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ON BOUNDEDNESS AND CONVERGENCE OF SOME BANACH SPACE VALUED RANDOM SERIES

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Abstract. Let (f_i) and (g_i) be sequences of independent symmetric random variables and (x_i) a sequence of elements from a Banach space. We prove that under certain assumptions the a.s. boundedness of the series $\sum x_i f_i$ implies the a.s. convergence of $\sum x_i g_i$ in every Banach space.

If f_i are identically distributed, $\mathrm{E}|f_i|$ is finite, g_i are identically distributed and non-degenerate, then the above implication fails in c_0.

If f_i are equidistributed and there is a sequence (a_n) such that

$$a_n^{-1} \sum_{i=1}^{n} |f_i| \to 1 \text{ in probability,}$$

then there is a sequence (x_i) in c_0 such that $\sum x_i f_i$ is a.s. bounded, but does not converge a.s.

In particular, if f_i are p-stable with $\mathrm{E}\, e^{itf_n} = e^{-|t|^p}$, then for $p < 1$ the a.s. boundedness of the series implies its a.s. convergence, but for $p \geqslant 1$ it fails.

The origin of this paper is the following Garling's question:

Let $(\eta_i)_{i \in N}$ be a sequence of p-stable random variables (r.v.) with characteristic function $e^{-|t|^p}$, $p \in (0, 2)$, and (x_i) a sequence in a Banach space E. If the series $\sum_{i \in N} \eta_i x_i$ is a.s. bounded, then is it a.s. convergent?

Some general results are obtained; it turns out that the answer is positive for $p \in (0, 1)$ and negative for $p \in [1, 2)$.

1. Preliminaries. We begin with some known facts.

1.1. Definition. Let (ϱ_i) and (ξ_i) be two sequences of independent symmetric real-valued r.v. The sequence (ϱ_i) is *dominated* by (ξ_i) if there exist constants K and L such that for every t and i

$$P(|\varrho_i| > t) \leqslant K P(L |\xi_i| > t).$$

The forthcoming theorem is an easy corollary to a result stated in [3]. The proof in the sequel with a better constant than in [3] is due to S. Kwapień and seems to be new.

1.2. THEOREM. *Let $X_1, X_2, \ldots, X_n$ be independent symmetric E-valued r.v. Then for every $t \in R$*

$$P(\|\sum_{i=1}^{n} a_i X_i\| > t) \leqslant 2P(\max_i |a_i| \, \|\sum_{i=1}^{n} X_i\| > t).$$

Proof. We can assume that $0 \leqslant a_1 \leqslant \ldots \leqslant a_n = 1$. Put $a_0 = 0$, $b_k = a_k - a_{k-1}$ for $k = 1, 2, \ldots, n$, $S_k = \sum_{i=k}^{n} X_i$. Then

$$\sum_{i=1}^{n} a_i X_i = \sum_{k=1}^{n} b_k S_k, \qquad \sum_{k=1}^{n} b_k = 1.$$

Consequently, if $\|\sum_{i=1}^{n} a_i X_i\| > t$, then $\max_k \|S_k\| > t$. Therefore we have

$$P(\|\sum_{i=1}^{n} a_i X_i\| > t) \leqslant P(\max_k \|S_k\| > t) \leqslant 2P(\|S_1\| > t),$$

which completes the proof.

1.3. THEOREM (E. Rychlik, oral communication). *If (ϱ_i) is dominated by (ξ_i) with constants K and L, where $K \in N$, then for every $x_1, x_2, \ldots, x_n \in E$ and $t \in R$*

$$P(\|\sum_{i \leqslant n} \varrho_i x_i\| > t) \leqslant 2K\, P(KL \|\sum_{i \leqslant n} \xi_i x_i\| > t).$$

Proof. We may assume without loss of generality that $L = 1$. Let ψ_i^k $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, K)$ be r.v. such that

(i) $P(\psi_i^k = 1) = 1 - P(\psi_i^k = 0) = 1/K$,

(ii) $\psi_i^1 + \ldots + \psi_i^K = 1$ for $i = 1, 2, \ldots, n$,

(iii) $\psi_1^k, \ldots, \psi_n^k, \varrho_1, \ldots, \varrho_n$ are independent for fixed k.

