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ASYMPTOTIC BEHAVIOR FOR THE SURVIVING BROWNIAN MOTION ON THE SIERPIŃSKI GASKET WITH POISSON OBSTACLES

BY

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Abstract. A Brownian motion on the Sierpiński gasket gets absorbed at the boundary of a cloud of balls with centers distributed according to an independent Poisson law. The aim of this paper is to investigate the asymptotic behavior of the probability that up to time t the process in question has traveled far provided it has not been absorbed.

1. INTRODUCTION

The results of this paper are a continuation of [6]. We consider Poisson cloud of points $\mathcal{N}$ falling onto the Sierpiński gasket $\mathcal{G}$. It is defined on some probability space $(\Omega, \mathcal{M}, P)$ and has intensity $\nu d\mu$ ($\nu > 0$ is a fixed positive parameter, μ is the x^{d_f}-Hausdorff measure on the gasket). Let $(Z_t)_{t \geqslant 0}$ denote the Brownian motion on the gasket. Assume that the Brownian motion in question and the Poisson cloud are independent.

The Poisson points are understood as centers of balls with fixed radius $a > 0$; the Brownian motion gets absorbed at the boundary of these balls (this corresponds to Dirichlet boundary conditions imposed on the boundary of the balls).

In [6] it is proved that

$$(1) \qquad -C\nu^{2/(d_s+2)} \leqslant \liminf_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{d_s/(d_s+2)}} \leqslant \limsup_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{d_s/(d_s+2)}} \leqslant -D\nu^{2/(d_s+2)},$$

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where C and D are some positive constants, $Z_{[0,t]}$ denotes the Brownian trajectory from time 0 to t, and d_s $(= (2\log 3)/(\log 5))$ is the spectral dimension of the gasket.

$E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]$ corresponds to the Brownian motion evolving in the environment with random Poisson obstacles ("traps"). When $a = 0$, the obstacles reduce to single points. As the Brownian motion on $\mathcal{G}$ is point-recurrent, this assumption is not a qualitative change. It is elementary to see that

$$E_x[\exp\{-\nu\mu(Z_{[0,t]})\}] = \boldsymbol{P}\otimes P_x[T > t],$$

where T denotes the hitting time of the obstacles. A similar relation holds for obstacles with positive radius:

$$\boldsymbol{P}\otimes P_x[T > t] = E_x[\exp\{-\nu\mu(Z^a_{[0,t]})\}],$$

where $Z^a_{[0,t]}$ is the Wiener sausage modeled on the trajectory from time 0 to t. By introducing minor changes into the proof of (1) we get

$$(2) \qquad -C\nu^{2/(d_s+2)} \leqslant \liminf_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z^a_{[0,t]})\}]}{t^{d_s/(d_s+2)}}$$

$$\leqslant \limsup_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z^a_{[0,t]})\}]}{t^{d_s/(d_s+2)}} \leqslant -D\nu^{2/(d_s+2)}.$$

These inequalities are a gasket counterpart of the famous Wiener sausage asymptotics due to Donsker and Varadhan (see [2]).

The goal of this work is to investigate the asymptotic behavior of the probability that up to time t the process has traveled "far" (at distance $\sim t^\alpha$) provided it has not been killed. The results we obtain are similar to those in the Euclidean space (see Sznitman [9]). We were able to modify the methods from the Euclidean case to the present setting.

To state the results we need the following

DEFINITION 1. For $t, x, \alpha > 0$, $z \in \mathcal{G}$ let us define

$$F_\alpha(t, z, x) = \boldsymbol{P}\otimes P_z[T > t, \sup_{s\leqslant t} d(Z_s, Z_0) \geqslant xt^\alpha].$$

In this paper we investigate the asymptotics of F_α. Since different phenomena prevail for α in various regimes, the asymptotics will depend on α (which was also true in the Euclidean space). We obtain the following:

1. If $\alpha \in (0, d_s/(d_s+2))$, then

$$(3) \qquad -2\nu^{2/(d_s+2)} \leqslant \liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}}$$

$$\leqslant \limsup_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \leqslant -C_1\,\nu^{2/(d_s+2)}.$$

2. If $\alpha = d_s/(d_s+2)$, then

$$-2\nu^{2/(d_s+2)} - D_1 x \leqslant \liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}}$$

$$\leqslant \limsup_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \leqslant -(C_1 \nu^{2/(d_s+2)} + \tfrac{1}{3} a^{d_f-1} \nu x).$$

3. If $\alpha \in (d_s/(d_s+2), 1)$, then

$$(4) \quad -D_1 x \leqslant \liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \leqslant \limsup_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \leqslant -C_2 \nu x a^{d_f-1}.$$

4. If $\alpha = 1$, then

$$-D_1 x - D_2 x^{d_w/(d_w-1)} \leqslant \liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^\alpha}$$

$$\leqslant \limsup_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \leqslant -C_2 \nu x a^{d_f-1} - C_3 x^{d_w/(d_w-1)},$$

where C_1, C_2, C_3, D_1 and D_2 are arbitrary positive constants, $d_w = \log_2 5$ is the dimension of the walk, $d_s = (2\log 3)/(\log 5)$ is the spectral dimension of the gasket.

The critical values of the parameter α are gasket counterparts of the critical values in $\boldsymbol{R}^d$ with spectral dimension replacing the Hausdorff dimension d_f (as in all problems of this sort; in $\boldsymbol{R}^d$ there is no distinction between them, $d_s = d_f = d$).

The lower and the upper bounds are dealt with separately. Let us now describe how the bounds are obtained.

To get the lower estimate we impose some additional properties on the process and then estimate the resulting probability. To get the proper asymptotics one makes the process rush through a long "cylindrical tube" of length proportional to t^α, and then stay in a "big ball" with radius of order $t^{1/(d_f+d_w)}$ (d_w is the dimension of the walk, $\log 5/\log 2$). The difficulty one had to deal with is the absence of translation invariance of state-space and consequently the lack of any Girsanov-type formula which was used in the Euclidean space. Instead we discretize the problem and then use some hitting time estimates. This is done in Section 3.

