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EXISTENCE AND NON-EXISTENCE OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC EQUATIONS

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Abstract. We consider the one-dimensional stochastic equation

$$X_t = x_0 + \int_0^t b(X_s)\, d\langle M\rangle_s + \int_0^t \sigma(X_s)\, dM_s$$

for a continuous local martingale M with square variation $\langle M\rangle$ and measurable drift and diffusion coefficients b and σ. The main purpose of this paper is to derive a necessary condition for the existence of a solution X starting from x_0. As a result, we construct a diffusion coefficient σ such that the above stochastic equation has no solution X whatever the initial value x_0 and the non-zero, say, continuous drift coefficient b might be.

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1. Introduction. In the present paper we study the one-dimensional stochastic equation

$$X_t = x_0 + \int_0^t b(X_s)\, d\langle M\rangle_s + \int_0^t \sigma(X_s)\, dM_s, \quad t < S_\infty(X), \tag{1.1}$$

where $b, \sigma\colon \mathbf{R} \to \mathbf{R}$ are Borel (or only Lebesgue) measurable functions, $x_0 \in \mathbf{R}$, and M is a continuous local martingale with square variation process $\langle M\rangle$. Here $S_\infty(X)$ denotes the explosion time of X defined by $S_\infty(X) = \sup_{m \geqslant 1} S_m(X)$, where

$$S_m(X) = \inf\{t \geqslant 0\colon |X_t| \geqslant m\}, \quad m \geqslant 1.$$

We always assume that the continuous local martingale M is not trivial, which means that $P(\langle M\rangle_\infty > 0) > 0$, where $\langle M\rangle_\infty = \sup_{t \geqslant 0} \langle M\rangle_t$. The continuous local martingale M is characterized by its distribution μ on the space $\big(C([0, +\infty)), \mathscr{C}([0, +\infty))\big)$ of continuous functions $x\colon [0, +\infty) \to \mathbf{R}$ equipped with the σ-algebra generated by the coordinate mappings.

A process X on a filtered probability space $(\Omega, \mathscr{F}, P, \boldsymbol{F})$ is a (weak) solution to equation (1.1) if there can be found a continuous local martingale $(M, \boldsymbol{F})$ with *prescribed* distribution μ such that (1.1) is satisfied.

Let us introduce the sets

$$E_\sigma = \{x \in \boldsymbol{R}: \int_{x-\varepsilon}^{x+\varepsilon} \sigma^{-2}(y)\,dy = +\infty, \ \forall \varepsilon > 0\}, \tag{1.2}$$

where we put $\sigma^{-2}(y) = +\infty$ if $\sigma(y) = 0$, and

$$N_\sigma = \{x \in \boldsymbol{R}: \sigma(x) = 0\}. \tag{1.3}$$

If $M = B$ is a Brownian motion, it was shown in [4] that, for all initial values $x_0 \in \boldsymbol{R}$, there exists a solution $(X, \boldsymbol{F})$ to equation (1.1) but without drift ($b \equiv 0$) if and only if the condition

$$E_\sigma \subseteq N_\sigma \tag{1.4}$$

is satisfied. Using space transformation, in [4] this existence result was transferred to stochastic equations with generalized drift

$$X_t = x_0 + \int_{\boldsymbol{R}} L^X(t, a)\,v(da) + \int_0^t \sigma(X_s)\,dB_s, \qquad t < S_\infty(X), \tag{1.5}$$

where $L^X(t, a)$ is the (right) local time of the continuous semimartingale $(X, \boldsymbol{F})$ up to $S_\infty(X)$, and v is a set function which is a finite signed measure on every interval $[-N, N]$, $N \geqslant 1$, such that $v(\{x\}) < \frac{1}{2}$ for all $x \in \boldsymbol{R}$. We notice that every solution $(X, \boldsymbol{F})$ to equation (1.5) is stopped after first reaching E_σ, i.e.,

$$X_t = X_{t \wedge D_{E_\sigma}}, \qquad t \geqslant 0, \ P\text{-a.s.}, \tag{1.6}$$

where D_{E_σ} denotes the first entry time of X into E_σ (cf. [5], Proposition (4.34) (iv)). The condition on v, however, is not quite satisfactory because, e.g., the measure $v(dx) = c\sigma^{-2}(x)\,dx$ does not fulfil it whenever $E_\sigma \neq \emptyset$. This would exclude constant drift functions $b \equiv c$ in equation (1.1) (see below). Therefore, equation (1.5) was also extended to the case where v is only a finite signed measure *locally* on the open set E_σ^c (see [5], Remark (4.40) (ii)). But in this case not every solution $(X, \boldsymbol{F})$ of equation (1.5) satisfies (1.6) and the existence criterion now takes the following form:

For every initial value $x_0 \in \boldsymbol{R}$ there exists a solution $(X, \boldsymbol{F})$ of equation (1.5) *such that the boundary condition* (1.6) *is satisfied if and only if the inclusion* (1.4) *holds.*

An important special case is obtained if we assume that the drift measure v is given by $v(dx) = b(x)\sigma^{-2}(x)\,dx$, where $b\sigma^{-2}$ is locally integrable in E_σ^c and, as we always agree, $0 \cdot +\infty = 0$. In this way, we come back to our stochastic equation (1.1) which we are mainly interested in. More precisely, we have the following slight modification of [5], Theorem (4.53) (1), (2).