We prove that

$$P(\|\sum_i \varrho_i x_i\| > t) \leqslant K\, P(K \|\sum_i \varrho_i \psi_i^1 x_i\| > t) \leqslant 2K\, P(K \|\sum_i \xi_i x_i\| > t).$$

The first inequality can be rewritten in the form

$$(*) \qquad P(\|\sum_i \varrho_i \psi_i^1 x_i + \ldots + \sum_i \varrho_i \psi_i^K x_i\| > t) \leqslant \sum_{j=1}^{K} P\left(\|\sum_i \varrho_i \psi_i^j x_i\| > \frac{t}{K}\right).$$

Now it is obvious that if the event on the left-hand side takes place, then some of K events on the right-hand side must take place. Therefore (*) holds.

The second inequality is a consequence of 1.1. We prove that

$$P\left(\left\|\sum_i \varrho_i \psi_i^1 x_i\right\| > t\right) \leqslant 2P\left(\left\|\sum_i \xi_i x_i\right\| > t\right).$$

We have

$$P(|\varrho_i \psi_i^1| > t) = \frac{1}{K} P(|\varrho_i| > t) \leqslant P(|\xi_i| > t).$$

Then it is not hard to see that there are r.v. φ_i' and ξ_i' on a probability space $(\Omega', \mathscr{F}', P')$ such that

(i) $|\varphi_i'| \leqslant 1$,

(ii) the sequences $(\xi_i)_{i\leqslant n}$ and $(\xi_i')_{i\leqslant n}$ are identically distributed,

(iii) the sequences $(\varphi_i' \xi_i')_{i\leqslant n}$ and $(\varrho_i \psi_i^1)_{i\leqslant n}$ are identically distributed.

Let $(\varepsilon_i)_{i\leqslant n}$ be a Bernoulli sequence on a probability space $(\Omega'', \mathscr{F}'', P'')$. Then

$$P\left(\left\|\sum_i \varrho_i \psi_i^1 x_i\right\| > t\right) = P\left(\left\|\sum_i \varphi_i' \varepsilon_i \xi_i' x_i\right\| > t\right) = P' \times P''\left(\left\|\sum_i \varphi_i' \varepsilon_i \xi_i' x_i\right\| > t\right)$$

$$\leqslant 2P' \times P''\left(\max_i |\varphi_i'| \left\|\sum_i \varepsilon_i \xi_i' x_i\right\| > t\right) \leqslant 2P\left(\left\|\sum_i \xi_i x_i\right\| > t\right).$$

The proof is completed.

As a simple consequence we obtain

1.4. THEOREM (Jain and Marcus [2]). *If (ϱ_i) is dominated by (ξ_i), $(x_i) \subset E$, then the convergence of $\sum \xi_i x_i$ in L^p for some $p \in [0, \infty)$ implies the convergence of $\sum \varrho_i x_i$ in L^p.*

1.5. Remark. If (ϱ_i) and (ξ_i) are sequences of i.i.d. r.v. and the assertion of Theorem 1.4 holds for $p = 0$ and every Banach space E, then (ϱ_i) is dominated by (ξ_i).

2. The main result.

2.1. THEOREM. *Assume that (ϱ_i) and (ξ_i) satisfy the following assumptions:*

(i) *(ϱ_i) is dominated by (ξ_i),*

(ii) *for every $\alpha > 0$ there exist constants K and L such that* (i) *holds and $KL < \alpha$.*

Then for every Banach space E and $(x_i) \subset E$ the a.s. boundedness of $\sum \xi_i x_i$ implies the a.s. convergence of $\sum \varrho_i x_i$.

