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ON WEAK CONVERGENCE OF ONE-DIMENSIONAL DIFFUSIONS WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. We investigate stability, with respect to convergence of coefficients, of one-dimensional Itô diffussions as well as one-dimensional diffusions corresponding to second order divergence form operators. We assume that the coefficients are measurable, uniformly bounded and that the diffusion coefficients are uniformly positive.

1. Introduction. In the present paper we investigate stability, with respect to convergence of coefficients, of one-dimensional diffusions $(X, P_x^{a,b})$ and $(X, Q_x^{a,b})$ corresponding to the operators

$$L(a, b) = \frac{1}{2} a \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x} \quad \text{and} \quad \mathscr{L}(a, b) = \frac{1}{2} \frac{\partial}{\partial x}\left(a \frac{\partial}{\partial x}\right) + b \frac{\partial}{\partial x},$$

respectively, starting at time 0 from $x \in \boldsymbol{R}$. Here and subsequently we assume that a, b: $[0, T] \times \boldsymbol{R} \to \boldsymbol{R}$ are real measurable functions such that

$$(1.1) \qquad (*)\ \lambda \leqslant a(t, x) \leqslant \Lambda, \quad (**)\ |b(t, x)| \leqslant \Lambda, \quad \lambda, \Lambda = \text{const} > 0,$$

for all $(t, x) \in [0, T] \times \boldsymbol{R}$, so that the measures $P_x^{a,b}$ and $Q_x^{a,b}$ are uniquely determined by $L(a, b)$ and $\mathscr{L}(a, b)$ (see [12], Exercise 7.3.3, for the non-divergence case, and [7], [10] for the divergence case).

Let $\mathscr{A}(\lambda, \Lambda)$ and $\mathscr{B}(\Lambda)$ denote the classes of all functions satisfying $(*)$ and $(**)$, respectively, and suppose that $\{a_n\} \subset \mathscr{A}(\lambda, \Lambda)$ and $\{b_n\} \subset \mathscr{B}(\Lambda)$. Then $\{P^n \equiv P_x^{a_n,b_n}\}$ and $\{Q^n \equiv Q_x^{a_n,b_n}\}$ are relatively compact in the topology of weak convergence. We will show that the limits are all again measures corresponding to some $L(a, b)$ or $\mathscr{L}(a, b)$. Characterization of weak convergence of measures in terms of convergence of coefficients presents a more delicate problem. To see this, following [11] let us consider two special cases. If a_n and b_n do not depend on t, then $P^n \Rightarrow P_x^{a,b}$ (and $Q^n \Rightarrow Q_x^{a,b}$) with $a = 1/A$, $b = B/A$ iff

$$(1.2) \qquad 1/a_n \to A \quad \text{and} \quad b_n/a_n \to B \quad \text{locally weakly in } \boldsymbol{L}_2(\boldsymbol{R})$$

(see Remark 4.2). On the other hand, if a_n and b_n do not depend on x, then $P^n = Q^n \Rightarrow P_x^{a,b}$ iff

$$(1.3) \qquad a_n \to a \quad \text{and} \quad b_n \to b \quad \text{weakly in } L_2(0, T)$$

(see Remark 4.2). Therefore, if the coefficients depend on both t and x, then the conditions (1.2) and (1.3) provide conflicting clues. To overcome this difficulty we impose some regularity assumptions on the dependence of diffusion coefficients on the time variable. More precisely, we prove that if $\{a_n\} \subset \mathscr{A}(\lambda, \Lambda)$ and $\{b_n\} \subset \mathscr{B}(\Lambda)$,

$$(1.4) \qquad 1/a_n \to A \quad \text{and} \quad b_n/a_n \to B \quad \text{locally weakly in } L_2((0, T)\times \mathbf{R})$$

and either the condition

$$(1.5) \qquad \forall (R > 0) \sup_{(t,x)\in(0,T)\times(-R,R)} \sup_{n\geq 1} \left|\frac{\partial}{\partial t}\int_0^x \frac{1}{a_n(t, y)}dy\right| < \infty$$

considered in [6] or the following condition introduced in [2] is fulfilled:

$$(1.6) \quad \forall (K \subset (0, T)\times \mathbf{R},\ K-\text{compact}) \lim_{h\to 0} \sup_{n\geq 1} \int_K |a_n(t+h, x) - a_n(t, x)|\, dt dx = 0,$$

then $P^n \Rightarrow P_x^{a,b}$ and $Q^n \Rightarrow Q_x^{a,b}$ with $a = 1/A$ and $b = B/A$.

From the results of [6] it follows that under (1.4) and (1.5)

$$\mathscr{L}(a_n, b_n) \to \mathscr{L}(1/A, B/A)$$

in the sense of G-convergence of parabolic operators, whereas in [2] (see Remark 4.2) it is proved that

$$\mathscr{L}(a_n, 0) \to \mathscr{L}(1/A, 0)$$

if (1.4) and (1.6) are satisfied. On the other hand, it is known that G-convergence of generators is equivalent to weak convergence of the corresponding diffusions (see [7]), so in the case of diffusions with divergence form generators the novelty of our work consists in extending the one-dimensional result of [2] to operators with non-zero first order terms. In the case of Itô diffusions we weaken the known sufficient conditions which state that, for general time-dependent coefficients, $P^n \Rightarrow P_x^{a,b}$ if $a_n \to a$ and $b_n \to b$ locally in $L_2((0, T)\times \mathbf{R})$ or $a_n \to a$, $b_n \to b$ locally weakly in $L_2((0, T)\times \mathbf{R})$ and for each $t \in [0, T]$ the functions $a_n(t, \cdot)$, $n \in \mathbf{N}$, are equicontinuous (see Remark 4.2).

We will use the following notation:

$$\Omega_T^R = (0, T)\times(-R, R), \qquad \Omega_T = (0, T)\times \mathbf{R}.$$

$C([0, T]; \mathbf{R})$ is the space of real continuous functions on $[0, T]$ equipped with the topology of uniform convergence, the Borel σ-field $\mathscr{C}$, and the canonical process X. $\mathscr{L}[Y \mid P]$ is the law of Y under P and "$\Rightarrow$" denotes weak convergence of measures. $L_p(\Omega_T^R)$ is the Banach space of measurable functions on Ω_T^R having the finite norm

$$\|u\|_{p,\Omega_T^R} = \Big(\int\int_{\Omega_T^R} |u(t, x)|^p \, dt\, dx\Big)^{1/p}$$

and $W_2^{1,0}(\Omega_T^R)$ is the Hilbert space consisting of the elements u of $L_2(\Omega_T^R)$ having generalized derivatives $\partial u/\partial t$ square-integrable on Ω_T^R. The scalar product is defined by the equality

$$(u, v) = \int\int_{\Omega_T^R} \left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + uv\right)(t, x)\, dt\, dx.$$

2. Diffusions corresponding to divergence form operators. In this section we will use some analytical results concerning G-compactness and G-convergence of parabolic operators to prove our main theorem on convergence of diffusions corresponding to divergence form operators. It is worth pointing out that in particular we will use results of [6], whose proofs are based on some estimates on solutions to equations in non-divergence form. These estimates were obtained from estimates of [3] on solutions of equations in divergence form by means of a change of variables which transforms parabolic equations in non-divergence form to equations in divergence form. In the next section we will go in the opposite direction. Namely, we will use a similar transformation to obtain results concerning convergence of Itô diffusions from the following theorem.

THEOREM 2.1. *Let* $\{a_n\} \subset \mathscr{A}(\lambda, \Lambda)$, $\{b_n\} \subset \mathscr{B}(\Lambda)$ *and let* $y_n \to y$.