THEOREM 1.1. *Suppose that, for measurable real functions b and σ, the function $b\sigma^{-2}$ is locally integrable in E_σ^c. For every initial value $x_0 \in \mathbf{R}$ then there exists a solution $(X, \mathbf{F})$ of*

$$(1.7)\qquad X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s, \qquad t < S_\infty(X),$$

such that the boundary condition (1.6) *is satisfied if and only if the following inclusion holds*:

$$(1.8)\qquad E_\sigma \subseteq N_\sigma \cap N_b.$$

Indeed, if $x_0 \in E_\sigma$, then for every solution $(X, \mathbf{F})$ of (1.7) with $X_0 = x_0$ satisfying (1.6) we have $X_t \equiv x_0$, which necessarily implies $x_0 \in N_\sigma$ as well as $x_0 \in N_b$. Conversely, the sufficiency of condition (1.8) follows from [5], Theorem (4.53) (1). To sketch the idea of the proof, we first observe that (1.8) ensures the existence criterion (1.4) for equation (1.5) with generalized drift v, $v(dx) = b(x)\sigma^{-2}(x)\,dx$, and the boundary condition (1.6). Hence, for every initial value $x_0 \in \mathbf{R}$, there exists a solution $(X, \mathbf{F})$ of equation (1.5) satisfying (1.6), which, moreover, is fundamental, i.e.,

$$\int_0^{D_{E_\sigma}} \mathbf{1}_{N_\sigma}(X_s)\,ds = 0 \;\; P\text{-a.s.}$$

(see [5], (4.35), (4.40) (ii)). Using this property, condition (1.8), and the occupation time formula we easily compute

$$\int_{\mathbf{R}} L^X(t, a)\,v(da) = \int_0^t b(X_s)\,ds \;\; P\text{-a.s.}$$

Thus $(X, \mathbf{F})$ is the desired solution of (1.7). ■

Let us emphasize that all existence results reviewed above for Brownian motion remain true for arbitrary (non-trivial) continuous local martingales M as a driving process. However, we will not deal with this extension in the present paper.

Now the question arises what happens if we drop the boundary condition (1.6). It turns out that then the situation becomes quite different:

(a) *There may exist solutions of* (1.7) *which do not satisfy the boundary condition* (1.6).

(b) *Condition* (1.8) *is sufficient but not necessary for the existence of solutions* $(X, \mathbf{F})$ *of equation* (1.7) *for arbitrary initial values* $x_0 \in \mathbf{R}$.

Furthermore, we observe that the restriction $E_\sigma \subseteq N_b$ (i.e., $b(x) = 0$ for all $x \in E_\sigma$) is too hard and excludes many interesting situations. What happens if we drop this condition? Naturally, we come back to condition (1.4): $E_\sigma \subseteq N_\sigma$. However, Rutkowski [10] showed:

(c) *The condition* $E_\sigma \subseteq N_\sigma$ *is neither necessary nor sufficient for the existence of a solution* (X, F) *to equation* (1.1) (*and hence also to equation* (1.7)) *with arbitrary initial value* $x_0 \in \mathbf{R}$.

To illustrate the situation in greater detail, let us examine several examples closely related to examples given by Rutkowski [10].

Example 1.2. First we consider equation (1.1) for bounded and continuous coefficients b and σ. Then there always exist non-exploding solutions with arbitrary initial value $x_0 \in \mathbf{R}$; see Skorohod [12] for $M = B$ or Jacod and Mémin [6]. This result is obtained approximating the coefficients uniformly by Lipschitz functions. We notice that the condition $E_\sigma \subseteq N_\sigma$ always holds but, obviously, $E_\sigma \subseteq N_b$ (and hence also (1.8)) need not be satisfied. This together with Theorem 1.1 proves assertion (b).

Example 1.3. Again we consider equation (1.1) for bounded and continuous b and σ. Additionally, for some $a \in \mathbf{R}$ we assume $b(a) > 0$ and

$$\int_a^{a+\varepsilon} \sigma^{-2}(y)\,dy = +\infty \qquad \text{for all } \varepsilon > 0. \tag{1.9}$$

In particular, this implies $a \in E_\sigma \subseteq N_\sigma$, This condition on σ is satisfied if, for example, σ is Lipschitz continuous and $a \in N_\sigma$. Let (X, F) be a solution of equation (1.1). According to Theorem 2.3 (ii) below we see that the point a is *non-sticky*, i.e.,

$$\int_0^\infty \mathbf{1}_{\{a\}}(X_s)\,d\langle M\rangle_s = 0 \quad P\text{-a.s.} \tag{1.10}$$

For the initial value $x_0 = a$, (1.10) shows that X does not satisfy the boundary condition (1.6).

Example 1.4. Let (X, F) be a solution to equation (1.1) for coefficients b and σ, and $a \in \mathbf{R}$ as in Example 1.3. In view of (1.10), X also solves

$$X_t = x_0 + \int_0^t \tilde{b}(X_s)\,d\langle M\rangle_s + \int_0^t \sigma(X_s)\,dM_s, \qquad t \geqslant 0,$$

where $\tilde{b} = \mathbf{1}_{\mathbf{R}\setminus\{a\}}\, b$. If we choose b such that $E_\sigma \setminus \{a\} \subseteq N_b$, we may conclude that for the coefficients $\tilde{b}$ and σ the condition (1.8), $E_\sigma \subseteq N_\sigma \cap N_{\tilde{b}}$, holds but our solution X for $x_0 = a$ does not satisfy the boundary condition (1.6), and hence X is different from the solution in Theorem 1.1. This proves (a).

Example 1.5. Again, let (X, F) be a solution to equation (1.1) for coefficients b and σ, and $a \in \mathbf{R}$ as in Example 1.3. In view of (1.10), for $\tilde{\sigma} = \sigma + \mathbf{1}_{\{a\}}$, X also solves

$$X_t = x_0 + \int_0^t b(X_s)\,d\langle M\rangle_s + \int_0^t \tilde{\sigma}(X_s)\,dM_s, \qquad t \geqslant 0.$$

For these coefficients b and $\tilde{\sigma}$, condition (1.4) fails. But we have seen that, for all initial values $x_0 \in \mathbf{R}$, a solution to the above equation does exist. Thus the first part of (c) is correct.

