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# SELF-SIMILAR PROCESSES AS WEAK LIMITS OF A RISK RESERVE PROCESS

BY

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*Abstract.* Self-similar processes are closely connected with limit theorems for identical and in general strongly dependent variables. Moreover, since they allow heavy-tailed distributions and provide an additional "adjusting" parameter $H$, they appear to be interesting in the area of risk models. In this paper we prove that only self-similar processes with stationary increments appear naturally as weak limits of a risk reserve process, and conversely every finite mean $H$-self-similar process with stationary increments for $0 < H \leqslant 1$ can result as the weak approximation. A lower bound for general self-similar processes with drift is also provided.

## 1. INTRODUCTION

The traditional approach in the *collective risk theory* is to consider a model of the risk business of an insurance company, and to study the probability of ruin, i.e. the probability that the risk business ever will be below some specific (negative) value. The classical risk process $R$ is defined by

$$R(t) = u + ct - \sum_{k=1}^{N(t)} Y_k, \tag{1}$$

where $u \geqslant 0$ denotes the initial capital, $c$ is a positive real constant, $N = (N(t))_{t \geqslant 0}$ is a point process independent of $(Y_k)$, and $(Y_k)_{k=1}^{\infty}$ forms a sequence of i.i.d. random variables, having the common distribution function $F$, with $F(0) = 0$, mean value $\mu$, and variance $\sigma^2$. $N(t)$ is to be interpreted as the number of claims on the company during the interval $(0, t]$. At each point of $N$ the company has to pay out a stochastic amount of money, and the company receives (deterministically) $c$ units of money per unit time. The constant $c$ is called the *premium income rate.*

However, in reality, claims are mostly modelled by heavy-tailed distributions like e.g. Pareto. Moreover, the independence of $Y_k$'s seems unrealistic

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since a correlation between claims is being observed. Therefore, in our approach we do not restrict ourselves to independent $Y_k$'s with $EY_k^2 < \infty$. We merely assume that $E|Y_k| < \infty$.

Already in 1940 Hadwiger compared a discrete-time risk process with diffusion. This can be viewed, though theoretically not comparable with modern approach, as the first treatment of diffusion approximations in the risk theory. A more modern version, based on weak convergence, is due to Iglehart [6]. The idea is to let the number of claims grow in a unit time interval and to make the claim sizes smaller in such a way that the risk process converges weakly to a diffusion. We shall consider weak approximations where the idea is to approximate the risk process with a self-similar process with stationary increments and drift. While the classical theory of diffusion approximation requires either short-tailed or independent claims, these assumptions can be dropped in our approach.

**1.1. Preliminaries.** Stochastic processes $X = (X(t))_{t\in T}$ in this paper are always assumed to be defined for $t \in T$, where $T = [0, \infty)$ or $\mathbf{R}$. The process $X$ is said to be *degenerate* if $X(t) = X(0)$ a.s. for any $t \in T$, and *non-degenerate* otherwise. By $(X(t)) \stackrel{d}{=} (Y(t))$ we mean the equality of all finite-dimensional distributions. Sometimes we simply write $X(t) \stackrel{d}{=} Y(t)$. By $X_n(t) \stackrel{d}{\Rightarrow} Y(t)$ we also mean the convergence of all finite-dimensional distributions as $n \to \infty$. Similarly, $X_n \stackrel{d}{\to} X$ denotes the convergence in distribution of random variables $X_n$ to $X$.

An integer-valued stochastic process $(N(t))_{t\geq 0}$ with $N(0) = 0$ a.s., $N(t) < \infty$ for each $t < \infty$ and non-decreasing realizations is called a *point process* (see Grandell [5]).

