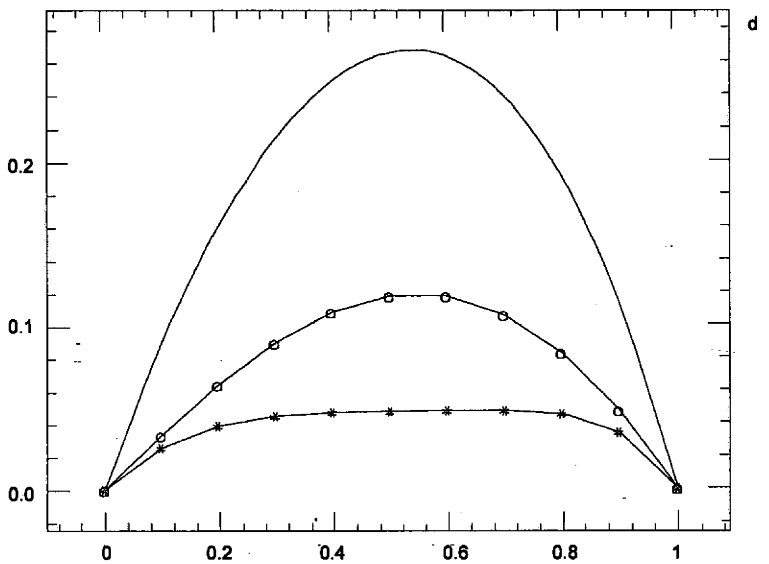


Fig. 4 (continued)

c — the same as in Fig. 2b for the exit time at the right

d — plot of the mean exit time from (0, 1), $E(\tau_{01}(x))$, (—), of the second order moment, $E(\tau_{01}^2(x))$, (o), and of the variance, $\mathrm{Var}(\tau_{01}(x))$, (*), as a function of the starting point $x \in [0, 1]$

An analogous calculation can be done for $\pi_0(x)$ and (2.25) holds under the condition $2\lambda - 4\mu + \frac{1}{2} \geqslant 0$.

Then, if the parameters λ and μ satisfy the two constraints given above, we can use Theorem 2.7, and we can obtain the first two moments of the first exit time through each of the ends of [0, 1]. However, the computations are heavy and the integral involved cannot be calculated explicitly; thus the quantities have been numerically computed. In Figs. 1, 2 and 3 some graphs are reported of the exit probabilities and of the moments of the first exit time, as a function of the initial point $x \in [0, 1]$, for some different values of the coefficients λ and μ. Notice that in the symmetric case $\lambda = \mu$ (Fig. 1), the moments of the first exit time at the left starting from x are approximately equal to those of the first exit time at the right starting from $1-x$, up to errors due to the numerical computations (let us consider that the formulas involve improper integral). In the nonsymmetric case (i.e. $\lambda \neq \mu$), while appreciable differences cannot be detected between the computed values of $E(\tau_0(1))$ and $E(\tau_1(0))$, the second moments and the variances of the exit times at the two ends show a qualitative different behavior (see Fig. 2b and 2c). In fact, if $\lambda \neq \mu$, the equilibrium point of the drift, $\lambda/2\mu$, is not equal to 1/2, but it is shifted to the left. Then, as one expects, if the process starts at $x = 1/2$, for instance, the probability to exit at the left is greater than the exit probability at the right; indeed, the point $\bar{x}$ for which $\pi_0(\bar{x}) = \pi_1(\bar{x}) = 1/2$ is shifted to the right ($\bar{x} \approx 0.7$) (see Fig. 2a).

In Fig. 3, the exit probabilities and the moments of first exit times at the right are shown in the symmetric case, for a set of decreasing values of $\lambda = \mu$ from 0.25 to 0 (step 0.05). As λ decreases, the exit time also decreases.

In the case when $b(x)$ is a polynomial of degree greater than one, we can see in an analogous way that when the coefficients of $b(x)$ lie in a certain range, the condition (2.25) and its analogue on the behavior of $\pi_0(x)$ and $\pi_1(x)$ are satisfied, and then Theorem 2.7 can be applied.

4.3. The Ornstein–Uhlenbeck process. We consider the SDE

$$dX(t) = -bX\,dt + \sigma dB_t, \tag{4.26}$$

where b and σ are positive constants, and we look for the moments of the first-passage time through the ends, for instance, of the interval [0, 1].

Also in this case, some integrals cannot be found explicitly, since the error function is involved. However, $\phi(x) = \exp(b/a^2)x^2$, and then

$$\psi(x) = \int_0^x \exp(b/a^2)s^2\,ds = x + O(x^3), \quad x \to 0,$$

so it is easy to see that

$$\pi_1(x) = \frac{x + O(x^3)}{\psi(1)}, \quad \pi_0(x) = 1 - \pi_1(x). \tag{4.27}$$

Then, by using also the fact that $\pi_1(0) = 0$ and $\pi_1(1) = 1$, one easily finds that $\pi_0(x) = O(1-x)$ as $x \to 1$, and $\pi_1(x) = O(x)$ as $x \to 0$. Thus the condition (2.25) and its analogue are satisfied, and Theorem 2.7 can be applied. In Fig. 4, the graphs of $\pi_0(x)$ and $\pi_1(x)$ are plotted as a function of the starting point $x \in (0, 1)$ for $b = \sigma = 1$; the graphs of the moments of the first-passage time through each of the ends of (0, 1) are also plotted.

APPENDIX

A hypergeometric function F is defined as

$$F(a, b, c; x) = 1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 + \ldots$$

The Jacobi polynomials $P_i^{\gamma,\delta}(y)$ are given by

$$P_i^{\gamma,\delta}(y) = \frac{1}{2^i}\sum_{m=0}^{i}\binom{i+\gamma}{m}\binom{i+\delta}{i-m}(y-1)^{i-m}(y+1)^m.$$

For more details see e.g. [1].

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