EXAMPLE 1.6. Now we consider the equation

$$X_t = x_0 + \int_0^t \mathbf{1}_{\{a\}}(X_s)\, d\langle M\rangle_s + \int_0^t \sigma(X_s)\, dM_s, \qquad t \geqslant 0, \tag{1.11}$$

where the diffusion coefficient σ and the point $a \in \mathbf{R}$ are chosen such that conditions (1.4) and (1.9) are fulfilled. For any solution $(X, \mathbf{F})$, we have the relation (1.10) in view of Theorem 2.3 (ii) below and, consequently,

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dM_s, \qquad t \geqslant 0,$$

an equation without drift. If $M = B$ is a Brownian motion, it is known that every solution X of this equation satisfies the boundary condition (1.6) (cf. [5], Proposition (4.34) (iv)). By time change it can easily be seen that this remains true for general M. If we choose $x_0 = a$ as an initial value, this implies $X_t \equiv a$ for all $t \geqslant 0$, which, however, contradicts (1.10) unless $P(\langle M\rangle_\infty = 0) = 1$. Thus we have shown that it is not true that (1.11) has a solution for all initial values, though the condition $E_\sigma \subseteq N_\sigma$ is satisfied. This verifies the second part of (c). Of course, in this example the condition (1.8) of Theorem 1.1 fails to hold.

In Example 1.6 we have seen how discontinuous drift can disturb existence of solutions while in Example 1.5 with "nicer" drift b the "bad" diffusion $\tilde{\sigma}$ cannot prevent existence, contrary to equations without drift.

The main purpose of this paper is to establish necessary conditions for the existence of solutions to equation (1.1). From this we shall see that σ can be chosen bad enough such that equation (1.1) has no solution whatever the initial value $x_0 \in \mathbf{R}$ and the non-zero continuous drift coefficient b might be. Moreover, the continuity of b is not important and can be weakened considerably.

To this end, as Theorem 1.1 shows, we have to look for diffusion coefficients σ such that $E_\sigma \neq \emptyset$. However, it turns out that if E_σ only consists of isolated points (i.e., if E_σ is denumerable without accumulation points) and if, e.g., b is continuous such that $N_b = \emptyset$, then there always exist (at least local) solutions starting at an arbitrary initial value $x_0 \in \mathbf{R}$. The continuity of b can also be replaced by the condition that b is non-negative (or non-positive) and such that b^{-1} is locally integrable. We will not deal with this problem in the present paper, referring the reader to the forthcoming paper [1].

It is left as an open problem what happens if E_σ is a more general *nowhere dense* subset of $\mathbf{R}$, perhaps, such that E_σ^c has arbitrarily small Lebesgue measure. We shall investigate the opposite case when E_σ has inner points. Our main example deals with the worst case: We will construct diffusion coefficients σ with $E_\sigma = \mathbf{R}$. In Section 2, we shall study the behaviour of solutions $(X, \mathbf{F})$

of equation (1.1), which is of interest in its own right. In Section 3, we collect some knowledge on solutions of (1.1) without diffusion ($\sigma \equiv 0$) for later applications. In Section 4, we state necessary conditions for the existence of solutions of (1.1). Finally, in Section 5 we deal with the non-existence of solutions of (1.1) and construct highly singular diffusion coefficients for which existence of solutions fails.

2. Properties of solutions. Let m be an arbitrary non-negative measure on $(\boldsymbol{R}, \mathscr{B}(\boldsymbol{R}))$, where $\mathscr{B}(\boldsymbol{R})$ denotes the σ-algebra of Borel subsets of the real line $\boldsymbol{R}$. We introduce the sets

$$E_m^+ = \{x \in \boldsymbol{R}\colon\ m([x, x+\varepsilon)) = +\infty,\ \forall \varepsilon > 0\},$$

$$E_m^- = \{x \in \boldsymbol{R}\colon\ m((x-\varepsilon, x]) = +\infty,\ \forall \varepsilon > 0\},$$

and

$$E_m = E_m^+ \cup E_m^- .$$

Obviously, E_m^+ (respectively, E_m^-) is closed in the right (respectively, left) topology of $\boldsymbol{R}$. The set E_m is closed and m is a locally finite measure on E_m^c.

We now define the non-negative measures μ and ν by

$$\mu(A) = \int_A (|b(y)|+1)\,\sigma^{-2}(y)\,dy, \qquad A \in \mathscr{B}(\boldsymbol{R}),$$

$$\nu(A) = \int_A |b(y)|\,\sigma^{-2}(y)\,dy, \qquad A \in \mathscr{B}(\boldsymbol{R}).$$

Clearly, we have the inclusions $E_\nu^+ \subseteq E_\mu^+$, $E_\nu^- \subseteq E_\mu^-$, $E_\nu \subseteq E_\mu$. In accordance with Section 1, in the case where $b \equiv 0$, we denote the sets E_μ^+, E_μ^-, and E_μ simply by E_σ^+, E_σ^-, and E_σ, respectively. Obviously, $E_\sigma^+ \subseteq E_\mu^+$, $E_\sigma^- \subseteq E_\mu^-$, $E_\sigma \subseteq E_\mu$. We note that σ^{-2} is locally integrable on E_σ^c.

Let $(X, \boldsymbol{F})$ be a solution of equation (1.1) for the drift and diffusion coefficients b and σ with given initial value $x_0 \in \boldsymbol{R}$. By $L^X(t, a)$ we denote the (right) local time spent in a up to time t by the continuous semimartingale $(X, \boldsymbol{F})$ up to $S_\infty(X)$. This is a continuous increasing process in $t < S_\infty(X)$, right-continuous and left-hand limited in $a \in \boldsymbol{R}$, with $L^X(0, a) = 0$ and such that the occupation time formula

$$\int_0^t f(X_s)\,d\langle X\rangle_s = \int_{\boldsymbol{R}} f(a)\,L^X(t, a)\,da, \qquad t < S_\infty(X),\ P\text{-a.s.}, \tag{2.1}$$

for all bounded or non-negative measurable functions f holds (see, e.g., [7], Chapter VI). Here $\langle X\rangle$ denotes the square variation process of the continuous local martingale part (up to $S_\infty(X)$) of X which is defined on $[0, S_\infty(X))$. The left-hand limit of the local time will be denoted by $L_-^X(t, a)$. To begin with, we study the local time $L^X(t, a)$ for $a \in N_\sigma$. We remember that N_σ is the set of zeros of σ (cf. (1.3)). The Lebesgue measure on $\boldsymbol{R}$ is denoted by λ.