To get the upper bound, we need a process on a compact state-space, but with no restrictions regarding its size. Therefore the approach from [6] – projecting the process onto the unit triangle – would not work. Instead we require the process stay up to time t in a "big ball" – of size comparable to $C \cdot t^{1/(d_f+d_w)}$. The exponential contribution of this assumption is asymptotically insignificant (see Theorem 2). Moreover, we were able to adapt the notion of "clearings" (where the process moves freely) and "forest" (where it risks to be killed with a big probability) from [9] and without any change in the exponential asymptotics one can assume that it kept clear from the forest for a reasonably long time and did not enter it deeper than $t^{1/(d_f+d_w)}$ too often. After we assume this, the proof of the upper bound goes like in [6] and [9].

Before we get down to any estimates, we must single out the behavior in ν — this is done by an appropriate scaling beforehand (see Section 4.1). Also, not all the numbers are permitted in the scaling (which is, basically, binary) — we again have to substitute some close binary number for the number we would like to have for the scaling factor.

Let me finally thank Professor Alain-Sol Sznitman for conjecturing the gasket counterpart of the Euclidean space-results and for encouraging discussions.

2. A SURVEY OF THE PROPERTIES OF THE BROWNIAN MOTION ON THE SIERPIŃSKI GASKET

Let us summarize the notation used and list the properties of the Brownian motion on the Sierpiński gasket we shall make use of (see [1]).

Let

$$a_0 = (0, 0), \quad a_1 = (0, 1), \quad a_2 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}), \quad V_0 = \{a_0, a_1, a_2\},$$

where a_0, a_1, a_2 are the vertices of an equilateral triangle of unit size. Let $\mathcal{I}_0$ be this equilateral triangle. We define inductively

$$V_{M+1} = V_M \cup \{2^M a_1 + V_M\} \cup \{2^M a_2 + V_M\}$$

and we put

$$\mathcal{G}_0 = \bigcup_{M=0}^{\infty} V_M \cup \bigcup_{M=0}^{\infty} \bar{V}_M,$$

where $\bar{V}_M$ denotes the symmetric image of V_M in the symmetry with respect to the y-axis. Now we let

$$\mathcal{G}_M = 2^M \mathcal{G}_0, \; M \in \mathbf{Z}, \quad \text{and} \quad \mathcal{G}_\infty = \bigcup_{M \leqslant 0} \mathcal{G}_M.$$

$\mathcal{G}_\infty$ is called the *(infinite) Sierpiński pre-gasket.* Its closure (in the Euclidean topology) is the *2-dimensional Sierpiński gasket* and it will be denoted by $\mathcal{G}$.

More notation: a *$\mathcal{G}_M$-triangle* is the closed set of points in $\mathcal{G}$ that lie inside an equilateral triangle, which is the translation of $2^M \mathcal{I}_0$ and whose vertices are the three neighboring points in $\mathcal{G}_M$. The collection of all closed $\mathcal{G}_M$-triangles will be denoted by $\mathcal{T}_M$.

The gasket can be endowed with the natural shortest path metric (which better suits our purposes): for $x, y \in \mathcal{G}_\infty$ define $d(x, y)$ to be the infimum over the Euclidean length of all paths joining x and y on the gasket.

It extends uniquely (the limit procedure) to the whole gasket $\mathscr{G}$. This metric in equivalent to the Euclidean metric on the plane, in fact

$$|x-y| \leqslant d(x, y) \leqslant 2|x-y|.$$

By B_M we shall denote the closed ball in the gasket metric, of radius 2^M, centered at zero, and by $\mathscr{F}_M$ we denote the intersection

$$B_M \cap \{(x, y) \in \boldsymbol{R}^2 \colon x \geqslant 0\}.$$

The Sierpiński gasket supports the following characteristic numbers:

$$d_f = \frac{\log 3}{\log 2} = 1.58496\ldots \text{ (fractal dimension of } \mathscr{G}\text{)},$$

$$d_s = \frac{2\log 3}{\log 5} = 1.36521\ldots \text{ (spectral dimension of } \mathscr{G}\text{)},$$

$$d_w = \frac{2d_f}{d_s} = \frac{\log 5}{\log 2} = 2.32193\ldots \text{ (dimension of the walk).}$$

These numbers fulfill:

$$\frac{d_f}{d_f+d_w} = \frac{d_s}{d_s+2}, \qquad \frac{d_w}{d_f+d_w} = \frac{2}{d_s+2}. \tag{5}$$

Let μ_M be the measure which puts mass $(\frac{2}{3})3^{-M}$ at each point in $\mathscr{G}_{-M}$. Now we state the following (Lemma 1.1 of [1]):

LEMMA 1. 1. *There exists a unique measure* μ *on* $(\boldsymbol{R}^2, \mathscr{B}(\boldsymbol{R}^2))$, *supported on* $\mathscr{G}$ *such that* $\mu(\Delta_M) = 3^{-M}$ *for all* $\Delta \in \mathscr{T}_M$, $M \in \boldsymbol{Z}$;
2. $\{\mu_M\}$ *converges to* μ *in the vague topology*;
3. μ *is a multiple of the Hausdorff* x^{d_f}*-measure on* $\mathscr{G}$;
4. $\mu(\mathscr{F}_0) = 1$.

Barlow and Perkins [1] give a construction of the process Z_t, called the *Brownian motion on the Sierpiński gasket* (the construction of the Brownian motion on the Sierpiński gasket was carried on earlier by Goldstein [4] and Kusuoka [5], but in [1] very precise estimates on the transition density were given, therefore we choose the approach from that paper). It is a strongly Markov Feller process which has a continuous symmetric density $p(t, x, y)$, satisfying (Theorem 1.5 of [1])

$$\text{(6)} \quad ct^{-d_s/2}\exp\{-c(d(x, y)t^{-1/d_w})^{d_w/(d_w-1)}\} \leqslant p(t, x, y) \leqslant ct^{-d_s/2}\exp\{-c(d(x, y)t^{-1/d_w})^{d_w/(d_w-1)}\}$$

(here and in the sequel, lower case c denotes a generic positive constant).

