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MAX-SEMISTABLE HEMIGROUPS: STRUCTURE, DOMAINS OF ATTRACTION AND LIMIT THEOREMS WITH RANDOM SAMPLE SIZE

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Abstract. Let (X_n) be a sequence of independent real valued random variables. A suitable convergence condition for affine normalized maxima of (X_n) is given in the semistable setup, i.e. for increasing sampling sequences (k_n) such that $k_{n+1}/k_n \to c > 1$, which enables us to obtain a hemigroup structure in the limit. We show that such hemigroups are closely related to max-semiselfdecomposable laws and that the norming sequences of the convergence condition can be chosen such that the limiting behaviour for arbitrary sampling sequences can be fully analysed. This in turn enables us to obtain randomized limits as follows. Suppose that (T_n) is a sequence of positive integer valued random variables such that T_n/k_n or T_n/n converges in probability to some positive random variable D, where we do not assume (X_n) and (T_n) to be independent. Then weak limit theorems of randomized extremes, where the sampling sequence (k_n) is replaced by random sample sizes (T_n), are presented. The proof follows corresponding results on the central limit theorem, containing the verification of an Anscombe condition.

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1. INTRODUCTION

Let $X_1, X_2, \ldots$ be independent, not necessarily identically distributed real valued random variables and $M_n = \max(X_1, \ldots, X_n)$. Assume that for some increasing sequence (k_n) of natural numbers and norming constants $a_n > 0$ and $b_n \in \mathbf{R}$ the affine normalized maxima $a_n^{-1}(M_{k_n} - b_n)$ converge in distribution to some nondegenerate limit. As argued in [9], the most general framework, in which satisfying results for weak convergence can be obtained, is to assume

$k_{n+1}/k_n \to c \geqslant 1$ for the growth of the sampling sequence. If additionally the sequence (X_n) of random variables is identically distributed, the cases $c = 1$ and $c > 1$ refer to max-stable and max-semistable limits, respectively; see e.g. [4] and [5]. For non-identically distributed random variables and $k_n = n$ the limit is max-selfdecomposable; see [9]. Moreover, in [16] a functional limit law

$$P\{M_{\lfloor nt \rfloor} \leqslant a_n x + b_n\} \to G_t(x) \tag{1.1}$$

for all $t > 0$, where G_t is a nondegenerate probability distribution function (pdf) and convergence holds for all points of continuity x of G_t, is assumed to obtain certain selfsimilar extremal processes in the limit. Hüsler [6] has obtained similar results for multivariate extremes by assuming additionally an infinitesimality condition to avoid cases where the maxima are dominated only by some of the random variables. Up to now, neither the max-semistable case nor the max-semiselfdecomposable one for non-identically distributed random variables has been considered in the literature. In the next section we present a suitable convergence condition, which naturally extends (1.1) to semistable situations, i.e. to situations where limits of normalized maxima can only be obtained along sampling sequences (k_n) with the growth condition $k_{n+1}/k_n \to c > 1$. We show that the structure of the limits fits into a hemigroup setting with a close connection to max-semiselfdecomposability. We restrict our considerations to one-dimensional extremes. Moreover, the norming sequences fulfil an embedding property that enables us to determine the limiting behaviour for arbitrary sampling sequences as well as for random sample sizes. The latter is understood as follows.

Let (T_n) be a sequence of positive integer valued random variables defined on the same probability space as (X_n) and consider the randomized maxima $M_{T_n} = \max(X_1, \ldots, X_{T_n})$, where we do not assume (X_n) and (T_n) to be independent. Under the condition $T_n/n \to D$ in probability, where $D > 0$ is an arbitrary positive random variable with distribution ϱ, randomized limit theorems were obtained for domains of attraction of max-stable distributions; see [2], [12] or [4]. In particular, if (X_n) is an i.i.d. sequence and

$$P\{M_n \leqslant a_n x + b_n\} \to G(x)$$

for all points of continuity x of a nondegenerate pdf G, then G is max-stable and under the above-mentioned assumptions on the random sample sizes (T_n) we have

$$P\{M_{T_n} \leqslant a_n x + b_n\} \to \int_0^\infty G^t(x)\, d\varrho(t), \tag{1.2}$$

where here and in what follows, according to weak topology, convergence of pdf's is meant pointwise for every point of continuity of the limit. For more general results on extremal processes of triangular arrays of rowwise i.i.d. variables we refer to [15]. A comparison of our methods and results to those of [15] is given in Section 4.