LEMMA 2.1. *We have* $L^X(t, a) = 0$, $a \in N_\sigma$, λ-*a.e.*, $t < S_\infty(X)$, *P-a.s.*

Proof. Because of the occupation time formula (2.1), for $t < S_\infty(X)$ we get P-a.s.

$$\int_{\mathbf{R}} \mathbf{1}_{N_\sigma}(a) L^X(t, a)\, da = \int_0^t \mathbf{1}_{N_\sigma}(X_s)\, d\langle X\rangle_s = \int_0^t \mathbf{1}_{N_\sigma}(X_s)\sigma^2(X_s)\, d\langle M\rangle_s = 0,$$

and hence

$$\mathbf{1}_{N_\sigma} L^X(t, \cdot) = 0 \ \lambda\text{-a.e.},$$

which proves the lemma. ∎

Now we investigate the behaviour of X in the set E_μ. As a first step, we obtain

LEMMA 2.2. (i) *For every* $a \in E_\mu^+$, $L^X(t, a) = 0$, $t < S_\infty(X)$, *P-a.s.*
(ii) *For every* $a \in E_\mu^-$, $L_-^X(t, a) = 0$, $t < S_\infty(X)$, *P-a.s.*

Proof. For proving (i), for fixed $t > 0$ let $A = \{L^X(t, a) > 0,\ t < S_\infty(X)\}$ and assume that $P(A) > 0$. Because of the right continuity of $L^X(t, \cdot)$ there exist a random variable $\varepsilon > 0$ and $n \geqslant 1$ such that

$$\{L^X(t, a+y) \geqslant n^{-1},\ \forall y \in [0, \varepsilon),\ t < S_\infty(X)\}$$

has strictly positive probability. On this set, we obtain

$$n^{-1}\mu([a, a+\varepsilon)) = n^{-1}\int_a^{a+\varepsilon} (|b(y)|+1)\sigma^{-2}(y)\, dy$$
$$\leqslant \int_a^{a+\varepsilon} (|b(y)|+1)\sigma^{-2}(y) L^X(t, y)\, dy \leqslant \int_{\mathbf{R}} (|b(y)|+1)\sigma^{-2}(y) L^X(t, y)\, dy.$$

Using Lemma 2.1 and the occupation time formula (2.1), we now estimate

$$n^{-1}\mu([a, a+\varepsilon)) \leqslant \int_{\mathbf{R}} (|b(y)|+1)\mathbf{1}_{N_\sigma^c}(y)\sigma^{-2}(y) L^X(t, y)\, dy$$
$$= \int_0^t (|b(X_s)|+1)\mathbf{1}_{N_\sigma^c}(X_s)\sigma^{-2}(X_s)\, d\langle X\rangle_s$$
$$= \int_0^t (|b(X_s)|+1)\mathbf{1}_{N_\sigma^c}(X_s)\sigma^{-2}(X_s)\sigma^2(X_s)\, d\langle M\rangle_s \leqslant \int_0^t |b(X_s)|\, d\langle M\rangle_s + \langle M\rangle_t,$$

the last term being finite for $t < S_\infty(X)$ P-a.s. Hence $\mu([a, a+\varepsilon)) < +\infty$ for some $\varepsilon > 0$ and, consequently, $a \in \mathbf{R}\backslash E_\mu^+$. This proves $P(A) = 0$ for any $a \in E_\mu^+$, and thus (i). Statement (ii) can be shown analogously. ∎

THEOREM 2.3. *Let* (X, F) *be a solution to equation* (1.1) *with drift and diffusion coefficients* b *and* σ. *We then have the following properties*:

(i) *For any* $a \in (E_\mu^+ \cap \{b \geq 0\}) \cup (E_\mu^- \cap \{b \leq 0\})$,

$$L^X(t, a) = L_-^X(t, a) = 0, \quad t < S_\infty(X) \ P\text{-a.s.} \tag{2.2}$$

(ii) *For any* $a \in (E_\mu^+ \cap \{b > 0\}) \cup (E_\mu^- \cap \{b < 0\}) \cup N_\sigma^c$,

$$\int_0^t \mathbf{1}_{\{a\}}(X_s)\, d\langle M\rangle_s = 0, \quad t < S_\infty(X) \ P\text{-a.s.} \tag{2.3}$$

Proof. According to Theorem VI.1.7 in Revuz and Yor [7] we get

$$L^X(t, a) - L_-^X(t, a) = 2\int_0^t \mathbf{1}_{\{a\}}(X_s)\, b(X_s)\, d\langle M\rangle_s \tag{2.4}$$

for $t < S_\infty(X)$ P-a.s. We now assume $a \in E_\mu^+$ and $b(a) \geq 0$. In view of Lemma 2.2, $L^X(t, a) = 0$ and from (2.4) we observe that also $L_-^X(t, a) = 0$ holds. This proves (2.2) for $a \in E_\mu^+ \cap \{b \geq 0\}$. If $a \in E_\mu^- \cap \{b \leq 0\}$, the proof is analogous. For $a \in (E_\mu^+ \cap \{b > 0\}) \cup (E_\mu^- \cap \{b < 0\})$, (2.3) follows from (2.2) and (2.4) since $b(a) \neq 0$ in this case. Finally, if $a \in N_\sigma^c$, the occupation time formula (2.1) yields

$$\int_0^t \mathbf{1}_{\{a\}}(X_s)\, d\langle M\rangle_s = \int_0^t \mathbf{1}_{\{a\}}(X_s)\, \sigma^{-2}(X_s)\, d\langle X\rangle_s = \int_{\mathbf{R}} \mathbf{1}_{\{a\}}(y)\, \sigma^{-2}(y)\, L^X(t, y)\, dy = 0$$