The process admits a discrete scaling: for $\Gamma \in \mathscr{B}(\mathscr{G})$,

$$P_x[Z_t \in \Gamma] = P_{2x}[\tfrac{1}{2}Z_{5t} \in \Gamma].$$

In particular, for the process starting from the origin,

$$\mathscr{L}(Z_t) = \mathscr{L}(\tfrac{1}{2} Z_{5t}).$$

All this translates into terms of density as (Theorem 7.8 of [1])

$$p(t, 2x, 2y) = \tfrac{1}{3} p(t/5, x, y). \tag{7}$$

The last properties we need are the following sample path and hitting time estimates from [1]:

- For all $x \in \mathscr{G}$ and all $t, \delta \in (0, \infty)$

$$P_x[\sup_{s \leqslant t} d(Z_s, Z_0) \geqslant \delta] \leqslant c \exp\{-c(\delta t^{-1/d_w})^{d_w/(d_w-1)}\}. \tag{8}$$

- If T^ϱ is the hitting time of the ϱ-grid (the grid with mesh $r = 2^\varrho$), then for $\lambda > 0$ and $z \in \mathscr{G}_\varrho$

$$E_z[\exp\{-\lambda T^\varrho\}] \geqslant \exp\{-c 5^{1+\varrho}\}. \tag{9}$$

3. ASYMPTOTIC LOWER BOUND

This section is devoted to establishing the lower-bound asymptotics for F_α.

THEOREM 1. *Let $x \in \mathbf{R}$, $z \in \mathscr{G}$. Then:*

1. *If $\alpha \in (0, d_s/(d_s+2))$, then*

$$\liminf_{t \to \infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \geqslant -2\nu^{2/(d_s+2)}. \tag{10}$$

2. *If $\alpha = d_s/(d_s+2)$, then*

$$\liminf_{t \to \infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \geqslant -2\nu^{2/(d_s+2)} - D_1 x. \tag{11}$$

3. *If $\alpha \in (d_s/(d_s+2), 1)$, then*

$$\liminf_{t \to \infty} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \geqslant -D_1 x. \tag{12}$$

4. *If $\alpha = 1$, then*

$$\liminf_{t \to \infty} \frac{F_\alpha(t, z, x)}{t^\alpha} \geqslant -D_1 x - D_2 x^{d_w/(d_w-1)}, \tag{13}$$

where D_1 and D_2 are arbitrary positive constants.

Proof. The idea that lies behind the proof is as follows: we shall force the process to move to the end of a long cylindrical "tube" in a relatively small time, and then to rest in a "big ball" attached to this "tube," its sizes being properly balanced. We should also assume that no Poisson point fell onto the a-neighborhood of this set.

To begin with, let us define the set where the process will be living up to time t. Let

$$\beta \in \left(0,\, \alpha \wedge \frac{1}{d_f + d_w}\right) \tag{14}$$

be a fixed number. Define l to be the union of two half-lines, meeting at the origin (see Fig. 1):

$$\boldsymbol{R}^2 \supset l \stackrel{\text{def}}{=} \begin{cases} (x_1, x_2)\colon\ x_2 = \frac{\sqrt{3}}{2} x_1 & \text{for } x_1 \geq 0, \\ (x_1, x_2)\colon\ x_2 = -\frac{\sqrt{3}}{2} x_1 & \text{for } x_1 \leq 0. \end{cases}$$

Let $z \in \mathscr{G}$ be fixed. Then $d = \operatorname{dist}(z, l)$ is well defined and finite. Let $\bar{z} \in l$ be the point on l which realizes this distance (if there are more than one such points, pick the one that lies closer to the origin). As d is finite and does not vary in t, and we will be interested in the long-time asymptotics, we can without loss of generality assume that z itself lies on l, and hence $z = \bar{z}$.

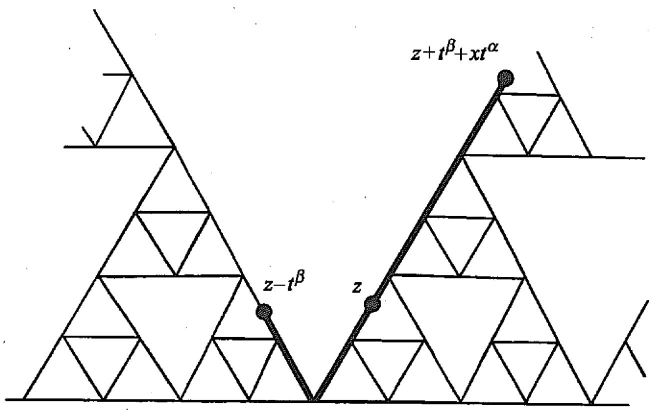

Fig. 1. The set l_t

l_t will then be a "subinterval" of l (the distance is now measured along l, see Fig. 1) such that

$$l \supset l_t \stackrel{\text{def}}{=} (z - t^{\beta},\, z + t^{\beta} + xt^{\alpha}).$$

Suppose that $r = 2^{\varrho}$ is a positive number (ϱ — an integer) and define

$$U_t \stackrel{\text{def}}{=} \{x \in \mathscr{G}\colon\ \operatorname{dist}(x, l_t) \leq r\}.$$

In the sequel U_t will be called the *cylinder*.

To the cylinder U_t we attach the ball (in gasket sense), centered at the point $l \ni p(t) \stackrel{\text{def}}{=} z + xt^\alpha$, with radius 2^n, where n is the unique integer satisfying

$$(15) \qquad 2^n \leqslant (t/\nu)^{1/(d_f+d_w)} < 2^{n+1}.$$

By $V_t \subset \mathscr{G}$ we denote the resulting set (the union of the cylinder and the ball), and by $V_t^a \subset \mathscr{G}$ its a-neighborhood.

For sufficiently large t the Hausdorff measure of V_t^a will not exceed

$$\mu(U_t^a) + \mu\big(B(p(t), 2^n + a)\big)$$
$$\leqslant \left(\frac{2t^\beta + xt^\alpha}{2^{[\log_2(r+a)]}} + 4\right)(2^{[\log_2(r+a)]+1})^{d_f} + (2^n)^{d_f}\mu\big(B\big(p(t)/2^n, 1 + a/2^n\big)\big)$$
$$\leqslant 6(2t^\beta + xt^\alpha)(r+a)^{d_f-1} + 4\cdot 3(r+a)^{d_f} + (2+\varepsilon)(t/\nu)^{d_f/(d_f+d_w)}$$

(ε corresponds to having radius $1 + a/2^n$ rather than 1 — a disturbance which can be made arbitrarily small by letting $t \to \infty$, which we will do anyway).