(i) *If* $Q_{y_n}^{a_n,b_n} \Rightarrow \tilde{Q}$ *in* $C([0, T]; \boldsymbol{R})$, *then there exist* $\tilde{\Lambda} > 0$, *and* $a \in \mathscr{A}(\lambda, \Lambda)$, $b \in \mathscr{B}(\tilde{\Lambda})$ *such that* $\tilde{Q} = Q_y^{a,b}$. *Moreover, if* $b_n = 0$ *for* $n \in \boldsymbol{N}$, *then* $b = 0$.

(ii) *Assume additionally that* $\{a_n\}$ *satisfies* (1.5) *or* (1.6). *If* $1/a_n \to A$, $b_n/a_n \to B$ *weakly in* $L_2(\Omega_T^R)$ *for every* $R > 0$, *then* $Q_{y_n}^{a_n,b_n} \Rightarrow Q_y^{a,b}$ *in* $C([0, T]; \boldsymbol{R})$, *where* $a = 1/A$ *and* $b = B/A$.

Proof. (i) By [13], Theorem 22, there exist bounded measurable functions a, b and a subsequence $\{n'\} \subset \boldsymbol{N}$ such that $\mathscr{L}(a_{n'}, b_{n'})$ strongly G-converges to $\mathscr{L}(a, b)$ in Ω_T. In particular, by Theorems 21 and 18 in [13], $\mathscr{L}(a_{n'}, 0)$ strongly G-converges to $\mathscr{L}(a, 0)$ in Ω_T, which forces $b = 0$ in the case $b_n = 0$ for $n \in \boldsymbol{N}$. Moreover, $\lambda - \mathscr{L}(a_{n'}, 0)$ are in the class $\sigma(\lambda/2, \sqrt{\Lambda/2})$ defined in [13]; hence by [13], Theorem 26,

$$\lambda - \mathscr{L}_t(a, 0) \in \sigma(\lambda/2, \sqrt{\Lambda/2}),$$

which yields $a \in \mathscr{A}(\lambda, \Lambda)$ (alternatively one can use [9], Theorem 3, and [13], Theorem 21). Due to Theorem 5.2 and Remark 7.3 in [7], G-convergence of $\mathscr{L}(a_{n'}, b_{n'})$ to $\mathscr{L}(a, b)$ implies that $Q_{y_{n'}}^{a_{n'},b_{n'}} \Rightarrow Q_y^{a,b}$ in $C([0, T]; \boldsymbol{R})$. Thus $\tilde{Q} = Q_y^{a,b}$.

(ii) In view of Theorem 4.2, Lemma 4.1 and Remark 7.3 in [7] it is sufficient to show that, for every $t\in(0, T]$, $Q_n^{\cdot t}\varphi \to Q^{\cdot t}\varphi$ weakly in $L_2(\Omega_t)$ for all $\varphi\in C_0^\infty(\boldsymbol{R})$, where $\{Q_n^{st}\}$ and $\{Q^{st}\}$ denote semigroups of operators associated with $\mathscr{L}(a_n, b_n)$ and $\mathscr{L}(1/A, B/A)$, respectively. By the inequality (2.3) in [7], $\{Q_n^{\cdot t}\varphi\}$ is weakly relatively compact in $L_2(\Omega_t)$, so we only need to show that

$$Q_n^{\cdot t}\varphi \to Q^{\cdot t}\varphi \quad \text{weakly in } L_2(\Omega_t^R) \tag{2.1}$$

for any $R>0$. If $\{a_n\}$ satisfies (1.5), then analysis similar to that in the proofs of Theorems 2 and 3 in [6] shows that $Q_n^{\cdot t}\varphi \to Q^{\cdot t}\varphi$ uniformly on compacts in Ω_t^R, which implies (2.1). Now, suppose that the condition (1.6) is satisfied. Fix $R>0$ and define $\tilde{a}_n\colon \boldsymbol{R}^2\to\boldsymbol{R}$ by

$$\tilde{a}(s, x) = \begin{cases} a^{-1}(s, x) & \text{if } (s, x)\in\Omega_T,\\ A^{-1} & \text{otherwise,}\end{cases}$$

$$\tilde{a}_n(s, x) = \begin{cases} a_n^{-1}(s, x) & \text{if } (s, x)\in\Omega_T,\\ A^{-1} & \text{otherwise.}\end{cases}$$

By (1.6), for every $\varepsilon>0$ there is $\delta_\varepsilon>0$ such that

$$\int\int_{\Omega_T^R} |\tilde{a}_n(s+h, x)-\tilde{a}_n(s, x)|\,ds\,dx < \varepsilon$$

for $|h|\leqslant\delta_\varepsilon$. Given $\varepsilon>0$ put

$$a_\varepsilon(s, x) = \int_{\boldsymbol{R}} \tilde{a}(s-\tau, x)\varrho_\varepsilon(\tau)\,d\tau, \qquad a_{n,\varepsilon}(s, x) = \int_{\boldsymbol{R}} \tilde{a}_n(s-\tau, x)\varrho_\varepsilon(\tau)\,d\tau \tag{2.2}$$

for $(s, x)\in\bar{\Omega}_T$, where $\varrho_\varepsilon\in C^\infty(\boldsymbol{R})$ is a non-negative function such that $\int\varrho_\varepsilon(s)\,ds = 1$ and $\varrho_\varepsilon(s) = 0$ for $|s|>\delta_\varepsilon$. We check at once that for every $\varepsilon>0$

$$\sup_{(s,x)\in\Omega_T^R}\sup_{n\geqslant 1}\left|\frac{\partial}{\partial s}\int_0^x a_{n,\varepsilon}(s, y)\,dy\right| \leqslant \lambda^{-1}R\int_{\boldsymbol{R}}\left|\frac{d}{ds}\varrho_\varepsilon(s)\right|ds \tag{2.3}$$

and

$$\sup_{n\geqslant 1}\int\int_{\Omega_T^R}\left|\frac{1}{a_{n,\varepsilon}}(s, x)-a_n(s, x)\right|ds\,dx \leqslant A^2\varepsilon. \tag{2.4}$$

Moreover, by the dominated convergence theorem, $a_{n,\varepsilon}\to a_\varepsilon$ weakly in $L_2(\Omega_T^R)$. Write

$$b_\varepsilon = b/(aa_\varepsilon), \qquad b_{n,\varepsilon} = b_n/(a_n a_{n,\varepsilon}). \tag{2.5}$$

Let $\{Q_{n,\varepsilon}^{st}\}$ be the semigroup corresponding to $\mathscr{L}_t(1/a_{n,\varepsilon}, b_{n,\varepsilon})$ and let $\{Q_\varepsilon^{st}\}$ correspond to $\mathscr{L}_t(1/a_\varepsilon, b_\varepsilon)$. Since $\{a_{n,\varepsilon}\}_{n\in N}$ satisfies (2.3) and for every $\varepsilon>0$

$$1/(1/a_{n,\varepsilon}) = a_{n,\varepsilon}\to a_\varepsilon = 1/(1/a_\varepsilon), \qquad b_{n,\varepsilon}/(1/a_{n,\varepsilon}) = b_n/a_n\to B = b_\varepsilon/(1/a_\varepsilon) \tag{2.6}$$

weakly in $L_2(\Omega_T^R)$, it follows that $Q_{n,\varepsilon}^{\cdot t}\varphi \to Q_\varepsilon^{\cdot t}\varphi$ uniformly on compacts in Ω_t^R, and hence

$$(2.7)\qquad Q_{n,\varepsilon}^{\cdot t}\varphi \to Q_\varepsilon^{\cdot t}\varphi \quad \text{weakly in } L_2(\Omega_t^R),$$

because $\{Q_{n,\varepsilon}^{\cdot t}\varphi\}_{n\in N}$ is bounded in $L_2(\Omega_t^R)$, and hence weakly relatively compact. Observe now that

$$(2.8)\qquad 1/a_\varepsilon \to a, \quad b_\varepsilon \to b \quad \text{in } L_2(\Omega_T^R)$$

as $\varepsilon \to 0$ and that $u_\varepsilon = Q_\varepsilon^{\cdot t}\varphi - Q^{\cdot t}\varphi$ is a weak solution to the Cauchy problem

$$\frac{\partial u_\varepsilon}{\partial s} + \frac{\partial}{\partial x}\left(a_\varepsilon \frac{\partial u_\varepsilon}{\partial x}\right) = f_\varepsilon + \frac{\partial \tilde{f}_\varepsilon}{\partial x}, \quad \lim_{s\nearrow t} u_\varepsilon(s,\cdot) = 0 \quad \text{in } L_2(\boldsymbol{R}),$$

where

$$f_\varepsilon = (b-b_\varepsilon)\frac{\partial}{\partial x}(Q^{\cdot t}\varphi), \quad \tilde{f}_\varepsilon = \frac{1}{2}(a - 1/a_\varepsilon)\frac{\partial}{\partial x}(Q^{\cdot t}\varphi)$$