for all $t < S_\infty(X)$ P-a.s. ∎

Corollary 2.4. *If* (X, F) *is a solution to equation* (1.1) *without drift* (*i.e.*, $b \equiv 0$), *then for every* $a \in E_\sigma$

$$L^X(t, a) = L_-^X(t, a) = 0, \quad t < S_\infty(X), \ P\text{-a.s.}$$

Remark 2.5. In [9], Lemma 4.2, and [10], Lemma 4.4, Rutkowski has shown (2.2) for all $a \in N_\sigma$ and (2.3) for all $a \in N_\sigma \cap N_b^c$ if σ satisfies the so-called local time condition (LT) (see [8]). But under (LT), $N_\sigma \subseteq E_\sigma^+ \cap E_\sigma^-$ holds and, consequently, the results of Rutkowski are part of Theorem 2.3. Indeed, let $x_0 \in N_\sigma \setminus E_\sigma^+$ and choose $x_1 > x_0$ such that σ^{-2} is integrable over $[x_0, x_1]$. We consider the trivial solution X of equation (1.1) for $b \equiv 0$ with $X_t \equiv x_0$ and a non-trivial solution Y of the stochastic equation with reflecting barriers x_0 and x_1:

$$Y_t = x_0 + L^Y(t, x_0) - L^Y(t, x_1) + \int_0^t \sigma(Y_s)\, dM_s, \quad t \geq 0$$

(cf. Schmidt [11]). We then have $L^Y(t, x_0) > 0$, and hence $L^{Y-X}(t, 0) > 0$ for $t > 0$ sufficiently large with positive probability. This means that (LT) is not satisfied. Thus under (LT) we must have $N_\sigma \setminus E_\sigma^+ = \emptyset$. Similarly, $N_\sigma \setminus E_\sigma^- = \emptyset$, proving the inclusion $N_\sigma \subseteq E_\sigma^+ \cap E_\sigma^-$.

Theorem 2.6. *Let* (X, F) *be a solution to equation* (1.1) *with drift and diffusion coefficients* b *and* σ. *We then have*:

(i) $L^X(t, a) = 0$, $a \in E_\mu \cup N_\sigma$ λ-*a.e.*, $t < S_\infty(X)$, P-*a.s.*

(ii) *For every non-negative measurable function* f,

$$\int_0^t f(X_s)\mathbf{1}_{E_\mu \cup N_\sigma}(X_s)\,d\langle X\rangle_s = 0, \quad t < S_\infty(X),\ P\text{-a.s.}$$

Proof. There is an at most countable set of points $a \in \boldsymbol{R}$ (depending on t and ω) such that $L_-^X(t, a) \neq L^X(t, a)$. For (i), it now suffices to apply Lemma 2.1 and (2.2). Statement (ii) follows from (i) and the occupation time formula (2.1). ∎

3. Equations without diffusion. In this section, we consider the equation

$$Z_t = x_0 + \int_0^t b(Z_s)\,d\langle M\rangle_s, \quad t < S_\infty(Z), \tag{3.1}$$

which is equation (1.1) for $\sigma \equiv 0$. If $M = B$ is a Brownian motion, this is a (deterministic) ordinary differential equation. For any continuous (and bounded) function b, equation (3.1) has a (non-exploding) solution.

Our objective is to establish an occupation time formula for later application. To begin with, we briefly discuss the existence and uniqueness of solutions for only measurable but non-negative drift b. In the sequel, we always assume that $b \geqslant 0$. The case $b \leqslant 0$ can be handled analogously.

PROPOSITION 3.1. *Suppose that* b^{-1} *is locally integrable.*

(i) *Then, for every* $x_0 \in \boldsymbol{R}$, *there exists a solution* Z *to equation* (3.1) *with initial value* x_0 *such that the following condition is satisfied*:

$$\int_0^t \mathbf{1}_{N_b}(Z_s)\,d\langle M\rangle_s = 0, \quad t < S_\infty(Z),\ P\text{-a.s.} \tag{3.2}$$

(ii) *For every solution* Z *of equation* (3.1) *such that* (3.2) *holds and for every* $d \geqslant x_0$ *we have*

$$\sup_{0 \leqslant t < S_\infty(Z)} Z_t \geqslant d \quad \text{on } \Big\{\int_{x_0}^d b^{-1}(s)\,ds \leqslant \langle M\rangle_\infty\Big\}\ P\text{-a.s.} \tag{3.3}$$

Proof. Similarly to equations without drift (cf. [3]–[5]), the proof is given by time change. Let $x_0 \in \boldsymbol{R}$ and define

$$T_t = \int_0^t b^{-1}(x_0 + s)\,ds, \quad t \geqslant 0.$$

We consider the right inverse A of the continuous strictly increasing process T defined by $A_t = \inf\{s \geqslant 0\colon T_s > t\}$, $t \geqslant 0$. Now we set

$$Z_t = x_0 + (A \circ \langle M\rangle)_t, \quad t < \langle M\rangle^{-1}_{T_\infty -}, \tag{3.4}$$

where $\langle M\rangle^{-1}$ is the right inverse of the increasing process $\langle M\rangle$. It can easily be verified that Z is a solution to equation (3.1) with explosion time $S_\infty(Z) = \langle M\rangle^{-1}_{T_\infty -}$ and that Z satisfies (3.2). Explosion does not occur if and only if $\langle M\rangle_t < T_\infty$ for all $t \geqslant 0$ P-a.s. For proving (ii), we notice that every solution Z of equation (3.1) with initial value x_0 satisfying (3.2) is P-a.s. given by (3.4). Consequently, on $\{\int_{x_0}^{d} b^{-1}(s)\,ds \leqslant \langle M\rangle_\infty\}$ we get

$$\sup_{0 \leqslant t < S_\infty(Z)} Z_t = x_0 + A_{\langle M\rangle_\infty} \geqslant x_0 + A_{T_{d-x_0}} = d \quad P\text{-a.s.},$$

completing the proof of the proposition. ∎

Remark 3.2. (i) It can easily be seen that the local integrability of b^{-1} is also necessary for the existence of a non-trivial solution for arbitrary initial values $x_0 \in \boldsymbol{R}$ (cf. [3]).