The last quantity will be denoted by v_t. Recalling that

$$\frac{d_f}{d_f + d_w} = \frac{d_s}{d_s + 2},$$

we see that when $t \to \infty$, v_t asymptotically behaves as

$$(16) \qquad \begin{cases} 2(t/\nu)^{d_s/(d_s+2)} & \text{if } \alpha \in \big(0, d_s/(d_s+2)\big), \\ t^{d_s/(d_s+2)}\left[6x(r+a)^{d_f-1} + 2\nu^{-d_s/(d_s+2)}\right] & \text{if } \alpha = d_s/(d_s+2), \\ 6xt^\alpha(r+a)^{d_f-1} & \text{if } \alpha \in \big(d_s/(d_s+2), 1\big]. \end{cases}$$

We are now ready to get the estimates (10)–(13). Let now $s = At^\alpha \leqslant t$, with a positive A whose value will be chosen later on.

The event

$$\{T > t\} \cap \{\sup_{s \leqslant t} d(Z_s, Z_0) > xt^\alpha\}$$

will hold if we assume that the process does not exit V_t up to time t and moves in the following way: first it goes to the right end of the cylinder in time smaller than $s < t$ and then does not leave the ball up to time t, provided no Poisson points fell onto V_t^a.

Let $\mathscr{R}$ be the *right end* of the cylinder,

$$\mathscr{R} \stackrel{\text{def}}{=} \{x \in U_t \colon d(z, x) = t^\beta + xt^\alpha + r\}.$$

From the strong Markov property of the process and the definition of v_t we obtain

$$(17)\quad F_\alpha(t, z, x) = P\otimes P_z[T > t, \sup_{s\leqslant t} d(Z_s, Z_0) > xt^\alpha]$$
$$\geqslant \exp\{-vv_t\} P_z[T_{(U_t)^c} > T_{\mathscr{R}}, T_{\mathscr{R}} \leqslant s] \inf_{y\in\mathscr{R}} P_y[T_{B(p(t), 2^n)} > t],$$

where T_A denotes the hitting time of the set A.

If $y\in\mathscr{R}$, then $d(y, \partial B(p(t), 2^n)) \geqslant 2^n - r - t^\beta$, behaving asymptotically as 2^n (this follows from the way in which n and β were defined – from (14) and (15)). Consequently, using now the scaling of the principal eigenvalue of the Laplacian ($\lambda(2U) = \frac{1}{5}\lambda(U)$) we get ($c$ is some constant which can be made arbitrarily close to 1 by using t large enough)

$$\log \inf_{y\in\mathscr{R}} P_y[T_{B(p(t), 2^n)} > t] \geqslant -c\lambda(B(p(t), 2^n))\cdot t$$
$$= -\frac{c}{5^n}\lambda(B(p(t)/2^n, 1))\cdot t \geqslant -5c\, t^{d_s/(d_s+2)} v^{2/(d_s+2)}\cdot S_1,$$

where $S_1 = \sup_{\mathscr{B}(1)} \lambda(B) < \infty$, with $\mathscr{B}(1)$ standing for the collection of all open balls with radius 1.

To estimate

$$P_z[\mathscr{A}] = P_z[T_{(U_t)^c} > T_{\mathscr{R}}, T_{\mathscr{R}} \leqslant s],$$

we will analyze the journey of the process along the ϱ-grid (recall that $\varrho = \log r$). As the distance of z from the ϱ-grid is bounded by r, we can and will assume that z itself belongs to $\mathscr{G}_\varrho$.

To be more precise, we will be looking only at the moves along l. Let q be the "first" point from the ϱ-grid that lies on l, no closer than $\mathscr{R}$ along l (possibly $q\in\mathscr{R}$). Formally,

$$q = z + \left(\left[\frac{t^\beta + xt^\alpha}{r}\right] + 1\right)\cdot r$$

(the distance measured along l all the time, see Fig. 2).

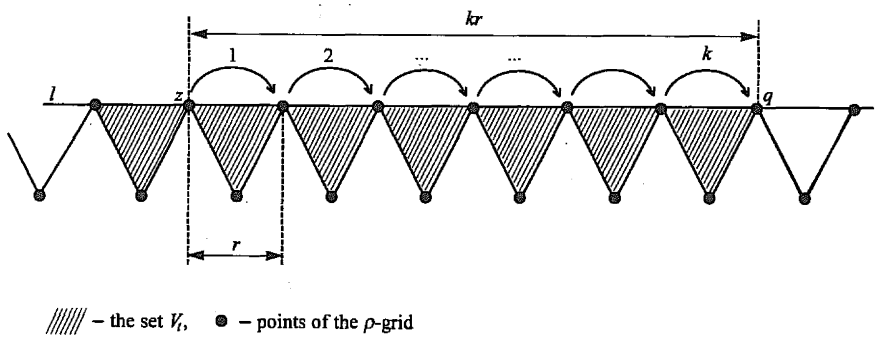

Fig. 2. Steps on the ϱ-grid

Let

$$k \stackrel{\text{def}}{=} \frac{d(z, q)}{r}$$

be the number of ϱ-triangles between z and q along l (Fig. 2). Let T_1^ϱ, T_2^ϱ, ... be consecutive hitting times of the ϱ-grid,

$$T_1^\varrho \stackrel{\text{def}}{=} \inf\{t \geqslant 0\colon\ Z_t \in \mathcal{G}_\varrho \backslash \{Z_0\}\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T_{i+1}^\varrho \stackrel{\text{def}}{=} \inf\{t \geqslant T_i\colon\ Z_t \in \mathcal{G}_\varrho \backslash \{Z_{T_i^\varrho}\}\}.$$

Observe that the event being investigated, $\mathcal{A}$, will hold if the k-th hitting time T_k^ϱ occurs at time smaller than s and at each step (by *step* we mean a step on the ϱ-grid) we pass to the next-door neighbor on the right (see Fig. 2).