(see the proof of (2.7) in [7] or Theorem III.4.5 in [5]). Consequently,

$$(2.9)\qquad Q_\varepsilon^{\cdot t}\varphi \to Q^{\cdot t}\varphi \quad \text{in } L_2(\Omega_t^R)$$

due to the energetic inequality and by the assumption that $f_\varepsilon = \tilde{f}_\varepsilon = 0$ on $\boldsymbol{R}^2 \setminus \Omega_t^R$. Finally, from (2.4) it follows that

$$(2.10)\qquad \lim_{\varepsilon\to 0} \sup_{n\geq 1} \left(\|(1/a_{n,\varepsilon}) - a_n\|_{2,\Omega_T^R} + \|b_{n,\varepsilon} - b_n\|_{2,\Omega_T^R}\right) = 0;$$

hence by the same manner as above we can see that

$$(2.11)\qquad Q_{n,\varepsilon}^{\cdot t}\varphi - Q_n^{\cdot t}\varphi \to 0 \quad \text{in } L_2(\Omega_t^R)$$

uniformly in $n \in N$. Combining (2.7) with (2.9) and (2.11) yields (2.1), and the proof is completed. ∎

3. Itô diffusions. We begin with introducing a transformation of the phase space which transforms diffusions corresponding to divergence form operators into Itô diffusions. This transformation is a counterpart of the transformation of variables used in [6].

LEMMA 3.1. *Let* $a \in \mathscr{A}(\lambda, \Lambda)$ *and* $b \in \mathscr{B}(\Lambda)$. *Set*

$$F(t,z) = \int_0^z \frac{1}{a(t,y)}\,dy, \quad (t,z) \in \bar{\Omega}_T,$$

and assume that there is $\Lambda_1 > 0$ *such that*

$$(3.1)\qquad \sup_{(t,z)\in\Omega_T} \left|\frac{\partial}{\partial t} F(t,z)\right| \leq \Lambda_1.$$

Then for every $x \in \boldsymbol{R}$

$$P_x^{a,b} = \mathscr{L}[F^{-1}(\cdot, X) \mid Q_y^{\alpha,\beta}],$$

where $y = F(0, x)$, $F^{-1}(t, \cdot)$ *is the inverse of* $F(t, \cdot)$, *and*

$$\alpha(t, z) = \frac{1}{a\big(t, F^{-1}(t, z)\big)}, \qquad \beta(t, z) = \left(\frac{b}{a} + \frac{\partial F}{\partial t}\right)\big(t, F^{-1}(t, z)\big) \tag{3.2}$$

for $(t, z) \in \bar{\Omega}_T$.

Proof. Let $\zeta \in C_0^\infty(\boldsymbol{R})$ be a non-negative function such that $\int_{\boldsymbol{R}} \zeta(x)\, dx = 1$, and define $\zeta_n(x) = n\zeta(nx)$ for $n \in \boldsymbol{N}$. Set

$$a_n(t, \cdot) = 1\Big/\left(\zeta_n * \frac{1}{a}(t, \cdot)\right), \qquad b_n(t, \cdot) = \zeta_n * b(t, \cdot)$$

($*$ denotes convolution); then

$$\alpha_n(t, z) = \frac{1}{a_n\big(t, F_n^{-1}(t, z)\big)}, \qquad \beta_n(t, z) = \left(\frac{b_n}{a_n} + \frac{\partial F_n}{\partial t}\right)\big(t, F_n^{-1}(t, z)\big) \tag{3.3}$$

for $(t, z) \in \bar{\Omega}_T$, where

$$F_n(t, z) = \int_0^z \frac{1}{a_n(t, y)}\, dy, \qquad (t, z) \in \bar{\Omega}_T. \tag{3.4}$$

Then $\{\alpha_n\} \subset \mathscr{A}(\lambda, \Lambda)$, $\{\beta_n\} \subset \mathscr{B}(\lambda^{-1}\Lambda + \Lambda_1)$ and, by the generalized Itô formula ([4], Theorem II.10.1), for each $n \in \boldsymbol{N}$

$$\mathscr{L}[F_n(\cdot, X) \mid P_x^{a_n, b_n}] = Q_{y_n}^{\alpha_n, \beta_n} \tag{3.5}$$

with $y_n = F_n(0, x)$. Moreover, $a_n \to a$ and $b_n \to b$ in $L_2(\Omega_T^R)$ for $R > 0$; hence using Krylov's estimates ([4], Theorem II.3.4) and arguing as in [12], Exercise 7.3.2, we prove that

$$P_x^{a_n, b_n} \Rightarrow P_x^{a, b} \quad \text{in } C([0, T]; \boldsymbol{R}). \tag{3.6}$$

We next show that

$$Q_{y_n}^{\alpha_n, \beta_n} \Rightarrow Q_y^{\alpha, \beta} \quad \text{in } C([0, T]; \boldsymbol{R}). \tag{3.7}$$

To this end, first observe that for every $(t, z) \in \bar{\Omega}_T$

$$\int_0^z \frac{1}{\alpha_n(t, y)}\, dy = \int_0^z a_n\big(t, F_n^{-1}(t, y)\big)\, dy = F_n^{-1}(t, z)$$

and that

$$0 = \frac{\partial}{\partial t}\Big(F_n\big(t, F_n^{-1}(t, z)\big)\Big)$$

$$= \frac{\partial F_n}{\partial t}\big(t, F_n^{-1}(t, z)\big) + \frac{\partial F_n}{\partial z}\big(t, F_n^{-1}(t, z)\big)\frac{\partial F_n^{-1}}{\partial t}(t, z);$$

hence for every $(t, z) \in \Omega_T$

$$\left|\frac{\partial}{\partial t}\int_0^z \frac{1}{\alpha_n(t, y)}dy\right| = \left|\frac{\partial F_n}{\partial t}(t, F_n^{-1}(t, z))/\alpha_n(t, z)\right|$$
$$\leqslant \lambda^{-1}\Lambda \sup_{(t,z)\in\Omega_T}\left|\frac{\partial}{\partial t}F_n(t, z)\right|.$$