We will generalize (1.2) to non-identically distributed random variables and semistable situations. These randomized limit theorems are presented in Section 3 where, unlike the proof of (1.2) in [2] or [4], we follow the ideas of corresponding results on randomized limits for normalized sums of independent random variables. A randomized central limit theorem has been proved in three consecutive steps: first for random sample sizes $T_n = \lfloor nD \rfloor$ with discrete random variable $D > 0$, and next for $T_n/n \to D$ in probability, again for discrete $D > 0$, both by Rényi [10]; the last step for arbitrary $D > 0$ was shown independently in [3] and [8]. Whereas in the first step a mixing property suffices, the last two steps require an Anscombe condition (see [1]) to be fulfilled. This condition is of independent interest and shall motivate our method of proof. Note that if $(Y_t)_{t>0}$ is a nondegenerate max-stable extremal process independent of D with corresponding pdf's $(G^t)_{t>0}$, then the limit in (1.2) is the pdf of Y_D. This asymptotic independence is a consequence of the mixing property we will derive for the given more general situation in Proposition 3.1.

2. SEMISTABLE DOMAIN OF ATTRACTION

As before, let (X_n) be a sequence of independent, not necessarily identically distributed real valued random variables and for all $0 \leqslant s < t$ and $n \in N$ define

$$M_n^{s,t} = \begin{cases} \max(X_{\lfloor ns \rfloor+1}, \ldots, X_{\lfloor nt \rfloor}) & \text{if } \lfloor nt \rfloor > \lfloor ns \rfloor, \\ 0 & \text{elsewhere.} \end{cases}$$

Moreover, let $F_n^{s,t}$ be the pdf of $M_n^{s,t}$. Suppose that for some increasing sequence (k_n) of natural numbers with $k_{n+1}/k_n \to c > 1$ and norming constants $a_n > 0$ and $b_n \in R$ we have

$$(2.1) \qquad P\{a_n^{-1}(M_{k_n}^{s_n,t_n} - b_n) \leqslant x\} = F_{k_n}^{s_n,t_n}(a_n x + b_n) \to G_{s,t}(x)$$

for all sequences $0 \leqslant s_n < t_n$ with $s_n \to s$, $t_n \to t$, $s < t$, and some nondegenerate pdf $G_{s,t}$.

PROPOSITION 2.1. *The limits $G_{s,t}$ in (2.1) continuously depend on the parameters $0 \leqslant s < t$ with respect to weak topology.*

Proof. Let $0 \leqslant s_m < t_m$ with $s_m \to s$, $t_m \to t$, $s < t$ and $\varepsilon > 0$ be arbitrary. Let $x \in R$ be a point of continuity of $G_{s,t}$ and choose $\delta > 0$ such that $x+\delta$ is also a point of continuity and $G_{s,t}(x+\delta) - G_{s,t}(x) < \varepsilon/6$. Choose $0 \leqslant \delta_m < \delta$ such that $x+\delta_m$ is a point of continuity of G_{s_m,t_m} and $G_{s_m,t_m}(x+\delta_m) - G_{s_m,t_m}(x) < \varepsilon/3$. Since

$$F_{k_n}^{s_m,t_m}(a_n(x+\delta_m)+b_n) \to G_{s_m,t_m}(x+\delta_m) \quad \text{as } n \to \infty,$$

choose $n_m \in N$ such that for all $n \geqslant n_m$ we have

$$|F_{k_n}^{s_m,t_m}(a_n(x+\delta_m)+b_n) - G_{s_m,t_m}(x+\delta_m)| < \varepsilon/3.$$

Without loss of generality we can choose $m \mapsto n_m$ to be strictly increasing. Then we have

$$\limsup_{m\to\infty} F_{k_{n_m}}^{s_m,t_m}(a_{n_m}(x+\delta_m)+b_{n_m}) \leqslant G_{s,t}(x+\delta)$$

and

$$\liminf_{m\to\infty} F_{k_{n_m}}^{s_m,t_m}(a_{n_m}(x+\delta_m)+b_{n_m}) \geqslant G_{s,t}(x)$$

such that for all sufficiently large $m \in N$ we get

$$\left|F_{k_{n_m}}^{s_m,t_m}(a_{n_m}(x+\delta_m)+b_{n_m})-G_{s,t}(x)\right| < \varepsilon/3.$$

Hence for sufficiently large $m \in N$ we obtain

$$\begin{aligned}|G_{s_m,t_m}(x)-G_{s,t}(x)| &\leqslant |G_{s_m,t_m}(x)-G_{s_m,t_m}(x+\delta_m)| \\ &\quad +\left|G_{s_m,t_m}(x+\delta_m)-F_{k_{n_m}}^{s_m,t_m}(a_{n_m}(x+\delta_m)+b_{n_m})\right| \\ &\quad +\left|F_{k_{n_m}}^{s_m,t_m}(a_{n_m}(x+\delta_m)+b_{n_m})-G_{s,t}(x)\right| < \varepsilon,\end{aligned}$$

which proves the continuity of $G_{s,t}$ on the set of parameters. ∎

In view of Proposition 2.1, equivalently to (2.1) one may say that the affine normalized maxima $a_n^{-1}(M_{k_n}^{s,t}-b_n)$ converge in distribution to some nondegenerate limit uniformly on compact subsets of $\{0 \leqslant s < t\}$. The following remarks relate this assumption on uniformly compact convergence to well-known convergence conditions in special situations.