By symmetry,

$$P_z[\mathcal{A}] \geqslant (\tfrac{1}{4})^k P_z[T_k^\varrho \leqslant s].$$

To estimate the probability that $\{T_k^\varrho \leqslant s\}$, we will use the following Tauberian theorem which can be found in [1] or [3]:

If Y is a nonnegative random variable, then

$$P(Y \leqslant t) \geqslant \frac{E[e^{-\lambda Y}] - e^{-\lambda t}}{1 - e^{-\lambda t}}$$

for all $\lambda, t > 0$.

This can be checked also elementarily.

By (9) we obtain

$$E_z[\exp\{-\lambda T_1^\varrho\}] \geqslant \exp\{-c\lambda^\gamma 5^{1+\varrho}\}$$

and using $\lambda = (2c \cdot 5^{1+\varrho}(k/s))^{1/(1-\gamma)}$ we get

$$P_z[T_k^\varrho \leqslant s] \geqslant \frac{(E[\exp\{-\lambda T_1^\varrho\}])^k - e^{-\lambda s}}{1 - e^{-\lambda s}} \geqslant \frac{1}{2} e^{-\lambda s/2}.$$

Recalling that

$$s = At^\alpha, \quad \gamma = \frac{1}{d_w} \quad \text{and} \quad k = \frac{d(z, q)}{r} \leftrightarrow \frac{t^\beta + xt^\alpha}{r}$$

we obtain

$$P_z[\mathcal{A}] \geqslant \frac{1}{4^k} \cdot \frac{1}{2} \exp\left\{-\frac{1}{2}\left[2c \cdot 5^{1+\varrho} \frac{t^\beta + xt^\alpha}{rAt^\alpha}\right]^{d_w/(d_w-1)} At^\alpha\right\},$$

whose logarithm will asymptotically behave as (recall that $\beta < \alpha$)

$$-t^\alpha\left[(\log 4)\frac{x}{r} + \frac{1}{2}[2c \cdot 5^{1+\varrho}]^{d_w/(d_w-1)} \left(\frac{x}{r}\right)^{d_w/(d_w-1)} A^{-1/(d_w-1)}\right].$$

Dividing now by $t^{d_s/(d_s+2)}$ (when $\alpha \in (0, d_s/(d_s+2))$) we get an expression that vanishes exponentially. When dividing by t^α (the case of $\alpha \in [d_s/(d_s+2), 1]$) we asymptotically contribute with

$$-\frac{x}{r}\log 4 - \frac{c}{A^{1/(d_w-1)}}\left(\frac{x}{r}\right)^{d_w/(d_w-1)} \tag{18}$$

For $\alpha < 1$ we can choose $A = (x/r)c^{d_w-1}$ (which, in order to get $At^\alpha \leqslant t$, may require looking at big times at once, correct in this setting) contributing with $-(x/r)(\log 4 - 1)$. For $\alpha = 1$ we are constrained to $A \leqslant 1$; therefore the best we can get is

$$-\frac{x}{r}\log 4 - \left(\frac{x}{r}\right)^{1+1/(d_w-1)} c. \tag{19}$$

Collecting (16)–(19) we obtain

1. for $\alpha \in (0, d_s/(d_s+2))$,

$$\liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \geqslant -2v^{2/(d_s+2)};$$

2. for $\alpha = d_s/(d_s+2)$,

$$\liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \geqslant -2v^{2/(d_s+2)} - D_1 x;$$

3. for $\alpha \in (d_s/(d_s+2), 1)$

$$\liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \geqslant -D_1 x;$$

4. for $\alpha = 1$,

$$\liminf_{t\to\infty} \frac{\log F_\alpha(t, z, x)}{t} \geqslant -D_1 x - D_2 x^{1+1/(d_w-1)}.$$

The constants are equal to

$$D_1 = 6(r+a)^{d_f-1} + \frac{1}{r}(\log 4 - 1), \qquad D_2 = \frac{c}{r^{d_w/(d_w-1)}}.$$

If we pick $r = 1$, we get exactly the statement of the theorem. The proof is complete. ∎

4. THE UPPER BOUND

To obtain the upper bound, we tend to somehow reduce our situation to the one that involves a process on a compact space. The usual (folding) projection (see [6]) will not yield much – one will not be able to control how far the process would go before time t, since the projected process will be confined to a small set, with size fixed once and forever. Instead, we follow the approach from [9] and request that the process does not leave a "big ball;" this procedure will add up an error that will be asymptotically negligible.

Observe that once we know that the process has survived up to time t, we can conclude that it has spent most of its time in the area where not too many obstacles were present. In this section, we shall define the areas where the process moves easily (the "clearings") and those where it risks to be killed (the "forest"). We shall see that one can reduce the whole situation to the one where the process is forced to stay in a big ball $\mathscr{P}$ and does not enter too deeply nor too frequently into the forest. As singling out the dependence on ν is desired for our purposes, we rescale the problem first.

4.1. Rescaling. The new space units we would like to have are $(t/\nu)^{1/(d_f+d_w)}$, so the resulting time units should be equal to $t^{d_s/(d_s+2)}\nu^{2/(d_s+2)}$ (recall that $d_f/(d_f+d_w) = d_s/(d_s+2)$ and $d_w/(d_f+d_w) = 2/(d_s+2)$). We face again (see [6]) the nuisance of the lack of continuous scaling: we are permitted to use only the numbers that are an integer power of 2. Instead, we rescale with a close admissible number and then estimate the error.

What we do is the following: if

$$2^n \leqslant (t/\nu)^{1/(d_f+d_w)} < 2^{n+1}, \quad \text{i.e.} \quad (t/\nu)^{1/(d_f+d_w)} = 2^n\theta(t) \text{ with } \theta(t)\in[1, 2),$$

then we use 2^n instead of $(t/\nu)^{1/(d_f+d_w)}$.