**1.2. Weak convergence of stochastic processes.** Let $D = D[0, \infty)$ be the space of *cadlag* functions, i.e. all real-valued functions that are right-continuous and have left-hand limits, on $[0, \infty)$. Endowed with the Skorokhod $J_1$ topology, $D$ is a Polish space, i.e. separable and metrizable with a complete metric. A stochastic process $X = (X(t))_{t\geq 0}$ is said to be *in D* if all its realizations are in $D$. A sequence $(X^{(n)})_{n\in\mathbf{N}}$ of stochastic processes in $D$ is said to *converge weakly* in the Skorokhod $J_1$ topology to a stochastic process $X$ if for every bounded continuous functional $f$ on $D$ it follows that

$$\lim_{n\to\infty} Ef(X^{(n)}) = Ef(X).$$

In this case one writes $X^{(n)} \Rightarrow X$. Weak convergence of $X^{(n)}$ to $X$ implies, for example, convergence of the finite-dimensional distributions provided that the limit process $X$ is continuous in probability, and that

$$\inf_{0\leq t\leq t_0} X^{(n)}(t) \stackrel{d}{\to} \inf_{0\leq t\leq t_0} X(t) \quad \text{for any } t_0 < \infty.$$

The latter stems from the fact that the function $i\colon D \to [-\infty, \infty]$ defined by $i(z) = \inf_{0 \leqslant s \leqslant t_0} z(s)$ is continuous on $D$ for any $z \in D$ and any $t_0 < \infty$. A sequence $(X^{(n)})_{n \in N}$ of stochastic processes in $D$ is said to *converge in probability* in the Skorokhod $J_1$ topology to a stochastic process $X$ if for all $\varepsilon > 0$

$$\lim_{n \to \infty} P[d(X^{(n)}, X) \geqslant \varepsilon] = 0,$$

where $d$ denotes the Skorokhod $J_1$ metric. We write $X^{(n)} \xrightarrow{P} X$. If $f(\cdot)$ is a deterministic function, then $X^{(n)} \xrightarrow{P} f$ if and only if $X^{(n)} \Rightarrow f$.

Hereafter we shall only consider processes in $D$ and continuous in probability.

**1.3. Self-similar processes.** A process $X = (X(t))_{t \geqslant 0}$ is *self-similar* (s.s.) (see Lamperti [7]) if for some $H > 0$

$$X(ct) \stackrel{d}{=} c^H X(t) \quad \text{for every } c > 0. \tag{2}$$

We call this $X$ an *H-self-similar* (s.s.) *process*. The parameter $H$ is called the *index* or the *exponent of the self-similarity*. $X$ is said to be *trivial* if $X(t) = t^H X(1)$ a.e., $t \geqslant 0$. The process $X = (X(t))_{t \geqslant 0}$ is said to have *stationary increments* (s.i.) if for any $b > 0$

$$\big(X(t+b) - X(b)\big) \stackrel{d}{=} \big(X(t) - X(0)\big).$$

We call $X$ simply an *s.i. process*.

**1.4. Model.** Let us specify in detail the assumptions in our model. We assume that the claims occur at jumps of a point process $(N(t))_{t \geqslant 0}$. While most work in the collective risk theory has assumed that $N(t)$ is a Poisson process, this restrictive assumption plays no role in our analysis. The successive claims $Y_k$ are supposed to form a sequence of identically distributed random variables, strongly dependent in general, with $EY_k = \mu > 0$. Furthermore, we assume that the initial risk reserve of the company is $u > 0$ and that the policyholders pay a gross risk premium of $c > 0$ per unit time. Thus the risk process is of the form (1).

One of the key problems of the collective risk theory concerns calculating the *ruin probability*, i.e. the probability that the risk process becomes negative. The ruin probability $\Phi(u, T)$ in finite time (or within finite horizon) of a company facing the risk process (1) is given by

$$\Phi(u, T) = P\big(R(t) < 0 \text{ for some } t \leqslant T\big), \quad 0 < T < \infty,\ u \geqslant 0.$$

Consequently, the ruin probability $\Phi(t)$ in infinite time can be defined as $\Phi(u) = \Phi(u, \infty)$. We also assume that the net profit condition $\lim_{t \to \infty} (ER(t))/t > 0$ holds.