(ii) The solution to equation (3.1) satisfying (3.2) is unique: As mentioned in the proof of Proposition 3.1, every solution Z to equation (3.1) with initial value x_0 satisfying (3.2) has the representation (3.4).

(iii) The condition of Proposition 3.1 that b^{-1} is locally integrable can be weakened: There can be given necessary and sufficient conditions for the existence and also for uniqueness of solutions to equation (3.1) for arbitrary initial value $x_0 \in \boldsymbol{R}$.

We now come to the occupation time formula.

PROPOSITION 3.3. *Suppose that* $b \geqslant 0$ *and let* Z *be a solution to equation* (3.1) *satisfying the condition* (3.2). *Then* Z *has a local time* $\mathscr{L}(t, a)$ *with respect to the increasing process* $\langle M\rangle$, *i.e., for every non-negative measurable function* f,

$$\int_0^t f(Z_s)\,d\langle M\rangle_s = \int_{\boldsymbol{R}} f(a)\,\mathscr{L}(t, a)\,da, \qquad t < S_\infty(Z),\ P\text{-a.s.}$$

Moreover, we have

$$\mathscr{L}(t, a) = \mathbf{1}_{[Z_0, Z_t]}(a)\,b^{-1}(a), \qquad a \in \boldsymbol{R},\ \lambda\text{-a.e.},\ t < S_\infty(Z),\ P\text{-a.s.}$$

Proof. By (3.2), for any non-negative measurable f,

$$\int_0^t f(Z_s)\,d\langle M\rangle_s = \int_0^t f(Z_s)\,b^{-1}(Z_s)\,b(Z_s)\,d\langle M\rangle_s = \int_0^t f(Z_s)\,b^{-1}(Z_s)\,dZ_s$$

and by time change in the integral (cf. [2], T IV.44) we get

$$\int_0^t f(Z_s)\,d\langle M\rangle_s = \int_{Z_0}^{Z_t} f(a)\,b^{-1}(a)\,da = \int_{\boldsymbol{R}} f(a)\,\mathbf{1}_{[Z_0, Z_t]}(a)\,b^{-1}(a)\,da$$

for every $t < S_\infty(Z)$ P-a.s. This proves the assertion. ∎

4. Necessary conditions for existence. Now we come back to the investigation of equation (1.1) and derive necessary conditions for the existence of solutions.

THEOREM 4.1. *Let b and σ be measurable functions. Suppose that b is non-negative and such that b^{-1} is locally integrable and $N_b = \emptyset$. Let be given an (in general) exploding solution (X, F) of equation* (1.1) *starting from $x_0 \in E_\mu^{(0)}$, the interior of E_μ, and denote the component of $E_\mu^{(0)}$ containing x_0 by (c, d). Then, as a necessary condition, we have*

$$\sigma(a) = 0 \quad \text{on } [x_0, d \wedge \sup_{t < S_\infty(X)} X_t) \ \lambda\text{-a.e., } P\text{-a.s.} \tag{4.1}$$

If, additionally, $\langle M \rangle_\infty = +\infty$ P-a.s., then

$$\sigma(a) = 0 \quad \text{on } [x_0, d), \ \lambda\text{-a.e.} \tag{4.2}$$

Proof. We remember the definition of E_μ at the beginning of Section 2. Let $D = D_{E_\mu^c}$ be the first entry time of X into E_μ^c. In view of (2.1) and Theorem 2.6 we get P-a.s.

$$\langle X \rangle_D = \int_0^D \mathbf{1}_{E_\mu}(X_s)\, d\langle X \rangle_s = \int_R \mathbf{1}_{E_\mu}(a)\, L^X(D, a)\, da = 0. \tag{4.3}$$

Hence the continuous local martingale part of X vanishes up to D and, consequently,

$$X_{t\wedge D} = Z_{t \wedge D}, \qquad t \geqslant 0, \tag{4.4}$$

where Z is the unique solution of equation (3.1), cf. Proposition 3.1 and Remark 3.2 (ii). We note that (3.2) holds automatically because $N_b = \emptyset$. On the other side, using (4.3), (4.4) and the occupation time formula in Proposition 3.3, we compute P-a.s.

$$\begin{aligned} 0 = \langle X \rangle_D &= \int_0^D \sigma^2(X_s)\, d\langle M \rangle_s = \int_0^D \mathbf{1}_{E_\mu \cap N_\sigma^c}(X_s)\, \sigma^2(X_s)\, d\langle M \rangle_s \\ &= \int_0^D \mathbf{1}_{E_\mu \cap N_\sigma^c}(Z_s)\, \sigma^2(Z_s)\, d\langle M \rangle_s = \int_R \mathbf{1}_{E_\mu \cap N_\sigma^c}(a)\, \sigma^2(a)\, \mathscr{L}(D, a)\, da \\ &= \int_R \mathbf{1}_{E_\mu \cap N_\sigma^c}(a)\, \sigma^2(a)\, \mathbf{1}_{[Z_0, Z_D]}(a)\, b^{-1}(a)\, da, \end{aligned}$$

where $Z_D = \sup_{t \geqslant 0} Z_t$ on $\{D = +\infty\}$. This implies

$$\mathbf{1}_{E_\mu \cap N_\sigma^c \cap [Z_0, Z_D]}\, \sigma^2 = 0 \ \ \lambda\text{-a.e. } P\text{-a.s.}$$

In view of $Z_0 = x_0$ we have

$$Z_D = d \wedge \sup_{0 \leqslant t < S_\infty(X)} X_t,$$

and $[x_0, d) \subseteq E_\mu$, which proves (4.1). Assertion (4.2) immediately follows from (4.1), (3.3) and (4.4). ∎