This way we will be studying the process starting from the point $z(t) = z/2^n$, evolving up to time $t/5^n$, among the obstacles with radius $a/2^n$, whose centers constitute the Poisson point process with intensity $\nu\cdot 3^n$. If we put

$$s = s(t) = t/5^n = t^{d_s/(d_s+2)}\nu^{2/(d_s+2)}[\theta(t)]^{d_w},$$

then t and s simultaneously go to $+\infty$. In terms of s, the new intensity ν_s (which now depends on time) equals $s/[\theta(t)]^{d_f+d_w}$ (and $s \geqslant \nu_s \geqslant s/15$), and the new radius of the obstacles is $\bar{a}s^{-1/d_f}(\theta(t))^{1+d_w/d_f}$ with $\bar{a} = a\nu^{1/d_f}$. The process is studied up to time s.

Once we are done with the scaling, we are ready to pursue the reduction step.

4.2. The reduction theorem. For an integer $N \geqslant 1$ and $s \geqslant 1$ we define

$$\mathscr{P} = B(0, 2^N\cdot 2^{[\log_2 s]}).$$

This ball can be partitioned into a number of smaller triangles from $\mathscr{T}_0$ meeting only at the vertices ($\mathscr{T}_0$ is the collection of the "gasket-triangles" with vertices from the 0-grid $\mathscr{G}_0$ – each smaller triangle has sides with length 1); number of those triangles equals $2\cdot 3^N\cdot 3^{[\log_2 s]}$. Without loss of generality we can and will assume that the Poisson point process does not charge the 0-grid $\mathscr{G}_0$ (vertices of the triangles).

We shall now adapt for our purposes the notions of "clearing-forest" from [9]. We will use the technique of enlarging the obstacles and then discretizing the possible obstacle set (see [8]). As the process has good recurrence properties (see Theorem 5 of [6]) we will not have to distinguish between the "good" (well surrounded) and "bad" obstacles to be neglected. The microscopic recurrence makes all the obstacles count for our purposes.

Let us pick one particular 0-triangle $\Delta \subset \mathscr{P}$. Fix a binary number $b = 2^{\beta}$. Let us write $\tilde{s}$ for the number $2^{[\log_2(s^{1/d_f})]}$, the largest binary number smaller than s^{1/d_f}. We then chop the sides of Δ into $\tilde{s}/b$ intervals (of length $b/\tilde{s}$ each), which yields $(\tilde{s}/b)^{d_f}$ smaller triangles.

Introduce now a binary number r and denote by $\mathrm{Cl}(\Delta)$ the event that "there is a clearing of size r in the triangle Δ," which means that the union of those small triangles where no Poisson points fall has relatively big measure:

$$\mathrm{Cl}(\Delta) = \{\omega\colon\ \mu(U_\Delta(\omega)) > \tfrac{1}{2}\omega(r)\},$$

where U_Δ is the (random) subset of Δ obtained by taking the union of those small triangles where no Poisson point falls, and

$$\omega(r) = \inf_{x\in\mathscr{G}} \mu(B(x, r)).$$

Let $B(\omega)$ denote the union of those 0-triangles Δ from $\mathscr{P}$ where clearing is present. $B(\omega)$ will be called the *clearing*, and $\mathscr{P}\backslash B(\omega) = F(\omega)$ the *forest*.

Define now the successive *excursion times* of the process $Z_t(w)$ (our Brownian traveler) at distance 1 into the forest as follows:

$$\begin{aligned}
A_0 &\equiv 0,\\
D_1 &= \inf\{t \geqslant 0\colon\ Z_t \in (B^1)^c\},\\
A_1 &= \inf\{t \geqslant D_1\colon\ Z_t \in B\} = D_1 + T_B \circ \theta_{D_1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
D_{n+1} &= \inf\{t \geqslant A_n\colon\ Z_t \in (B^1)^c\} = A_n + T_{(B^1)^c} \circ \theta_{A_n},\\
A_{n+1} &= \inf\{t \geqslant D_{n+1}\colon\ Z_t \in B\} = A_n + A_1 \circ \theta_{A_n} = D_{n+1} + T_B \circ \theta_{D_1},
\end{aligned}$$

where $B^1 = \{x \in \mathscr{G}\colon\ d(x, B) \leqslant 1\}$ is defined to be empty if B is empty, and T_B denotes the hitting time of the set B.

Finally, let N_s stand for the number of excursions completed by time s, and L_s for the fraction of time spent in the excursions up to time s:

$$\{N_s = k\} = \{A_k \leqslant s < A_{k+1}\} \qquad (A_0 \equiv 0),$$

$$L_s = s^{-1} \sum_{i \geqslant 1} (A_i \wedge s - D_i \wedge s).$$

We have now prepared the notions and notation for the following reduction theorem:

THEOREM 2. *For any given* $\eta > 0$ *and* $z \in \mathcal{G}$,

$$\overline{\lim_{N\to\infty}}\ \overline{\lim_{r\to 0}}\ \overline{\lim_{n_0\to\infty}}\ \overline{\lim_{b\to\infty}}\ \overline{\lim_{t\to\infty}} \frac{1}{s(t)} \log \boldsymbol{P}_t \otimes P_{z(t)} [\{T > s\} \cap (\{T_{\mathcal{P}} \leqslant s\} \cup \{\mu(B) > n_0\} \cup \{N_s \geqslant [\eta s]\} \cup \{L_s \geqslant \eta\})] = -\infty, \tag{20}$$

where $s(t)$ *and* $z(t)$ *were defined in Section* 4.1, *the limit in* r *and* b *is taken over binary numbers, and* $\boldsymbol{P}_t$ *is the law of the rescaled cloud.*

COMMENT 1. The theorem states that if the process is known to have survived up to time s, then, asymptotically up to exponential error it has stayed in the big ball, and the process kept clear from the forest for a reasonably long time. Excursions at distance 1 for the rescaled process correspond to the excursions at distance comparable with $t^{1/(d_f+d_w)}$ of the initial process. The constant η can be understood as a small number – in the next section it will be made going to zero.

The proof is similar to the proof of the reduction theorem from [9], so we omit it. ∎

4.3. Asymptotic upper bound. In this section we will derive an upper bound estimate for the probability that the Brownian motion gets at distance at least xt^{α} provided it has survived up to time t. These bounds will provide a counterpart to the lower bounds from Section 1.