## 2. GENERAL RESULTS

The main aim of this section is to show the following

STATEMENT 2.1. *The only processes that emerge in a "natural way" as weak approximations of the risk reserve process are H-s.s. processes with stationary increments with* $0 < H \leqslant 1$. *Conversely, every H-s.s. s.i. process X with* $EX(t) = 0$, *in D and with* $0 < H \leqslant 1$, *can serve as the weak approximation of some risk process.*

In order to justify this statement first we need the following proposition:

PROPOSITION 2.1 (Lamperti [7]). *Let* $(\zeta)_{k=1}^{\infty}$ *be a stationary sequence of* $\boldsymbol{R}$*-valued random variables with the partial sum process* $Y(t) = \sum_{k=1}^{[t]} \zeta_k$ *for* $t \geqslant 0$. *If*

$$\frac{1}{A(\lambda)} Y(\lambda t) \stackrel{d}{\Rightarrow} X(t) \quad \text{as } \lambda \to \infty,$$

*where reals* $(A(\lambda))_{\lambda \geqslant 0}$ *with* $A(\lambda) > 0$, $\lim_{\lambda \to \infty} A(\lambda) = \infty$, *and* $X(1) \neq 0$ *with positive probability, then there is an* $H > 0$ *such that*

$$A(\lambda) = \lambda^H L(\lambda),$$

*where L is a slowly varying function and X is an H-s.s. s.i. process. Conversely, all H-s.s. s.i. processes X with* $H > 0$ *can be obtained in this way.*

Remark. For the last statement take $A(\lambda) = \lambda^H$ and $\zeta_k = X(k) - X(k-1)$ for $k \in N$.

We have the following relationship between the moment condition and the parameter $H$ of s.s. s.i. processes. Let $X$ be a non-degenerate $H$-s.s. s.i. process, $H > 0$. In this case, if $E|X(t)| < \infty$, then $H \leqslant 1$. Moreover, if $0 < H < 1$, then $E|X(t)| = 0$ (cf. Maejima [8]), and if $H = 1$, we get $X(t) = tX(1)$ a.s., i.e. $X$ is trivial (see Vervaat [11]).

Combining that and the fact that weak convergence in the Skorokhod topology implies convergence with respect to finite-dimensional distributions we may assert the following

COROLLARY 2.1. *Let* $(Y_k)_{k \in N}$ *be a stationary sequence with common distribution function F and mean zero such that*

$$\frac{1}{\phi(n)} \sum_{k=1}^{[nt]} Y_k \Rightarrow X(t) \quad \text{as } n \to \infty$$

*for some reals* $(\phi(n))_{n \geqslant 0}$, $\phi(n) > 0$, $\lim_{n \to \infty} \phi(n) = \infty$, *and X is a non-degenerate stochastic process. Then for some* $0 < H \leqslant 1$ *the process X is H-s.s. s.i., and* $\phi$ *is of the form* $\phi(n) = n^H L(n)$ *for L being a slowly varying function. Conversely, every H-s.s. s.i. process X in D, of the mean* $EX(t) = 0$, *can be obtained this way.*

Proof. The last part of the assertion follows from the fact that the convergence in the converse part of Proposition 2.1 is in fact weak provided that $X$ is in $D$. ■

Now we can state the theorem that yields our statement.

THEOREM 2.1. *Let $(Y_k)_{k\in N}$ be a stationary sequence with common distribution function $F$ and mean $\mu > 0$. Assume that*

$$\frac{1}{\phi(n)}\sum_{k=1}^{[nt]}(Y_k-\mu)\Rightarrow X(t) \quad \text{as } n\to\infty \tag{3}$$

*for some non-degenerate process $X$ and reals $(\phi(n))_{n\geq 0}$, $\lim_{n\to\infty}\phi(n)=\infty$. Furthermore, let $(N^{(n)})_{n\in N}$ be a sequence of point processes such that*

$$\frac{N^{(n)}(t)-\lambda nt}{\phi(n)}\xrightarrow{P}0 \tag{4}$$

*and*

$$\frac{N^{(n)}(t)}{n}\xrightarrow{P}\lambda t \tag{5}$$

*in probability in the Skorokhod topology for some positive constant $\lambda$.*

*If*

$$\lim_{n\to\infty}\left(c^{(n)}-\lambda n\frac{\mu}{\phi(n)}\right)=c \tag{6}$$

*and*

$$\lim_{n\to\infty}u^{(n)}=u,$$

*then*

(i) *there exists an $0 < H \leqslant 1$ such that $X$ is $H$-s.s., s.i., $\phi$ is of the form $\phi(n)=n^H L(n)$ for $L$ being a slowly varying function, and*