Proof. Suppose that $\sum \varrho_i x_i$ does not converge a.s.; then it does not converge in probability. So we can find $\alpha > 0$ and $n_1 < m_1 < n_2 < m_2 < \dots$ such that $P\left(\left\|\sum_{n_k \leqslant i \leqslant m_k} \varrho_i x_i\right\| > \alpha\right) > \alpha$. Put

$$U_k^\varrho = \left\|\sum_{n_k \leqslant i \leqslant m_k} \varrho_i x_i\right\|, \quad U_k^\xi = \left\|\sum_{n_k \leqslant i \leqslant m_k} \xi_i x_i\right\|,$$

$$S_n = \sum_{i \leqslant n} \xi_i x_i, \quad M = \sup_n \|S_n\|.$$

Note that $\sup_k U_k^\xi \leqslant 2M$. Since $M < \infty$ a.s., there is λ such that $P(2M \leqslant \lambda) > 0$. Hence

$$0 < P(2M \leqslant \lambda) \leqslant P(\sup_k U_k^\xi \leqslant \lambda) = \prod_{k=1}^{\infty} (1 - P(U_k^\xi > \lambda)).$$

Therefore $\sum_k P(U_k^\xi > \lambda) < \infty$. By assumptions, (i) holds with K and L such that $\alpha/KL > \lambda$. It is easy to see that K can be chosen to be natural. Then 1.3 yields

$$\alpha < P(U_k^\varrho > \alpha) \leqslant 2K P(KLU_k^\xi > \alpha) \leqslant 2K P(U_k^\xi > \lambda).$$

But $P(U_k^\xi > \lambda) \to 0$ as $k \to \infty$, a contradiction. This completes the proof.

2.2. Remark. One can prove the following converse:

If (ϱ_i) *and* (ξ_i) *are sequences of i.i.d. r.v. and the assertion of Theorem* 2.1 *holds, then for every* $L > 0$ *there exists a constant* K *such that for every* t *and* i

$$P(|\varrho_i| > t) \leqslant K P(L|\xi_i| > t).$$

2.3. Corollary. *Let* $\eta, \eta_1, \eta_2, \ldots$ *be i.i.d. symmetric r.v. such that* $P(|\eta| > t) \sim t^{-p}$ *for* $t \to \infty$, $p \in (0, 1)$, *e.g. p-stable r.v. Let* $(x_i) \subset E$. *Then the a.s. boundedness of the series* $\sum \eta_i x_i$ *implies its a.s. convergence.*

Proof. Fix t_0 such that for $t > t_0$ and for some C

$$\frac{1}{C} t^{-p} \leqslant P(|\eta| > t) \leqslant Ct^{-p}.$$

If $0 < L \leqslant 1$, then for $t > t_0$ we have $C^{-1} L^p t^{-p} \leqslant P(L|\eta| > t)$, whence

$$C^2 L^{-p} P(L|\eta| > t) \geqslant Ct^{-p} \geqslant P(|\eta| > t).$$

So it suffices to take K such that $K \geqslant C^2 L^{-p}$ and $KP(L|\eta| > t_0) \geqslant 1$, e.g.

$$K = [\max(C^2, C^{-1} t_0^p) L^{-p}] + 1.$$

Then $KL \sim L^{1-p}$, whence KL can be made arbitrarily small, which completes the proof.

The following theorem answers Garling's problem in the negative for $p \in (1, 2)$.

2.4. Theorem. *Let* $\xi, \xi_1, \xi_2, \ldots$ *be i.i.d. symmetric r.v. and let* $\varrho, \varrho_1, \varrho_2, \ldots$ *be i.i.d. symmetric with* $P(\varrho = 0) < 1$. *If* $\mathrm{E}|\xi| < \infty$, *then there are a Banach space* E *and a sequence* $(x_i) \subset E$ *such that* $\sum \xi_i x_i$ *is a.s. bounded but* $\sum \varrho_i x_i$ *is not a.s. convergent.*

Proof. Assume $\mathrm{E}|\xi| = 1$ and put

$$q_n = P\left(\frac{1}{n}\sum_{i=1}^{n} |\xi_i| > 2\right).$$

By the weak law of large numbers we have $q_n \to 0$, so we can choose $n_1 < n_2 < \ldots$ such that

$$\sum_i q_{n_i} \leqslant \tfrac{1}{4}.$$

Put $m_i = n_1 + \ldots + n_i$ and let $E = (l^1_{n_1} \times l^1_{n_2} \times \ldots)_{c_0}$ be the set of all sequences (a_i) such that

$$\sum_{m_{k-1} < i \leqslant m_k} |a_i| \to 0 \quad \text{and} \quad \|(a_i)\| = \sup_k \sum_{m_{k-1} < i \leqslant m_k} |a_i|.$$