Since

$$\frac{\partial}{\partial t}F_n(t, z) = \int_{\boldsymbol{R}}\frac{\partial}{\partial t}(F(t, x-z)-F(t, -z))\zeta_n(z)\,dz$$

for $(t, z)\in\Omega_T$, it follows by the above that $\{\alpha_n\}$ satisfies (1.5). Now notice that for arbitrary but fixed $R > 0$ and for any bounded continuous f: $\boldsymbol{R}^2 \to \boldsymbol{R}$ with $\operatorname{supp} f \in \Omega_T^R$

$$\int\int_{\Omega_T^R}\frac{1}{\alpha_n}f(s, z)\,ds\,dz = \int\int_{\Omega_T}\frac{1}{a_n(s, z)}f(s, F_n(s, z))\frac{\partial F_n}{\partial z}(s, z)\,dsdz$$
$$= \int\int_{\Omega_T} f(s, F_n(s, z))\,dsdz,$$

which converges to

$$\int\int_{\Omega_T} f(s, F(s, z))\,dsdz = \int\int_{\Omega_T^R}\frac{1}{\alpha}f(s, z)\,dsdz,$$

since $F_n \to F$ pointwise in Ω_T^R. Furthermore, $F_n \to F$ in $L_2(\Omega_T^R)$ and $\{F_n\}$ is bounded in $W_2^{1,0}(\Omega_T^R)$, and hence weakly relatively compact in $W_2^{1,0}(\Omega_T^R)$. Therefore $\partial F_n/\partial t \to \partial F/\partial t$ weakly in $L_2(\Omega_T^R)$, and, consequently,

$$\int\int_{\Omega_T^R}\frac{\beta_n}{\alpha_n}f(s, z)\,dsdz = \int\int_{\Omega_T}\left(\frac{b_n}{a_n}+\frac{\partial F_n}{\partial s}\right)(s, z)\,f(s, F_n(s, z))\,dsdz$$
$$\to \int\int_{\Omega_T}\left(\frac{b}{a}+\frac{\partial F}{\partial s}\right)(s, z)\,f(s, F(s, z))\,dsdz = \int\int_{\Omega_T^R}\frac{\beta}{\alpha}f(s, z)\,dsdz,$$

because $b_n/a_n \to b/a$ in $L_2(\Omega_T^R)$ and $F_n \to F$ pointwise in Ω_T^R. On the other hand, $\{1/\alpha_n\}$ and $\{\beta_n/\alpha_n\}$ are weakly relatively compact in $L_2(\Omega_T^R)$; hence $1/\alpha_n \to 1/\alpha$ and $\beta_n/\alpha_n \to \beta/\alpha$ weakly in $L_2(\Omega_T^R)$. Thus, (3.7) follows from Theorem 2.1. Finally, let us define

$$\tilde{F}, \tilde{F}_n\colon C([0, T]; \boldsymbol{R}) \to C([0, T]; \boldsymbol{R})$$

by

$$\tilde{F}(\tilde{x})(t) = F(t, \tilde{x}_t), \qquad \tilde{F}_n(\tilde{x})(t) = F_n(t, \tilde{x}_t)$$

for $\tilde{x}\in C([0, T]; \boldsymbol{R})$, $t\in[0, T]$. It is easy to check that $\tilde{F}_n(\tilde{x}_n) \to F(\tilde{x})$ in $C([0, T]; \boldsymbol{R})$ as $\tilde{x}_n \to \tilde{x}$ in $C([0, T]; \boldsymbol{R})$, so $\tilde{F}_n \to F$ uniformly on compacts in

$C([0, T]; \boldsymbol{R})$. Therefore, taking into account (3.5)–(3.7) and applying the continuous mapping theorem we obtain

$$\mathscr{L}[F(\cdot, X) \mid P_x^{a,b}] = Q_y^{\alpha,\beta}.$$

This proves the lemma, because the inverse $\tilde{F}^{-1}$ of $\tilde{F}$ is easily seen to be continuous, and hence measurable. ■

The next lemma is based on Lemma 7.1.5 and Theorem 7.1.6 in [12], however differently from [12] we consider diffusions with non-zero drift term.

LEMMA 3.2. *Let $a \in \mathscr{A}(\lambda, \Lambda)$ and $b \in \mathscr{B}(\Lambda)$. For $T, R > 0$ define*

$$G_\Lambda^T f(s, x) = \int_s^T dt \int_{\boldsymbol{R}} f(t, y) \frac{1}{\sqrt{2\pi\Lambda(t-s)}} \exp\left(-\frac{|y-x|^2}{2\Lambda(t-s)}\right) dy$$

and

$$K_{a,b}^T f(s, x) = \frac{1}{2}(a(s, x) - \Lambda) \frac{\partial^2 G_\Lambda^T f}{\partial x^2}(s, x) + b(s, x) \frac{\partial G_\Lambda^T f}{\partial x}(s, x)$$

for $f \in C_0^\infty(\bar{\Omega}_T^R)$. Then:

(i) *For each $p \in [2, 6]$, $K_{a,b}^T$ admits a unique extension (again denoted by $K_{a,b}^T$) as a bounded operator on $L_p(\Omega_T^R)$ into itself and the extensions corresponding to different p's are consistent.*

(ii) *Assume that $0 < T \leqslant T_0 = \lambda^2/(32\Lambda^3)$. Then there is $\delta > 0$, depending only on λ, Λ, R such that $I - K_{a,b}^T$ admits a bounded inverse $(I - K_{a,b}^T)^{-1}$ on $L_p(\Omega_T^R)$ for $p \in [2, 2+\delta]$, $(I - K_{a,b}^T)^{-1}$ are consistently defined for $p \in [2, 2+\delta]$ and the norm of $(I - K_{a,b}^T)^{-1}$ on $L_p(\Omega_T^R)$ is less than or equal to $4\Lambda/\lambda$.*

(iii) *For every $(s, x) \in [0, T_0] \times \boldsymbol{R}$, $t \in (s, T_0]$ and $p \in [2, 2+\delta]$,*

$$E_{s,x} \int_s^t f(u, X_u)\, du = G_\Lambda^t \circ (I - K_{a,b}^t)^{-1} f(s, x)$$

for all $f \in L_p(\Omega_t)$ with $\operatorname{supp} f \in \bar{\Omega}_t^R$, where $E_{s,x}$ is the expectation sign with respect to the measure $P_{s,x}$ associated with $L(a, b)$ such that $P_{s,x}(X_t = x, t \in [0, s]) = 1$.

Proof. Observe that $u = G_\Lambda^T f$ is a classical solution to the Cauchy problem

$$\frac{\partial u}{\partial s} + \frac{\Lambda}{2} \frac{\partial^2 u}{\partial x^2} = -f \text{ on } \boldsymbol{R} \times [0, T), \quad \lim_{s \nearrow T} u(s, x) = 0, \ x \in \boldsymbol{R}.$$

Integrating by parts and taking into account that $\partial u/\partial x(T, \cdot) = 0$ we obtain

$$\left\| \frac{\partial u}{\partial x}(t, \cdot) \right\|_{2,\boldsymbol{R}}^2 = -\int_t^T ds \int_{\boldsymbol{R}} \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$= -2\int_t^T ds \int_{\boldsymbol{R}} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial s \partial x} dx = 2\int_t^T ds \int_{\boldsymbol{R}} \frac{\partial u}{\partial s} \frac{\partial^2 u}{\partial x^2} dx$$

for any $t \in [0, T)$. Hence

$$\sup_{0 \leqslant t \leqslant T} \left\| \frac{\partial u}{\partial x}(t, \cdot) \right\|_{2,\mathbf{R}}^2 \leqslant 2 \left\| \frac{\partial u}{\partial s} \right\|_{2,\Omega_T} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{2,\Omega_T} \tag{3.8}$$

$$\leqslant \left\{ \Lambda \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{2,\Omega_T} + 2 \|f\|_{2,\Omega_T} \right\} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{2,\Omega_T}$$

$$\leqslant 8\Lambda^{-1} \|f\|_{2,\Omega_T}^2,$$

because, by the inequality (0.4) in the Appendix in [12] and Exercise 7.3.3 in [12],

$$\left\| \frac{\partial^2 u}{\partial x^2} \right\|_{2,\Omega_T} \leqslant 2\Lambda^{-1} \|f\|_{2,\Omega_T}. \tag{3.9}$$

Therefore,

$$\left\| \frac{\partial u}{\partial x} \right\|_{2,\Omega_T} \leqslant (8T\Lambda^{-1})^{1/2} \|f\|_{2,\Omega_T}.$$