(ii) *it follows that*

$$Q^{(n)}(t)=u^{(n)}+c^{(n)}t-\frac{1}{\phi(n)}\sum_{k=1}^{N^{(n)}(t)}Y_k\Rightarrow Q(t)=u+ct-\lambda^H X(t) \tag{7}$$

*in the Skorokhod topology as $n\to\infty$.*

*Conversely, every $H$-s.s. s.i. process $X$ in $D$, with $EX(t)=0$ and $0<H\leqslant 1$, can be obtained by (7).*

Proof. Part (i) of the assertion is obvious by Corollary 2.1. In order to prove part (ii) let us recall the following Whitt theorem on random time change (for details see Whitt [12]). Let $(Z_n)_{n\in N}$, let $Z$ be processes in $D[0,\infty)$ and suppose that $Z_n \Rightarrow Z$. Let $(N_n)_{n\in N}$ be a sequence of processes with non-decreasing sample paths starting from 0 such that $N_n \Rightarrow \lambda I$, $\lambda > 0$. For each $n \in N$, $Z_n$ and $N_n$ are assumed to be defined on the same probability space. Then

$$Z_n(N_n) \Rightarrow Z(\lambda I). \tag{8}$$

Now let us rewrite the process $Q^{(n)}(t)$ in the following form:

$$Q^{(n)}(t) = u^{(n)} + c^{(n)}(t) - \frac{1}{\phi(n)} \sum_{k=1}^{N^{(n)}(t)} Y_k \tag{9}$$

$$= u^{(n)} + t\left(c^{(n)} - \lambda n \frac{\mu}{\phi(n)}\right) - \mu\left(\frac{N^{(n)}(t) - \lambda nt}{\phi(n)}\right) - \frac{1}{\phi(n)} \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu). \tag{10}$$

From assumptions (3), (5) and the Whitt theorem (8) we obtain

$$\frac{1}{\phi(n)} \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu) \Rightarrow \lambda^H X(t) \quad \text{ as } n \to \infty.$$

Since

$$u^{(n)} + t\left(c^{(n)} - \lambda n \frac{\mu}{\phi(n)}\right) - \mu\left(\frac{N^{(n)}(t) - \lambda nt}{\phi(n)}\right)$$

converges to $u + ct$ in probability in the Skorokhod topology, the proof of part (ii) is complete.

By Corollary 2.1, setting $Y_k = X(k) - X(k-1) + \mu$, $k = 1, 2, \ldots$, where $\mu > 0$, in order to conclude the converse part we merely have to construct a sequence $(N^{(n)})_{n\in N}$ that fulfills the conditions (4) and (5). To this end, set $N(t) = [nt]$ and $\phi(n) = n^H$. This completes the proof. ∎

Remarks. 1. The condition (4) implies (5) unless $H = 1$.

2. $H = 1$ corresponds to the case when $X$ is trivial.

3. In order to construct a "more realistic" sequence $(N^{(n)})_{n\in N}$ for $\frac{1}{2} < H \leqslant 1$ one can consider the case where the occurrence of the claims is described by a renewal process $N$:

$$N(t) = \max\{n: \sum_{k=1}^{n} T_k \leqslant t\}.$$

The interoccurrence times $(T_k)_{k\in N}$ are assumed to be independent, positive random variables with mean $1/\lambda$ and variance $\sigma^2$. We define

$$N^{(n)}(t) = N(nt).$$

Then for $\frac{1}{2} < H \leqslant 1$ and $\phi(n) = n^H$ the conditions (4) and (5) are fulfilled (see Furrer et al. [4]).

4. We could omit the point (10) and use just the previous relation (9) with slightly modified assumptions in order to state a more general result on the

weak convergence to s.s. s.i. processes. That is, it is enough to assume instead of (6) the condition $\lim_{n\to\infty} c^{(n)} = c$ and apply (5). Then, if we do not restrict $Y_k$ to variables with the finite mean, the resulting $H$-s.s. s.i. process $X$ may be quite general with infinite mean. Nevertheless, as a consequence, this would lead us to an artificial collective risk model interpretation of the final process $Q$ (cf. Section 3). Thus we do not intend to generalize this theorem.