Recall that for $t, x, \alpha > 0$ and $z \in \mathcal{G}$ we defined F_{α} by

$$F_{\alpha}(t, z, x) = \boldsymbol{P} \otimes P_x [T > t, \sup_{s \leqslant t} d(Z_s, Z_0) \geqslant xt^{\alpha}].$$

We will estimate

$$\frac{1}{t^{\alpha}} \log \boldsymbol{P} \otimes P_z [T > t, \sup_{u \leqslant t} d(Z_u, Z_0) \geqslant xt^{\alpha}] = \frac{\log F_{\alpha}(t, z, x)}{t^{\alpha}}.$$

Let us start with the case $\alpha = d_s/(d_s+2)$, which is more complicated than other cases and requires subtler methods. We are now in a position to prove

THEOREM 3. *For any* $z \in \mathcal{G}$, *and* $x \in \boldsymbol{R}^+$,

$$\limsup_{t\to\infty} \frac{1}{t^{d_s/(d_s+2)}} F_{d_s/(d_s+2)}(t, z, x) \leqslant -\left(C_1 \nu^{2/(d_s+2)} + \frac{1}{3} a^{d_f-1} \nu x\right),$$

where the constant C_1 *neither depends on* z *nor on* x *and is given by*

$$C_1 = 5 \inf_{U \in \mathcal{U}_0} [\lambda^0(U) + \tfrac{1}{15}\mu(U)]$$

($\mathcal{U}_0$ *is the collection of all open subsets of* $\mathcal{F}_0$, *and* λ^0 *the principal eigenvalue of the generator of the reflected Brownian motion on* $\mathcal{F}_0$, *killed upon coming to* ∂U*).*

Proof. Since the proof goes similarly to the one in [8], we give only the two parts that we had to change in the present setting.

First, rescaling: we begin by rescaling the whole situation in convenient time and space units, as at the beginning of Section 4.1: If

$$2^n \leqslant (t/v)^{1/(d_f+d_w)} < 2^{n+1},$$

i.e.,

$$(t/v)^{1/(d_f+d_w)} = 2^n \theta(t) \quad \text{with } \theta(t) \in [1, 2), \tag{21}$$

then 2^n will be taken as the new space unit.

This leads us to studying the process up to time $t/5^n \stackrel{\text{def}}{=} s(t)$, the Poisson obstacles get the new intensity

$$3^n v = \frac{s}{[\theta(t)]^{d_f+d_w}},$$

the new radius of obstacles is

$$\varrho = a/2^n = \bar{a}s^{-1/d_f}\theta^{1+d_w/d_f} \qquad (\bar{a} = av^{1/d_f}),$$

see Section 4.1.

The expression we want to bound from above will then be of the form

$$\frac{v^{2/(d_s+2)}[\theta(t)]^{d_w}}{s(t)} \log P_t \otimes P_{z(t)}[T > s, \sup_{u \leqslant s} d(Z_u, Z_0) > xs^{1-1/d_f} v^{1/d_f - 2/(d_s+2)}],$$

where P_t is the distribution of the new (rescaled) obstacles, and T is the entrance time into those obstacles.

Next, the lemma (a gasket counterpart of Lemma 3.2 from [9]):

LEMMA 2. *Let Φ: $[0, T] \to \mathscr{G}$ be a continuous function such that $d(\Phi(0), 0) \leqslant 1$. Let $\mathscr{P} = B(0, 2^N 2^{[\log_2 s]})$. Let n_0 be a fixed positive integer. Suppose that $\mathscr{B} \subset \mathscr{P}$ is a set constructed from no more than n_0 0-triangles. Let $l > 0$, $\varrho > 0$ be given. Then*

$$\mu(W^\varrho(\Phi) \cap (\mathscr{B}^{2l})^c) \geqslant \varrho^{d_f-1}\big(\tfrac{1}{2} \sup_{0 \leqslant t \leqslant T} d(\Phi(t), \Phi(0)) - 6n_0(4l+1)\big) - \varrho^{d_f}(12n_0-2),$$

where $W^\varrho(\Phi) = \{x \in \mathscr{G}: d(x, \{\Phi(t): t \in [0, T]\}) \leqslant \varrho\}$ is the sausage of radius ϱ modeled on the trajectory of Φ from time $t = 0$ to $t = T$.

Proof. Let us put $\Phi(0) = x_0$. Without loss of generality we can assume that

$$d(\Phi(T), x_0) = \sup_{0 \leqslant t \leqslant T} d(\Phi(t), x_0)$$

(reduce the interval if necessary).

The idea is to replace the given function by another one for which the parts of paths within $\mathscr{B}^{2l}$ (i.e. the paths that enter into the set whose measure is to be estimated) are noncomplicated.

Let $\mathscr{B} = \bigcup_{i=0}^{k} \Delta_i$, where $k \leqslant n_0$, $\Delta_1, \Delta_2, \ldots, \Delta_k$ being pairwise interior disjoint 0-triangles.

Define now

$$s_1 \stackrel{\text{def}}{=} \inf\{0 \leqslant v < T\colon\ \Phi(u) \in \mathscr{B}^{2l}\}$$

(with the convention that $\inf\emptyset = \infty$).

Then let t_1 be given by

$$t_1 \stackrel{\text{def}}{=} \sup\{s_1 \leqslant v \leqslant T\colon\ \Phi(v) \text{ and } \Phi(s_1) \text{ belong to the same } \Delta_i^{2l},\ i = 1, \ldots, k\}$$

for some $\Delta \in \mathscr{B}$ (note that there can be more than one triangle Δ_i such that $\Phi(s_1) \in \Delta_i^{2l}$). Repeat the procedure and define

$$s_2 = \inf\{t_1 \leqslant v < T\colon\ \Phi(v) \in \mathscr{B}^{2l}\},$$

and so on. Observe that necessarily $\Phi(s_1)$ and $\Phi(s_2)$ belong to $2l$-neighborhoods of two disjoint 0-triangles from $\mathscr{G}_0$. As $\mathscr{B}$ consists of at most n_0 triangles, after a finite number of steps – less than n_0 for sure – we exhaust all the possibilities. Therefore we obtain a finite sequence

$$0 \leqslant s_1 \leqslant t_1 \leqslant s_2 \leqslant \ldots \leqslant s_k \leqslant t_k \leqslant T, \qquad 0 \leqslant k \leqslant n_0,$$

such that for $u \notin \bigcup_{i \leqslant k} [s_i, t_i]$, $\Phi(u) \notin \mathscr{B}^{2l}$, and $\Phi(s_i)$ and $\Phi(t_i)$ belong to the same Δ^{2l} for some $\Delta \in \mathscr{M}$.