Clearly,

$$|K_{a,b}^T f(s, x)| \leqslant 2^{-1}(\Lambda - \lambda) \left| \frac{\partial^2 u}{\partial x^2}(s, x) \right| + \Lambda \left| \frac{\partial u}{\partial x}(s, x) \right|$$

for all $(s, x) \in \Omega_T$, so, by the above and the fact that $\operatorname{supp} f \in \bar{\Omega}_T^R$,

$$\|K_{a,b}^T f\|_{2,\Omega_T} \leqslant \left(1 - \lambda\Lambda^{-1} + (8T\Lambda)^{1/2}\right) \|f\|_{2,\Omega_T^R}. \tag{3.10}$$

By the inequality (II.3.2) in [5], we obtain

$$\left\| \frac{\partial u}{\partial x} \right\|_{6,\Omega_T} \leqslant 2^{1/3} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{2,\Omega_T}^{1/3} \sup_{0 \leqslant t \leqslant T} \left\| \frac{\partial u}{\partial x}(t, \cdot) \right\|_{2,\mathbf{R}}^{2/3}.$$

Hence taking into account (3.8), (3.9) and using Hölder's inequality we see that

$$\|K_{a,b}^T f\|_{6,\Omega_T^R} \leqslant C \|f\|_{6,\Omega_T^R} \tag{3.11}$$

for some $C > 0$ depending only on λ, Λ, R and T. Combining (3.10) with (3.11) and using the Riesz–Thorin interpolation theorem proves that $K_{a,b}^T$ admits a bounded extension on $L_p(\Omega_T^R)$ for $p \in [2, 6]$. Moreover, the extensions of $K_{a,b}^T$ are defined consistently for different p's, because if $f \in L_p(\Omega_T^R) \cap L_q(\Omega_T^R)$ for some $p, q \in [2, 6]$, then we can find $\{f_n\} \subset C_0^\infty(\bar{\Omega}_T^R)$ such that $f_n \to f$ in both $L_p(\Omega_T^R)$ and $L_q(\Omega_T^R)$. This proves (i).

Now, assume that $T \leqslant T_0$. Then from (3.10) we see that the norm of $K_{a,b}^T$ on $L_2(\Omega_T^R)$ does not exceed $1 - \lambda/(2\Lambda)$. Hence applying once again the Riesz–Thorin interpolation theorem we conclude that there is a $\delta > 0$ depending only on λ, Λ and R such that for each $p \in [2, 2+\delta]$ the norm of $K_{a,b}^T$ on $L_p(\Omega_T^R)$ is less than or equal to $1 - \lambda/(4\Lambda)$. Therefore $(I - K_{a,b}^T)^{-1}$ exists and is given by

$\sum_{n=0}^{\infty}(K_{a,b}^T)^n$. In particular, it follows that $(I-K_{a,b}^T)^{-1}$ has the norm not greater than $4\Lambda/\lambda$ and that $(I-K_{a,b}^T)^{-1}$ are consistently defined for different $p\in[2, 2+\delta]$. Finally, by Krylov's estimates ([4], Theorem II.3.4; see also [12], Exercise 7.3.3) there is a $C>0$ depending only on λ, Λ, T and R such that

$$E_{s,x}\int_s^T f(u, X_u)\,du \leqslant C\,\|f\|_{p,\Omega_t^R}$$

for $f\in L_p(\Omega_t)$ with $\operatorname{supp} f\subset \bar{\Omega}_t^R$, so to prove (iii) we can proceed as in the proof of equality (7.1.8) in [12]. ■

We can now state the analogue of Theorem 2.1 for Itô diffusions.

THEOREM 3.3. *Let* $\{a_n\}\subset\mathscr{A}(\lambda, \Lambda)$, $\{b_n\}\subset\mathscr{B}(\Lambda)$ *and let* $x_n\to x$.

(i) *If* $P_{x_n}^{a_n,b_n}\Rightarrow\tilde{P}$ *in* $C([0, T]; \mathbf{R})$, *then there exist* $\tilde{\Lambda}>0$ *and* $a\in\mathscr{A}(\lambda, \Lambda)$, $b\in\mathscr{B}(\tilde{\Lambda})$ *such that* $\tilde{P}=P_x^{a,b}$. *Moreover, if* $b_n=0$ *for* $n\in N$, *then* $b=0$.

(ii) *Assume additionally that* $\{a_n\}$ *satisfies* (1.5) *or* (1.6). *If* $1/a_n\to A$, $b_n/a_n\to B$ *weakly in* $L_2(\Omega_T^R)$ *for every* $R>0$, *then* $P_{x_n}^{a_n,b_n}\Rightarrow P_x^{a,b}$ *in* $C([0, T]; \mathbf{R})$, *where* $a=1/A$, $b=B/A$.

Proof. (ii) Write $P=P_x^{a,b}$, $P^n=P_{x_n}^{a_n,b_n}$ and let E and E^n denote the expectation signs with respect to P and P^n, respectively. $\{P^n\}$ is weakly relatively compact in $C([0, T]; \mathbf{R})$, so due to [12], Theorem 6.2.3, it suffices to show that for every $t\in[0, T]$ and $\varphi\in C_0^\infty(\mathbf{R})$

$$\lim_{n\to\infty} E^n\varphi(X_t) = E\varphi(X_t). \tag{3.12}$$

Obviously, (3.12) is satisfied for $t=0$, so we need only consider the case $t\in(0, T]$. First suppose that

$$\sup_{(s,z)\in\Omega_T}\sup_{n\geqslant 1}\left|\frac{\partial}{\partial s}F_n(s, z)\right| \leqslant \Lambda_1, \tag{3.13}$$

where F_n is defined by (3.4). Then $\{F_n\}$ is bounded in $W_2^{1,0}(\Omega_T^R)$ for each $R>0$. On the other hand, $F_n\to F$ in $L_2(\Omega_T^R)$, and hence, for each $R>0$, $\{\partial F_n/\partial t\}$ converges weakly in $L_2(\Omega_T^R)$ to the generalized derivative $\partial F/\partial t$. Moreover, from (3.13) it follows that there is a version of $\partial F/\partial t$ satisfying (3.1). Define now α_n, β_n by (3.3) and α, β by (3.2). Then as in the proof of Lemma 3.1 we show that $\{\alpha_n\}$ satisfies (1.5) and $1/\alpha_n\to 1/\alpha$, $\beta_n/\alpha_n\to\beta/\alpha$ weakly in $L_2(\Omega_T^R)$ for $R>0$. Observe also that $y_n=F_n(0, x_n)\to F(0, x)=y$ and $F_n^{-1}(t, \cdot)\to F^{-1}(t, \cdot)$ uniformly on compacts in $\mathbf{R}$. Hence, by Theorem 2.1, Lemma 3.1, and the continuous mapping theorem,

$$\mathscr{L}[X_t \mid P_{x_n}^{a_n,b_n}] = \mathscr{L}[F_n^{-1}(t, X_t) \mid Q_{y_n}^{\alpha_n,\beta_n}]$$

$$\Rightarrow \mathscr{L}[F^{-1}(t, X_t) \mid Q_y^{\alpha,\beta}] = \mathscr{L}[X_t \mid P_x^{a,b}]$$

in $\boldsymbol{R}$ for $t \in (0, T]$. This clearly forces (3.12), and the proof under the assumption (3.13) is complete. Suppose now that (1.5) is satisfied. For $n, k \in \boldsymbol{N}$ let

$$_k a(s, x) = \begin{cases} a(s, x) & \text{if } (s, x) \in \bar{\Omega}_T^k, \\ \lambda & \text{otherwise,} \end{cases}$$

$$_k a_n(s, x) = \begin{cases} a_n(s, x) & \text{if } (s, x) \in \bar{\Omega}_T^k, \\ \lambda & \text{otherwise.} \end{cases}$$