## 3. APPROXIMATION OF RUIN PROBABILITY

Collective risk theory has paid considerable attention to the ruin functional in infinite and finite time. The weak convergence of $Q^{(n)}$ to $Q$ implies, for example,

$$\inf_{0\leqslant t\leqslant t_0} Q^{(n)}(t) \xrightarrow{d} \inf_{0\leqslant t\leqslant t_0} Q(t) \quad \text{for any } t_0 < \infty,$$

and thus

$$\lim_{n\to\infty} P\{\inf_{0\leqslant t\leqslant t_0} Q^{(n)}(t) < 0\} = P\{\inf_{0\leqslant t\leqslant t_0} Q(t) < 0\} \tag{11}$$

if

$$P\{\inf_{0\leqslant t\leqslant t_0} Q(t) = 0\} = 0. \tag{12}$$

Therefore, we may approximate the finite-time ruin probability of a risk process by the ruin probability in finite time of the corresponding weak approximation if the condition (12) is satisfied. That condition is suggested by the author to be true for a wide class of s.s. s.i. processes and has already been known for the Brownian and Lévy motion.

THEOREM 3.1. *Consider a risk process* $R(t) = u+ct-\sum_{k=1}^{N(t)} Y_k$. *Denote the corresponding finite-time ruin probability by* $\Psi(u, T)$. *If the assumptions from Theorem* 2.1 *are satisfied for* $Y_k$, *the sequence* $N^{(n)}(t) = N(nt)$, $\phi(n) = n^H$, $0 < H < 1$, $P\{\inf_{0\leqslant t\leqslant t_0} Q(t) = 0\} = 0$ *and the relative safety loading* $\theta = c/(\lambda\mu)-1 > 0$, *then*

$$\Psi(u, T) \sim_{n\to\infty} P\{\inf_{0\leqslant s\leqslant T} (u+\theta\lambda\mu s-\lambda^H X_H(s)) < 0\}.$$

Proof. For each $n\in N$, we have

$$\Psi(u, T) = P\left\{\inf_{0\leqslant s\leqslant T}\left(u+cs-\sum_{k=1}^{N(s)} Y_k\right) < 0\right\}$$
$$= P\left\{\inf_{0\leqslant s\leqslant T}\left(\frac{u}{\phi(n)}+\frac{cs}{\phi(n)}-\frac{1}{\phi(n)}\sum_{k=1}^{N(s)} Y_k\right) < 0\right\}$$

$$= P\left\{\inf_{0\leqslant s\leqslant T/n}\left(\frac{u}{\phi(n)}+\frac{cns}{\phi(n)}-\frac{1}{\phi(n)}\sum_{k=1}^{N(ns)} Y_k\right)<0\right\}$$

$$= P\left\{\inf_{0\leqslant s\leqslant T/n}\left(\frac{u}{\phi(n)}+s\left(\frac{cn}{\phi(n)}-\frac{\lambda\mu n}{\phi(n)}\right)\right.\right.$$

$$\left.\left.-\mu\left(\frac{N(ns)-\lambda ns}{\phi(n)}\right)-\frac{1}{\phi(n)}\sum_{k=1}^{N(ns)}(Y_k-\mu)\right)<0\right\}.$$

Now assume that $T_0 = T/n$, $\theta_0 = (\theta\lambda\mu n)/\phi(n)$ and $u_0 = u/\phi(n)$ are constants, i.e. we increase $T$ and $u$ with $n$, and decrease at the same time the safety loading $\theta$ with $n$ (as $H < 1$). This means that a small safety loading is compensated by a large initial capital. Then we obtain

$$\Psi(u, T) = P\left\{\inf_{0\leqslant s\leqslant T_0}\left(u_0+\theta_0 s-\mu\left(\frac{N(ns)-\lambda nt}{\phi(n)}\right)-\frac{1}{\phi(n)}\sum_{k=1}^{N(ns)}(Y_k-\mu)\right)<0\right\}.$$

Applying Theorem 2.1 and (11) we obtain

$$\Psi(u, T) \to P\left\{\inf_{0\leqslant s\leqslant T_0}\left(u_0+\theta_0 s-\lambda^H X_H(s)\right)<0\right\}.$$