Corollary 4.2. *Suppose that $\langle M\rangle_\infty = +\infty$ P-a.s. and that b is as in the formulation of Theorem* 4.1. *Let $x_0 \in E_\sigma^{(0)}$, the interior of E_σ, and denote the component of $E_\sigma^{(0)}$ containing x_0 by (c, d). If there exists a solution (X, F) of equation* (1.1) *starting from x_0, then condition* (4.2) *is satisfied.*

Proof. The assertion follows from Theorem 4.1 and the inclusion $E_\sigma \subseteq E_\mu$. ∎

Corollary 4.3. *Suppose that $\langle M\rangle_\infty = +\infty$ P-a.s. and that b is as in the formulation of Theorem* 4.1. *Let $x_0 \in E_\nu^{(0)}$, the interior of E_ν, and denote the component of $E_\nu^{(0)}$ containing x_0 by (c, d). If there exists a solution (X, F) of equation* (1.1) *starting from x_0, then condition* (4.2) *is satisfied.*

Proof. We recall the definitions of ν and E_ν from the beginning of Section 2. The assertion now follows from the inclusion $E_\nu \subseteq E_\mu$. ∎

In the next corollary, F_b denotes the set E_ν for $\sigma \equiv 1$. In other words, F_b consists of all $x \in \mathbf{R}$ such that b is not integrable in an arbitrary neighbourhood of x.

Corollary 4.4. *Suppose that $\langle M\rangle_\infty = +\infty$ P-a.s. and that b is as in the formulation of Theorem* 4.1. *Additionally, we assume that σ is locally bounded. Let $x_0 \in F_b^{(0)}$, the interior of F_b, and denote the component of $F_b^{(0)}$ containing x_0 by (c, d). If there exists a solution (X, F) of equation* (1.1) *starting from x_0, then condition* (4.2) *is satisfied.*

Proof. Since σ is locally bounded, $F_b \subseteq E_\nu$ and the assertion follows from Corollary 4.3. ∎

5. Non-existence of solutions. For simplicity, from now on we assume that $\langle M\rangle_\infty = +\infty$ P-a.s. Moreover, the drift function b is always supposed to be non-negative and measurable and such that b^{-1} is locally integrable and $N_b = \emptyset$. From Theorem 4.1 we get the following result on non-existence of solutions.

Theorem 5.1. *Suppose that σ is a measurable function, $x_0 \in \mathbf{R}$, and (c, d) is an interval such that the following conditions are satisfied:*

$$x_0 \in (c, d) \subseteq E_\mu, \tag{5.1}$$

$$\lambda(N_\sigma^c \cap [x_0, d)) > 0. \tag{5.2}$$

Then there does not exist a solution (X, F) of equation (1.1) *starting from x_0.*

Now we state

Theorem 5.2. *Let σ be a measurable function such that the following conditions are satisfied:*

$$E_\mu = \mathbf{R}, \tag{5.3}$$

$$\lambda(N_\sigma^c \cap [n, +\infty)) > 0 \quad \textit{for all } n \geqslant 1. \tag{5.4}$$

Then, for every starting point $x_0 \in \mathbf{R}$, *there does not exist a solution* $(X, \mathbf{F})$ *of equation* (1.1).

Proof. The conditions of Theorem 5.2 imply those of Theorem 5.1 for every $x_0 \in \mathbf{R}$. ∎

COROLLARY 5.3. *Suppose that* σ *is a measurable function such that* (5.4) *and*

$$E_\sigma = \mathbf{R} \tag{5.5}$$

are satisfied. Then, for every $x_0 \in \mathbf{R}$, *there does not exist a solution* $(X, \mathbf{F})$ *of equation* (1.1) *starting from* x_0.

Proof. This follows from $E_\sigma \subseteq E_\mu$ and Theorem 5.2. ∎

COROLLARY 5.4. *Suppose that the condition*

$$F_b = \mathbf{R} \tag{5.6}$$

is satisfied. Then, for every locally bounded measurable function σ, *for which condition* (5.4) *holds, and for every* $x_0 \in \mathbf{R}$, *there does not exist a solution* $(X, \mathbf{F})$ *of equation* (1.1) *starting from* x_0.

Proof. We recall that F_b consists of all $x \in \mathbf{R}$ such that b is not integrable in an arbitrary neighbourhood of x. If σ is locally bounded, $F_b \subseteq E_\nu \subseteq E_\mu$, and the assertion follows from Theorem 5.2. ∎

Remark 5.5. (i) Under the assumptions of Theorem 5.1, Theorem 5.2, Corollary 5.3 and Corollary 5.4, respectively, there also do not exist *local* solutions.

(ii) Condition (5.3) is equivalent to $\lambda(\mathbf{R}\backslash E_\mu) = 0$. Analogously, (5.4) and (5.5) are equivalent to $\lambda(\mathbf{R}\backslash E_\sigma) = 0$ and $\lambda(\mathbf{R}\backslash F_b) = 0$, respectively.

(iii) If b is continuous and $N_b = \emptyset$, then all conditions on b are satisfied. Consequently, if σ fulfils (5.4) and (5.5), then there does not exist a solution $(X, \mathbf{F})$ of equation (1.1) whatever the starting point x_0 and the continuous drift b without zeros are. This rejects the following conjecture of Rutkowski [10] which was the starting point of the present paper:

If b *is a bounded and continuous function with* $N_b = \emptyset$ *and* σ *is an arbitrary Borel measurable function, then there exists a solution of equation* (1.1) *for every* $x_0 \in \mathbf{R}$.

We now give two examples which show that, indeed, there exist functions σ satisfying the conditions (5.4) and (5.5) and functions b such that b^{-1} is locally integrable, $N_b = \emptyset$, and $F_b = \mathbf{R}$.