The new function, $\Psi(t)$, will be defined as follows: $\Psi(t)$ agrees with $\Phi(t)$ outside $\bigcup_{i \leqslant k} [s_i, t_i]$, and on each interval $[s_i, t_i]$ the function $\Psi(\cdot)$ follows the trajectory between $\Phi(s_i)$ and $\Phi(t_i)$ which realizes the distance between them (if there is more than one such trajectory, choose any of them).

As for a path Ψ of length d (elementary check)

$$(d/2\varrho)\,\varrho^{d_f} \leqslant \mu(W^{\varrho}(\Psi)) \leqslant (d/\varrho+2)(2\varrho)^{d_f},$$

we have

$$\mu(W^{\varrho}(\Phi) \cap (\mathscr{B}^{2l})^c) \geqslant \mu((W^{\varrho})(\Phi) \cap (\mathscr{B}^{2l})^c) - n_0\left[\left(\frac{4l+1}{\varrho}+2\right)(2\varrho)^{d_f}\right], \tag{22}$$

$(4l+1)$ being the diameter of Δ^{2l} and n_0 being the maximal number of triangles involved; (22) in turn will be greater than (for the same reasons)

$$\begin{aligned} &\mu(W^{\varrho}(\Psi)) - 2n_0\left[\left(\frac{4l+1}{\varrho}+2\right)(2\varrho)^{d_f}\right] \\ &\qquad \geqslant \left(\frac{d(\Psi(T), x_0)}{2\varrho}+2\right)\varrho^{d_f} - 2n_0\left[\left(\frac{4l+1}{\varrho}+2\right)(2\varrho)^{d_f}\right] \\ &\qquad = \varrho^{d_f-1}\left[\tfrac{1}{2} d(\Psi(T), x_0) - 6n_0(4l+1)\right] - \varrho^{d_f}(12n_0-2) \end{aligned}$$

(where the inequality used the fact that the length of the path is greater than the distance between the initial and the end points). As the definition of Ψ preserved the value at the endpoint (i.e. $\Psi(T) = \Phi(T)$), the lemma holds. ∎

The conclusion of the proof relies on some estimates for the process with compact state-space. Recall that the ordinary projection (as in $\boldsymbol{R}^d$) is useless here – it destroys the Markov property. Therefore we use the "folding projection" of the Sierpiński gasket (see [6]), which transforms the Brownian motion on the infinite gasket onto the normally reflected Brownian motion on the unit gasket triangle (see [7]) and then proceed as in [9]. ∎

The reduction theorem and the above are needed only for the most delicate case $\alpha = d_s/(d_s+2)$. For different α we do not need such subtle methods – one gets a counterpart of the lower bounds as in Theorem 1 using fairly crude estimates. We get the following

THEOREM 4. *Let* $F_\alpha(t, z, x)$ *be as before. Then*:

1. *If* $\alpha \in (0, d_s/(d_s+2))$, *then*

$$\overline{\lim_{t\to\infty}} \frac{1}{t^{d_s/(d_s+2)}} \log F_\alpha(t, z, x) \leqslant -C_1 v^{2/(d_s+2)}.$$

2. *If* $\alpha \in (d_s/(d_s+2), 1)$, *then*

$$\overline{\lim_{t\to\infty}} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \leqslant -C_2 v x a^{d_f-1}.$$

3. *If* $\alpha = 1$, *then*

$$\overline{\lim_{t\to\infty}} \frac{\log F_\alpha(t, z, x)}{t^\alpha} \leqslant -C_2 v x a^{d_f-1} - C_3 x^{d_w/(d_w-1)}.$$

The constants C_1, C_2, C_3 *are positive and neither depend on* x *nor on* z.

Proof. For $\alpha \in (0, d_s/(d_s+2))$, we use the natural bound from [6]:

$$\overline{\lim_{t\to\infty}} \frac{\log F_\alpha(t, z, x)}{t^{d_s/(d_s+2)}} \leqslant \overline{\lim_{t\to\infty}} \frac{\log P_z[T > t]}{t^{d_s/(d_s+2)}} \leqslant -C_1 v^{2/(d_s+2)}$$

(C_1 is the same constant as in Theorem 3).

To get the estimate for $\alpha > d_s/(d_s+2)$, we first observe that

$$F_\alpha(t, z, x) \leqslant E_z[\exp\{-v\mu(Z^a_{[0,t]})\} \cdot 1\{\sup_{s\leqslant t} d(Z_s, Z_0) > xt^\alpha\}]$$

and

$$\mu(Z^a_{[0,t]}) \geqslant C_2 \sup_{s\leqslant t} d(Z_s, Z_0)$$

with some positive constant C_2 (see the proof of Lemma 2); therefore (use the estimate (8))

$$F_\alpha(t, z, x) \leqslant \exp\{-\nu C_2 x t^\alpha a^{d_f - 1}\} P_z[\sup_{s \leqslant t} d(Z_s, Z_0) > x t^\alpha]$$
$$\leqslant \exp\{-\nu C_2 x t^\alpha a^{d_f - 1}\} c \exp\{-c(x t^\alpha t^{-1/d_w})^{d_w/(d_w - 1)}\},$$

which, after taking the logarithm and dividing by t^α, contributes asymptotically with $-C_2 \nu x a^{d_f - 1}$ for $\alpha < 1$ and with $-C_2 \nu x a^{d_f - 1} - C_3 x^{d_w/(d_w - 1)}$ for $\alpha = 1$. The theorem is established. ■

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