By self-similarity this completes the proof. ■

## 4. RUIN PROBABILITIES FOR GENERAL SELF-SIMILAR PROCESSES

In the previous sections we showed that the process $Q$ defined in (7) can be looked as an approximation of a risk process. Our aim in this section is to investigate the probability that the process $Q$ reaches the level 0 before time $t$. In the Brownian case the probability can be calculated explicitly (see for instance Asmussen [1]):

$$P\left\{\sup_{0\leqslant s\leqslant t}\left(\lambda^{1/2}B(s)-cs\right)>u\right\} = \bar{\Phi}\left(\frac{u+ct}{\sqrt{\lambda t}}\right)+\exp\left(-\frac{2uc}{\lambda}\right)\Phi\left(\frac{-u+ct}{\sqrt{\lambda t}}\right),$$

where $\Phi$ is the standard normal distribution and $\bar{\Phi} = 1-\Phi$. Furrer et al. [4] and Michna [9] provide upper bounds where $X_H(t)$ is a standard symmetric $\alpha$-stable Lévy motion $Z_\alpha$ and a standard fractional Brownian motion $B_H$, respectively. Furrer et al. [4] prove that for positive numbers $u$, $c$ and $\lambda$

$$P\left\{\sup_{0\leqslant s\leqslant t}\left(\lambda^{1/\alpha}Z_\alpha(s)-cs\right)>u\right\} \leqslant 2\bar{G}\left(\frac{u}{(\lambda t)^{1/\alpha}}\right),$$

where $\bar{G} = 1-G$ and $G$ denotes the cumulative distribution function of a standard $S\alpha S$ variable. Michna [9] shows that for $\frac{1}{2} < H < 1$

$$P\{\sup_{0\leqslant s\leqslant t}(\lambda^H B_H(s)-cs) > u\} \leqslant \bar{\Phi}\left(\frac{u+ct}{(\lambda t)^H}\right)+\exp\left(-\frac{2uct}{(\lambda t)^{2H}}\right)\bar{\Phi}\left(\frac{u-ct}{(\lambda t)^H}\right).$$

Now let us state a theorem which yields a lower bound for the ruin probability of the process $Q$ for an arbitrary self-similar process $X_H$ with $H > 0$.

THEOREM 4.1. *Let $(X_H(t))_{t\geqslant 0}$ be an arbitrary self-similar process with the exponent $H > 0$. If $0 < H < 1$ and $t$ is sufficiently large, namely $(uH)/ct(1-H) < 1$, then*

$$P\{\sup_{0\leqslant s\leqslant t}(\lambda^H X_H(s)-cs) > u\} \geqslant \bar{G}\left[\left(\frac{u}{1-H}\right)^{1-H}\left(\frac{c}{\lambda H}\right)^H\right]; \tag{13}$$

*otherwise*

$$P\{\sup_{0\leqslant s\leqslant t}(\lambda^H X_H(s)-cs) > u\} \geqslant \bar{G}\left(\frac{u+ct}{(\lambda t)^H}\right), \tag{14}$$

*where $\bar{G} = 1-G$ and $G$ denotes the distribution function of $X_H(1)$.*

Proof. Since the process $X_H$ is $H$-s.s., we have

$$P\{\sup_{0\leqslant s\leqslant t}(\lambda^H X_H(s)-cs) > u\} = P\{\sup_{0\leqslant s\leqslant 1}(\lambda^H t^H X_H(s)-cts) > u\}.$$

Furthermore, it is obvious that

$$P\{\sup_{0\leqslant s\leqslant 1}(\lambda^H t^H X_H(s)-cts) > u\} \geqslant P\{\lambda^H t^H X_H(\tau)-ct\tau > u\}$$

for all $\tau\in(0, 1]$.

Eventually, applying one more time the definition of self-similarity we obtain

$$\begin{aligned} P\{\sup_{0\leqslant s\leqslant t}(\lambda^H X_H(s)-cs) > u\} &\geqslant P\left\{X_H(1) > \frac{u+ct\tau}{\lambda^H t^H \tau^H}\right\} \\ &= 1-G\left(\frac{u+ct\tau}{(\lambda t\tau)^H}\right) \end{aligned} \tag{15}$$

for all $\tau\in(0, 1]$, where $G$ stands for the distribution function of $X_H(1)$.