Then, for each $k \in \boldsymbol{N}$, $\{_k a_n\}_{n \in \boldsymbol{N}}$ satisfies (3.13), so by what has already been proved, $P_{x_n}^{_k a_n, b} \Rightarrow P_x^{_k a, b}$ in $C([0, T]; \boldsymbol{R})$ as $n \to \infty$ for $k \in \boldsymbol{N}$. Hence arguing as in the proof of Theorem 11.3.4 in [12] we show that $P_{x_n}^{a_n, b_n} \Rightarrow P_x^{a, b}$ in $C([0, T]; \boldsymbol{R})$. In turn assume that (1.6) is satisfied. Define a_ε, $a_{n,\varepsilon}$, b_ε, $b_{n,\varepsilon}$ by (2.2) and (2.5). From (2.3), (2.6) and what has already been proved it follows that $P_{x_n}^{(1/a_{n,\varepsilon}), b_{n,\varepsilon}} \Rightarrow P_x^{(1/a_\varepsilon), b_\varepsilon}$ in $C([0, T]; \boldsymbol{R})$, and, consequently,

$$\lim_{n \to \infty} E^{n,\varepsilon} \varphi(X_t) = E^\varepsilon \varphi(X_t) \tag{3.14}$$

for every $t \in (0, T]$, where $E^{n,\varepsilon}$ (E^ε) stands for the expectation sign with respect to $P_{x_n}^{(1/a_{n,\varepsilon}), b_{n,\varepsilon}}$ ($P_x^{(1/a_\varepsilon), b_\varepsilon}$). Using (2.8) and proceeding as in Exercises 7.3.2 and 7.3.3 in [12] we prove that $P_x^{(1/a_\varepsilon), b_\varepsilon} \Rightarrow P$ in $C([0, T]; \boldsymbol{R})$. Hence

$$\lim_{\varepsilon \to 0} E^\varepsilon \varphi(X_t) = E\varphi(X_t) \tag{3.15}$$

for $t \in (0, T]$. Finally, we will show that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} |E^{n,\varepsilon} \varphi(X_t) - E^n \varphi(X_t)| = 0. \tag{3.16}$$

By Itô's formula, we have

$$\begin{aligned} I^n &\equiv |E^{n,\varepsilon} \varphi(X_t) - E^n \varphi(X_t)| \\ &= \Big|E^{n,\varepsilon} \int_0^t g_{n,\varepsilon}(s, X_s)\, ds - E^n \int_0^t g_n(s, X_s)\, ds\Big| \\ &\leqslant \Big|E^{n,\varepsilon} \int_0^t (g_{n,\varepsilon} - g_n)(s, X_s)\, ds\Big| + \Big|E^{n,\varepsilon} \int_0^t g_n(s, X_s)\, ds - E^n \int_0^t g_n(s, X_s)\, ds\Big| \\ &\equiv I_1^n + I_2^n, \end{aligned}$$

where $g_{n,\varepsilon} = L(a_{n,\varepsilon}, b_{n,\varepsilon})\varphi$ and $g_n = L(a_n, b_n)\varphi$. By Krylov's estimates ([4], Theorem II.3.4), there is $C_1 > 0$ depending only on λ, Λ and T such that

$$I_1^n \leqslant C_1 \|g_{n,\varepsilon} - g_n\|_{2, \Omega_T^R}.$$

Hence, by (2.10),

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} I_1^n = 0. \tag{3.17}$$

To estimate I_2^n, we assume first that $T \leqslant T_0$, where $T_0 = \Lambda^2/(32\Lambda^3)$. Then, by Lemma 3.2,

$$I_2^n = G_\Lambda^t \circ H_{n,\varepsilon}^t g_n(0, x_n), \qquad H_{n,\varepsilon}^t = (I - K_{1/a_{n,\varepsilon}, b_{n,\varepsilon}}^t)^{-1} - (I - K_{a_n,b_n}^t)^{-1}.$$

By Schwarz's inequality,

$$I_2^n \leqslant (\pi\Lambda)^{-1/4} t^{1/4} \|H_{n,\varepsilon}^t g_n\|_{2,\Omega_T^R},$$

while from the identity

$$\begin{aligned} H_{n,\varepsilon}^t &= (I - K_{1/a_{n,\varepsilon}, b_{n,\varepsilon}}^t)^{-1} \circ (I - K_{a_n,b_n}^t) \circ (I - K_{a_n,b_n}^t)^{-1} \\ &\quad - (I - K_{1/a_{n,\varepsilon}, b_{n,\varepsilon}}^t)^{-1} \circ (I - K_{1/a_{n,\varepsilon}, b_{n,\varepsilon}}^t) \circ (I - K_{a_n,b_n}^t)^{-1} \\ &= (I - K_{1/a_{n,\varepsilon}, b_{n,\varepsilon}}^t)^{-1} \circ (K_{1/a_{n,\varepsilon}, b_{n,\varepsilon}}^t - K_{a_n,b_n}^t) \circ (I - K_{a_n,b_n}^t)^{-1} \end{aligned}$$

and Lemma 3.2 it follows that there is a $C_2 > 0$ depending only on λ, Λ, R such that

$$\|H_{n,\varepsilon}^t g_n\|_{2,\Omega_t^R} \leqslant C_2 \{\|(1/a_{n,\varepsilon}) - a_n\|_{2,\Omega_T^R} + \|b_{n,\varepsilon} - b_n\|_{2,\Omega_T^R}\} \|g_n\|_{2+\delta,\Omega_T^R}.$$

Therefore,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} I_2^n = 0 \tag{3.18}$$

by (2.10) and the boundedness of $\{g_n\}$ in $L_{2+\delta}(\Omega_T^R)$. Combining (3.17) with (3.18) we obtain (3.12) under the additional assumption that $T \leqslant T_0$. Now suppose that this is no longer so. For fixed $s \in [0, T)$ let μ_s denote the distribution of X_s under P and let μ_s^n denote the distribution of X_s under P^n, $n \in N$. Obviously, the general case will be proved once we prove that for every $s \in [0, T)$, if $\mu_s^n \Rightarrow \mu_s$, then (3.12) holds for $t \in [s, (s+T_0) \wedge T]$. So, fix $s \in [0, T)$, $t \in [s, (s+T_0) \wedge T]$ and assume that $\mu_s^n \Rightarrow \mu_s$. Let $P_{s,y}$ denote the diffusion measure associated with $L(a, b)$ such that $P_{s,y}(X_t = y, 0 \leqslant t \leqslant s) = 1$ and let $E_{s,y}$ stand for the expectation sign with respect to $P_{s,y}$. Similarly, for each $n \in N$ define $P_{s,y}^n$, $E_{s,y}^n$ on the basis of a_n, b_n. By the Markov property,

$$\begin{aligned} E^n \varphi(X_t) - E\varphi(X_t) &= \int_R E_{s,y}^n \varphi(X_t) \mu_s^n(dy) - \int_R E_{s,y} \varphi(X_t) \mu_s(dy) \\ &= \int_R \big(E_{s,y}^n \varphi(X_t) - E_{s,y} \varphi(X_t)\big) \mu_s^n(dy) \\ &\quad + \int_R \big(E_{s,y} \varphi(X_t)\big) \big(\mu_s^n(dy) - \mu_s(dy)\big) \\ &\equiv J_1^n + J_2^n. \end{aligned}$$

By what has already been proved, $\mathscr{L}[X_t \mid P_{s,y}^n] \Rightarrow \mathscr{L}[X_t \mid P_{s,y}]$ for every $y \in R$; hence $E_{s,\cdot}^n \varphi(X_t) \to E_{s,\cdot} \varphi(X_t)$ pointwise. Actually, the convergence is uniform on compact subsets of R, because by Theorem 3 in [3] (see also [6]), the functions $E_{s,\cdot}^n \varphi(X_t)$, $n \in N$, are equicontinuous on compact subsets of R. Therefore

$J_1^n \to 0$, since $\{\mu_s^n\}$ is tight. The convergence $J_2^n \to 0$ follows from the boundedness and continuity of $E_{s,\cdot}\,\varphi(X_t)$. The proof of (ii) is complete.