EXAMPLE 5.6. Let $\{r_1, r_2, \ldots\}$ be an enumeration of the non-negative rational numbers. We define

$$\rho(x) = \sum_{n=1}^{\infty} 2^{-n} \big||x| - r_n\big|^{-1/2}, \qquad x \in \mathbf{R}.$$

It can easily be verified that ρ is locally integrable. In particular, this implies

(5.7) $$\rho < +\infty \quad \lambda\text{-a.e.}$$

Now we set $\sigma(x) = \rho^{-1}(x)$, $x \in \mathbf{R}$, where $+\infty^{-1} = 0$. In view of (5.7), we obtain $\lambda(N_\sigma) = 0$, and hence condition (5.4). On the other side, for every $\varepsilon > 0$ we compute

$$\int_{a-\varepsilon}^{a+\varepsilon} \sigma^{-2}(x)\,dx = \int_{a-\varepsilon}^{a+\varepsilon} \rho^2(x)\,dx \geqslant \int_{a-\varepsilon}^{a+\varepsilon} 2^{-n}\big||x|-r_n\big|^{-1}\,dx = +\infty,$$

where n is chosen such that $r_n \in (|a|-\varepsilon, |a|+\varepsilon)$. This yields $a \in E_\sigma$, and hence $E_\sigma = \mathbf{R}$. Thus σ verifies (5.4) and (5.5).

EXAMPLE 5.7. Let ρ be defined as is Example 5.6. We now set

$$b = \mathbb{1}_{\{\rho < +\infty\}}\rho^2 + \mathbb{1}_{\{\rho = +\infty\}}.$$

We then have $b = \rho^2$ λ-a.e., and as above we see that $F_b = \mathbf{R}$. Obviously, $b(x) > 0$ for all $x \in \mathbf{R}$. We may assume $r_1 = 0$. Then

$$b^{-1}(x) = \Big(\sum_{n=1}^{\infty} 2^{-n}\big||x|-r_n\big|^{-1/2}\Big)^{-2} \leqslant 4|x| \quad \lambda\text{-a.e.}$$

This shows that b^{-1} is locally integrable. Hence b satisfies the conditions of Corollary 5.4 and, for every locally bounded measurable function σ such that (5.4) holds and for every $x_0 \in \mathbf{R}$, there does not exist a solution $(X, \mathbf{F})$ of equation (1.1) starting from x_0.

The next result shows that the conclusion of Example 5.7 (or, more generally, the conclusions of Theorem 5.2, Corollary 5.3 and Corollary 5.4) is not true if σ does not satisfy (5.4).

PROPOSITION 5.8. *Suppose that $n \geqslant 1$ and the diffusion coefficient σ are such that*

(5.8) $$\lambda\big(N_\sigma^c \cap [n, +\infty)\big) = 0$$

holds. Then, for all $x_0 \geqslant n$, there exists a solution $(X, \mathbf{F})$ of equation (1.1) starting from x_0. This solution is pathwise unique if $x_0 > n$.

Proof. Let $x_0 \geqslant n$ and consider the (pathwise unique) solution Z of equation (3.1) starting from x_0. Then Z is also a solution of equation (1.1). Indeed, for this it is sufficient to verify that

$$\int_0^t \sigma(Z_s)\,dM_s = 0, \qquad t < S_\infty(Z), \ P\text{-a.s.}$$

or, equivalently,

$$\int_0^t \sigma^2(Z_s)\,d\langle M\rangle_s = 0, \qquad t < S_\infty(Z), \ P\text{-a.s.}$$

Using Proposition 3.3 we obtain

$$\int_0^t \sigma^2(Z_s)\,d\langle M\rangle_s = \int_{\mathbf{R}} \sigma^2(a)\,\mathbf{1}_{[x_0,Z_t]}(a)\,b^{-1}(a)\,da = 0$$

on $\{t < S_\infty(Z)\}$ P-a.s. Let now $x_0 > n$ and (X, F) be an arbitrary solution of equation (1.1) starting from x_0. First we notice that (5.8) implies $(n, +\infty) \subseteq E_\sigma \subseteq E_\mu$. As in the proof of Theorem 4.1, we can show that $X_{t\wedge D} = Z_{t\wedge D}$, $t \geqslant 0$, P-a.s. Here Z is as above and D is the first exit time of X from $(n, +\infty)$. Since Z is non-decreasing, we observe that $D = S_\infty(Z) = S_\infty(X)$ and, consequently, $X = Z$. But Z is pathwise unique, and the assertion follows. ∎

Finally, we give the following concluding remarks:

Remark 5.9. (i) If $x_0 \in E_\mu^c$, then there always exists a local solution (X, F) of equation (1.1) up to D_{E_μ} starting from x_0. Here D_{E_μ} denotes the first entry time of X into E_μ. To see this, let (c, d) be the component of E_μ^c containing x_0 and set $\tilde{b}$ and $\tilde{\sigma}$ equal to b and σ on (c, d), respectively, and equal to zero otherwise. By Theorem 1.1 (which also holds for arbitrary M) there exists a solution (X, F) of equation (1.1) with coefficients $\tilde{b}$ and $\tilde{\sigma}$ starting from x_0. This is the desired local solution with coefficients b and σ.

(ii) If $x_0 \in E_\mu^{(0)}$, the interior of E_μ, and (c, d) denotes the component of $E_\mu^{(0)}$ containing x_0, then the necessary condition (4.2) or, equivalently, $\lambda(N_\sigma^c \cap [x_0, d)) = 0$ is also sufficient for the existence of a local solution (X, F) of equation (1.1) up to $D_{E_\mu^c}$ starting from x_0 (cf. the proof of Proposition 5.8).

(iii) The problem under which conditions there exists a local solution of equation (1.1) starting from $x_0 \in \partial E_\mu$ remains open. In the special case when E_μ is a denumerable set of isolated points (hence $\partial E_\mu = E_\mu$) this problem is completely solved in [1]. If this problem could also be solved in the general situation, then, pasting together the local solutions, we obtain a global solution and, perhaps, necessary and sufficient conditions for this. However, the problem seems to be not quite simple if the boundary ∂E_μ of E_μ has accumulating points or even is of strictly positive Lebesgue measure.

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