In order to find the best possible lower bound for the ruin probability in finite time we are to find minimum of the function $f(\tau) = (u+ct\tau)/(\lambda t\tau)^H$ on the interval (0, 1]. To this end we calculate the derivative of the expression: $(u\tau^{-H}+c t\tau^{1-H})$ and find out that it is equal to 0 for

$$\tau = \tau_0 = \begin{cases} (uH)/ct(1-H) & \text{if } H < 1, \\ \infty & \text{if } H \geqslant 1. \end{cases}$$

Hence, if $\tau_0 < 1$, then the minimum of the function $f$ on (0, 1] is

$$f(\tau_0) = \left(\frac{u}{1-H}\right)^{1-H}\left(\frac{c}{\lambda H}\right)^{H};$$

otherwise

$$f(1) = \frac{u+ct}{(\lambda t)^{H}}.$$

This proves the theorem. ∎

Remark. The condition $0 < H < 1$ corresponds to the case when $X_H(t)$ is non-trivial, has stationary increments and finite first moment for each $t$.

Since the lower bound (13) does not depend explicitly on $t$, it can serve as well as a bound for the ruin probability for $Q$ in infinite time. Furthermore, the bound defined in (14) tends to $\bar{G}(c/\lambda)$ when $H = 1$ and to $\bar{G}(0)$ when $H > 1$ as $t \to \infty$. Therefore, we may claim the following

COROLLARY 4.1. *Let $(X_H(t))_{t\geqslant 0}$ be an arbitrary self-similar process with the exponent $H > 0$. Then we have*

$$(16)\quad P\{\sup_{s\geqslant 0}(\lambda^H X_H(s)-cs) > u\} \geqslant \begin{cases} \bar{G}\left[(u/(1-H))^{1-H}(c/(\lambda H))^{H}\right] & \text{if } H < 1, \\ \bar{G}(c/\lambda) & \text{if } H = 1, \\ \bar{G}(0) & \text{if } H > 1, \end{cases}$$

*where $\bar{G} = 1-G$ and $G$ denotes the distribution function of $X_H(1)$.*

Remarks. 1. The lower bound (16) has already been obtained by Norros [10] for a special case when $X$ is an FBM ($H < 1$), in the storage model setting.

2. Duffield and O'Connell [3] using the result from Norros [10] showed that the bound is in fact accurate in the logarithmic sense (the case when $X$ is an FBM).

Considering specific cases when $X_H$ is a standard Gaussian or a standard $S\alpha S$ process, and letting the initial risk reserve $u$ become large we obtain the following results:

COROLLARY 4.2. *If $X_H$ is Gaussian with $X_H(1)$ being a standard normal variable and $0 < H < 1$, then*

$$P\{\sup_{s\geqslant 0}(\lambda^H X_H(s)-cs) > u\} \geqslant \exp\left\{-\frac{1}{2}\left(\frac{u}{1-H}\right)^{2-2H}\left(\frac{c}{\lambda H}\right)^{2H}\right\}.$$

Proof. Recall the elementary relation

$$1-\Phi(x) \sim x^{-1}f(x) \sim \exp(-x^2/2) \quad \text{for } x\to\infty,$$

where $\phi$ and $f$ stand for a distribution and density function of the standard normal distribution, respectively. ∎

COROLLARY 4.3. *If $X_H$ is standard $S\alpha S$ and $0 < H < 1$, then*

$$P\{\sup_{s\geqslant 0}(\lambda^H X_H(s)-cs) > u\} \geqslant \left(\frac{1-H}{u}\right)^{\alpha(1-H)}\left(\frac{\lambda H}{c}\right)^{\alpha H}.$$

Proof. This stems from the fact that the tail probabilities of a standard $S\alpha S$ distribution behave like $C_\alpha x^{-\alpha}$, where $C_\alpha$ is constant.

The construction of the lower bound (13) in the proof of Theorem 4.1 suggests that the bound should be quite a good estimate. For instance, the bound in Corollary 4.2 for $H = \frac{1}{2}$ gives in fact an exact result for the Brownian motion. Michna [9] shows that the lower bound (14) yields a good approximation of the ruin probability for the fractional Brownian motion, when $u$ is large.

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