Note that E is isometric to a subspace of c_0. Put $x_k = (1/n_i) e_k$ for $m_{i-1} < k \leqslant m_i$, where e_k is the k-th unit vector. If (ε_i) is a Bernoulli sequence, then $\sum \varepsilon_i x_i$ does not converge a.s. because

$$\Big\| \sum_{m_{i-1} < k \leqslant m_i} \varepsilon_k x_k \Big\| = 1.$$

Hence, by Theorem 1.4, $\sum \varrho_i x_i$ does not converge a.s. It remains to show that $\sum \xi_i x_i$ is a.s. bounded. Let S_n be the n-th partial sum, $M = \sup_n \|S_n\|$. Then we have

$$\begin{aligned} P(\sup_{i \leqslant k} \|S_i\| > 2) &\leqslant P(\sup_{i \leqslant m_k} \|S_i\| > 2) \leqslant 2P(\|S_{m_k}\| > 2) \\ &= 2P\left(\left(\frac{1}{n_1}\sum_{i \leqslant n_1} |\xi_i| > 2\right) \cup \ldots \cup \left(\frac{1}{n_k}\sum_{m_{k-1} < i \leqslant m_k} |\xi_i| > 2\right)\right) \\ &\leqslant 2\sum_i q_{n_i} \leqslant \frac{1}{2}. \end{aligned}$$

Hence $P(M > 2) \leqslant \frac{1}{2}$, and then $P(M < \infty) = 1$. This completes the proof.

The following theorem gives a negative answer to Garling's question for $p = 1$.

2.5. Theorem. *Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. symmetric r.v. such that*

$$(**) \qquad \frac{\mathrm{E}|\xi| \cdot I_{(|\xi| \leqslant t)}}{t P(|\xi| > t)} \to \infty \quad \text{as } t \to \infty.$$

Then there are a Banach space E and a sequence $(x_i) \subset E$ such that $\sum \xi_i x_i$ is a.s. bounded but does not converge a.s.

Proof. If (**) holds, then there is $(a_n)_{n\in N}$ such that

$$\frac{1}{a_n}\sum_{i\leqslant n}|\xi_i| \to 1 \text{ in probability}$$

(cf. [1]). Let E be as in the proof of Theorem 2.4. Further reasoning is quite similar: put

$$q_n = P\left(\frac{1}{a_n}\sum_{i\leqslant n}|\xi_i| > 2\right),$$

choose $n_1 < n_2 < \ldots$ such that $\sum_i q_{n_i} \leqslant \frac{1}{4}$, and put $x_k = (1/a_{n_i})e_k$ for $m_{i-1} < k \leqslant m_i$. It is clear that $\sum \xi_i x_i$ is a.s. bounded, but does not converge a.s. since

$$P\Big(\Big\|\sum_{m_{i-1}<k\leqslant m_i}\xi_k x_k\Big\| > \tfrac{1}{2}\Big) \to 1 \quad \text{as } i\to\infty.$$

This completes the proof.

2.6. Remark. The a.s. boundedness of $\sum \xi_i x_i$, where ξ_i are 1-stable r.v., implies the convergence of $\sum \varepsilon_i x_i$, which is in contrast with the case of $p > 1$.

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Added in proof. Let (X_i) be a sequence of independent E-valued r.v. and (θ_i) i.i.d. real r.v. Assume that for every i and $\varepsilon > 0$ there are $y_1, \ldots, y_k \in E$ such that

$$d\Big(\mathscr{L}(X_i), \mathscr{L}\big(\sum_{j\leqslant k}\theta_j y_j\big)\Big) < \varepsilon,$$

where d is the Prokhorov distance. If the a.s. boundedness of $\sum x_i\theta_i$ implies its a.s. convergence, the same holds for $\sum X_i$. Typical examples are p-stable or semistable symmetric r.v. if $p < 1$.

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