(i) For $n, k \in N$ set $a_{n,k} = \xi_k * a_n$, $b_{n,k} = \xi_k * b_n$, where $\xi_k = k^2 \xi(kt, kx)$, $\xi \in C_0^\infty(R^2)$ is a non-negative function such that $\int\int_{R^2} \xi(t, x)\,dt\,dx = 1$, and when computing convolutions we regard a_n's b_n's as defined on the whole R^2 by extending them to be zero outside $[0, T] \times R$. Then $\{a_{n,k}\}_{k \in N} \subset \mathscr{A}(\lambda, \Lambda)$, $\{b_{n,k}\}_{k \in N} \subset \mathscr{B}(\Lambda)$ and, for each $n \in N$ and $R > 0$, $a_{n,k} \to a_n$, $b_{n,k} \to b_n$ in $L_2(\Omega_T^R)$ as $k \to \infty$; hence $P_x^{a_{n,k}, b_{n,k}} \Rightarrow P_x^{a_n, b_n}$ in $C([0, T]; R)$. Therefore, for each $n \in N$ we can choose $k = k(n) \in N$ such that $\varrho(\tilde{P}_x^n, P_x^{a_n, b_n}) \leqslant 1/n$, where $\tilde{P}_x^n$ is a measure corresponding to $L(a_{n,k(n)}, b_{n,k(n)})$ such that $\tilde{P}^n(X_0 = x) = 1$ and ϱ is the Prokhorov metric on the set of probability measures on $\mathscr{C}$. On the other hand, $a_{n,k(n)}, b_{n,k(n)}$, $n \in N$, are smooth functions having bounded derivatives; hence by Theorem 3 of [1] there is a subsequence $\{n'\} \subset N$ and $a \in \mathscr{A}(\lambda, \Lambda)$, $b \in \mathscr{B}(\Lambda)$ such that $\varrho(P_x^{n'}, P_x^{a,b}) \to 0$ as $n' \to 0$. Moreover, $b = 0$ if $b_n = 0$ for $n \in N$. By the above, $\varrho(P_x^{a_n, b_n}, P_x^{a,b}) \to 0$. In particular, $E_x^n \varphi(X_t) \to E\varphi(X_t)$ for every $t \in [0, T]$ and $\varphi \in C_0^\infty(R)$, where E_x^n stands for the expectation sign with respect to $P_x^{a_n, b_n}$. Furthermore, by Theorem 3 of [3], the functions $E_\cdot^n \varphi(X_t)$, $n \in N$, are equibounded and equicontinuous; hence $E^n \varphi(X_t) - E_x^n \varphi(X_t) \to 0$ as $n \to \infty$. Thus, (3.12) is satisfied, and (i) is proved. ∎

4. Remarks and comments. In this section we gather remarks concerning assumptions (1.4)–(1.6) and give examples of applications of the preceding theorems.

THEOREM 4.1. *Assume that* $\{a_n\} \subset \mathscr{A}(\lambda, \Lambda)$, $\{b_n\} \subset \mathscr{B}(\Lambda)$ *and that* $x_n \to x$, $y_n \to y$.

(i) *If* $a_n \to a$ *in* $L_2(\Omega_T^R)$ *and* $b_n \to b$ *weakly in* $L_2(\Omega_T^R)$ *for* $R > 0$, *then*

$$P_{x_n}^{a_n, b_n} \Rightarrow P_x^{a,b} \quad \text{and} \quad Q_{y_n}^{a_n, b_n} \Rightarrow Q_y^{a,b}.$$

(ii) *If* a_n *are functions of* x *only,* $1/a_n \to A$ *weakly in* $L_2(-R, R)$ *and* $b_n/a_n \to B$ *weakly in* $L_2(\Omega_T^R)$ *for* $R > 0$, *then*

$$P_{x_n}^{a_n, b_n} \Rightarrow P_x^{1/A, B/A} \quad \text{and} \quad Q_{y_n}^{a_n, b_n} \Rightarrow Q_y^{1/A, B/A}.$$

(iii) *If* a_n *are functions of* t *only,* $a_n \to a$ *weakly in* $L_2(0, T)$ *and* $b_n \to b$ *weakly in* $L_2(\Omega_T^R)$ *for* $R > 0$, *then*

$$P_{x_n}^{a_n, b_n} \Rightarrow P_x^{a,b} \quad \text{and} \quad Q_{y_n}^{a_n, b_n} \Rightarrow Q_y^{a,b}.$$

Proof. If $a_n \to a$ locally in $L_2(\Omega_T)$ or a_n are functions of x only, then (1.6) is satisfied, so (i) and (ii) follow immediately from Theorems 2.1 and 3.3. As for (iii), observe that Theorem 11.3.3 of [12] is applicable. ∎

Remark 4.2. (a) If a_n are functions of t only, then (1.6) implies that $\{a_n\}$ is relatively compact in $L_2(0, T)$. Therefore the assertion (iii) of Theorem 4.1 does not follow from Theorems 2.1 and 3.3. This is the main weakness of our results.

(b) The conditions (1.5) and (1.6) are in general not comparable. Clearly, (1.6) does not imply (1.5). The following example shows that the opposite implication is also false. For $n = 1, 2, \ldots$ define a_n: $[0, \pi] \times \boldsymbol{R}$ by

$$a_n(t, x) = \begin{cases} 2/\big(2-\cos\big(n(1+t)/x\big)\big) & \text{if } t \in [0, \pi],\ x > 0, \\ 1 & \text{if } t \in [0, \pi],\ x \leqslant 0. \end{cases}$$

Then for any $h > 0$

$$\begin{aligned} I^n(h) &\equiv \int_0^\pi \int_0^1 |a_n(t+h, x) - a_n(t, x)|\, dt\, dx \\ &\geqslant \frac{2}{9} \int_0^\pi \int_0^1 \left| \cos\frac{n(1+t+h)}{x} - \cos\frac{n(1+t)}{x} \right| dt\, dx \\ &= \frac{4}{9} \int_0^1 \left| \sin\frac{nh}{2x} \right| dx \int_0^\pi \left| \sin\frac{2n(1+t)+nh}{2x} \right| dt. \end{aligned}$$

Observe that

$$\begin{aligned} \int_0^\pi \left| \sin\frac{2n(1+t)+nh}{2x} \right| dt &\geqslant \frac{x}{n} \int_0^{[x/n]\pi} \left| \sin\left(t + \frac{n(2+h)}{x}\right) \right| dt \\ &= \frac{x}{n} \left[\frac{x}{n}\right] \int_0^\pi |\sin t|\, dt \geqslant 1 \end{aligned}$$

($[y]$ denotes the greatest integer less than or equal to y) for $n \in N$, $x \in (0, 1]$ and $h > 0$, and

$$\begin{aligned} \int_0^1 \left| \sin\frac{nh}{2x} \right| dx &= \int_1^\infty \frac{1}{x^2} \left| \sin\frac{nhx}{2} \right| dx \geqslant \frac{1}{4} \int_1^2 \left| \sin\frac{nhx}{2} \right| dx \\ &\geqslant \frac{1}{4} \left[\frac{nh}{2\pi}\right] \int_1^{1+2\pi/(nh)} \sin\frac{nhx}{2}\, dx = \frac{2}{nh} \left[\frac{nh}{2\pi}\right]. \end{aligned}$$

By the above, $I^n(h) \geqslant 8\,[nh/(2\pi)]/(9nh)$, and hence

$$\lim_{h \searrow 0} \sup_{n \geqslant 1} I^n(h) = 4/(9\pi).$$

Accordingly, $\{a_n\}$ does not satisfy (1.6). On the other hand, for every $t \in [0, T]$, $x > 0$ and $n \in N$,

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^x \cos\frac{n(1+t)}{y}\, dy &= \frac{\partial}{\partial t} \int_{n(1+t)/x}^\infty \frac{n(1+t)\cos y}{y^2}\, dy \\ &= n \int_{n(1+t)/x}^\infty \frac{\cos y}{y^2}\, dy - \frac{x}{1+t} \cos\frac{n(1+t)}{x}. \end{aligned}$$

Therefore $\{a_n\}$ satisfies (1.5), because

$$\left| n \int_{n(1+t)/x}^{\infty} \frac{\cos y}{y^2}\, dy \right| \leqslant n \int_{n(1+t)/x}^{\infty} \frac{1}{y^2}\, dy = \frac{x}{1+t}.$$

Let us remark also that $1/a_n \to 1$ weakly in $L_2(\Omega_T^R)$ for $R > 0$, so $\{P_x^{a_n,0}\}$ and $\{Q_x^{a_n,0}\}$ converge weakly to the Wiener measure starting at x.

(c) If a_n and b_n do not depend on t, then $P_x^{a_n,b_n} \Rightarrow P_x^{1/A,B/A}$ iff (1.2) is satisfied (see [8] and [11] for the case of diffusions with no drift terms). Similarly, $Q_y^{a_n,b_n} \Rightarrow Q_y^{1/A,B/A}$ iff (1.2) is satisfied. The "if" part is a special case of Theorem 7.2 in [7]. Since $\{1/a_n\}$ and $\{b_n/a_n\}$ are weakly relatively compact in $L_2(-R, R)$ for every $R > 0$, to prove the "only if" part we only need to show that if a, α, b, β do not depend on t, satisfy (1.1) and $Q_y^{a,b} = Q_y^{\alpha,\beta}$ for every $y \in \boldsymbol{R}$, then $a = \alpha$, $b = \beta$ a.e. To see this, for fixed $\varphi \in C_0^\infty(\boldsymbol{R})$ set

$$u(t, y) = \int \varphi\big(X_t(\omega)\big)\, dQ_y^{a,b}(\omega) = \int \varphi\big(X_t(\omega)\big)\, dQ_y^{\alpha,\beta}(\omega), \qquad t \geqslant 0,\ y \in \boldsymbol{R}.$$

Then for every $t > 0$ and $\psi \in C_0^\infty(\boldsymbol{R})$

$$\int\!\!\int_{\Omega_t} \left(\frac{1}{2} a \frac{\partial u}{\partial y} \frac{d\psi}{dy} - b \frac{\partial u}{\partial y} \psi\right)(s, y)\, ds\, dy = \int\!\!\int_{\Omega_t} \left(\frac{1}{2} \alpha \frac{\partial u}{\partial y} \frac{d\psi}{dy} - \beta \frac{\partial u}{\partial y} \psi\right)(s, y)\, ds\, dy.$$

Differentiating both sides of the above equality with respect to t and then letting $t \searrow 0$ we obtain

$$\text{(4.1)} \qquad \int_{\boldsymbol{R}} \left(\frac{1}{2} a \frac{d\varphi}{dy} \frac{d\psi}{dy} - b \frac{d\varphi}{dy} \psi\right)(y)\, dy = \int_{\boldsymbol{R}} \left(\frac{1}{2} \alpha \frac{d\varphi}{dy} \frac{d\psi}{dy} - \beta \frac{d\varphi}{dy} \psi\right)(y)\, dy,$$

because $[0, T] \ni t \mapsto u(t, \cdot) \in W_2^1(\boldsymbol{R})$ is weakly continuous (see [10], Theorem II.3.8). Since (4.1) holds true for arbitrary $\varphi, \psi \in C_0^\infty(\boldsymbol{R})$, the desired conclusion follows.

(d) If a_n and b_n are functions of t only, then $P_x^{a_n,b_n} = Q_x^{a_n,b_n} \Rightarrow P_x^{1/A,B/A} = Q_x^{1/A,B/A}$ iff (1.3) is satisfied. The "if" part is the very special case of Theorem 11.3.3 in [12]. The "only if" part follows from the "if" part and the relative weak compactness of $\{1/a_n\}$ and $\{b_n/a_n\}$ in $L_2(0, T)$, because if $P_x^{a,b} = P_x^{\alpha,\beta}$, where a, b, α, β satisfy (1.1) and do not depend on the space variable, then $a = \alpha$, $b = \beta$ a.e.

(e) It is known that $P_x^{a_n,b_n} \Rightarrow P_x^{a,b}$ if $a_n \to a$ and $b_n \to b$ in $L_2(\Omega_T^R)$ for $R > 0$ (see [12], Exercise 7.3.2) or $a_n \to a$ and $b_n \to b$ weakly in $L_2(\Omega_T^R)$ for $R > 0$ and there is a non-decreasing function δ: $(0, \infty) \to (0, \infty)$ such that $\lim_{h \searrow 0} \delta(h) = 0$ and

$$\sup_{n \geqslant 1} |a_n(t, x) - a_n(t, y)| \leqslant \delta(|x - y|), \qquad t \in [0, T],\ x, y \in \boldsymbol{R}$$

(see [12], Theorem 11.3.3). Theorem 4.1 sharpens these results. In particular, it is worth pointing out that in the second case we do not assume that a_n and a are continuous in x.

(f) In view of Theorem 5.2 in [7], part (i) of Theorem 2.1 states that the sequence $\{\mathscr{L}(a_n, b_n)\}$ is G-compact, while from part (ii) it follows that if (1.4) and (1.5) or (1.4) and (1.6) are satisfied, then $\{\mathscr{L}(a_n, b_n)\}$ G-converges to $\mathscr{L}(1/A, B/A)$.

(g) If (1.6) is satisfied and $1/a_n \to A$ locally weakly in $L_2(\Omega_T)$, then from every subsequence $\{n'\} \subset N$ one can choose a further subsequence $\{n''\}$ such that, for almost every $t \in [0, T]$, $1/a_{n''}(t, \cdot) \to A(t, \cdot)$ locally weakly in $L_2(\boldsymbol{R})$. Therefore, combining Theorem 5.6 of [2] with Theorem 17 of [14], we conclude that under (1.4), if $1/a_n \to A$ locally weakly in $L_2(\Omega_T)$, then $\{\mathscr{L}(a_n, 0)\}$ G-converges to $\mathscr{L}(1/A, 0)$, hence that $Q_{y_n}^{a_n,0} \Rightarrow Q_y^{1/A,0}$, by Theorem 5.2 of [7]. Theorem 2.1 (ii) extends this result to diffusions corresponding to divergence form operators with non-zero first order terms.

(h) By Theorem 29 of [13], if $1/a_n(t, \cdot) \to A(t, \cdot)$, $b_n(t, \cdot)/a_n(t, \cdot) \to B(t, \cdot)$ locally weakly in $L_2(\boldsymbol{R})$ for every $t \in [0, T]$ and

$$\lim_{h\to 0}\sup_{n\geq 1}\sup_{(t,x)\in\Omega_T^R}\big(|a_n(t+h, x)-a_n(t, x)|+|b_n(t+h, x)-b_n(t, x)|\big) = 0 \tag{4.2}$$

for $R > 0$, then $\{\mathscr{L}(a_n, b_n)\}$ G-converges strongly to $\mathscr{L}(1/A, B/A)$, hence G-converges, and so $Q_{y_n}^{a_n,b_n} \Rightarrow Q_y^{1/A,B/A}$, as remarked in [7], Theorem 7.2. We do not impose any regularity assumptions on the coefficients b_n and, moreover, (1.6) is weaker than (4.2). Note, however, that strong G-convergence is in general essentially stronger than G-convergence.

(i) If $a_n(t, x) = a(t, nx)$, $n \in N$, where $a(t, x)$ is a function periodic in x with the period independent of t, then $\{a_n\}$ satisfies (1.6) (see Remark 5.12 in [2]).

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