Remark 2.2. To compare (2.1) with the functional limit law (1.1) let us exceptionally assume $k_n = n$ in contrast to the growth condition on the sampling sequence. According to Weissman [17] the norming constants in (1.1) can be chosen such that for some $\alpha \in R$ and all $t > 0$ we have

$$\text{(2.2)} \qquad \frac{a_{\lfloor nt\rfloor}}{a_n} \to t^\alpha \quad \text{and} \quad \frac{b_{\lfloor nt\rfloor}-b_n}{a_n} \to \begin{cases}\log t & \text{if } \alpha = 0,\\ 0 & \text{if } \alpha \neq 0.\end{cases}$$

Especially, the sequence (a_n) can be chosen to vary regularly, and thus the first convergence holds uniformly on compact subsets of $\{t > 0\}$ according to standard regular variation techniques; see e.g. [14]. The same techniques can be applied to see that also the latter convergence holds uniformly on compact subsets of $\{t > 0\}$. In view of (1.1) we obtain for every sequence $t_n \to t > 0$

$$\begin{aligned}P\{a_n^{-1}(M_{\lfloor nt_n\rfloor}-b_n) \leqslant x\} &= P\left\{a_{\lfloor nt_n\rfloor}^{-1}(M_{\lfloor nt_n\rfloor}-b_{\lfloor nt_n\rfloor}) \leqslant \frac{a_n}{a_{\lfloor nt_n\rfloor}}\left(x-\frac{b_{\lfloor nt_n\rfloor}-b_n}{a_n}\right)\right\} \\ &\to G_{0,t}(x) = \begin{cases}G_1(t^{-\alpha}x) & \text{if } \alpha \neq 0,\\ G_1(x-\log t) & \text{if } \alpha = 0.\end{cases}\end{aligned}$$

As shown in [16] this is sufficient for (2.1) to be fulfilled with $G_{s,t} = G_{0,t}/G_{0,s}$, where $G_{s,t}(x)$ is defined to be zero when $G_{0,t}(x) = 0$ even if $G_{0,s}(x) = 0$. In the sequel we will derive similar results on the norming sequences as above, but we need uniformly compact convergence to achieve the results in a more general semistable setup.

Remark 2.3. Suppose (X_n) is additionally identically distributed with common pdf F. Then (2.1) is already fulfilled if F belongs to the domain of attraction of some nondegenerate max-semistable pdf G, i.e.

$$P\{a_n^{-1}(M_{k_n} - b_n) \leqslant x\} = F^{k_n}(a_n x + b_n) \to G(x).$$

In particular, for all sequences $0 \leqslant s_n < t_n$ with $s_n \to s$, $t_n \to t$ and $s < t$ we have

$$F_{k_n}^{s_n,t_n}(a_n x + b_n) = \big(F^{k_n}(a_n x + b_n)\big)^{(\lfloor k_n t_n \rfloor - \lfloor k_n s_n \rfloor)/k_n} \to G^{t-s}(x).$$

Especially we obtain $G_{s,t} = G^{t-s}$ for all $0 \leqslant s < t$. This shows that our results can be applied to the special case of identically distributed random variables. For a characterization of the domain of attraction we refer to [7].

We now turn to the structure of the limits, which is determined by semistability and closely related to max-semiselfdecomposable distributions.

PROPOSITION 2.4. *Under the above-given assumptions we have*

$$G_{s,t} = G_{s,r} \cdot G_{r,t} \quad \textit{and} \quad G_{s,t}(x) = G_{cs,ct}(c^{\alpha} x + \beta)$$

for all $0 \leqslant s < r < t$ *and* $x \in \mathbf{R}$, *where* $\alpha, \beta \in \mathbf{R}$ *are determined by*

$$\frac{a_{n+1}}{a_n} \to c^{\alpha} \quad \textit{and} \quad \frac{b_{n+1} - b_n}{a_n} \to \beta.$$

Proof. Since $F_{k_n}^{s,t} = F_{k_n}^{s,r} \cdot F_{k_n}^{r,t}$ for all $0 \leqslant s < r < t$ and sufficiently large $n \in \mathbf{N}$, the first assertion follows by passing to limits. Furthermore, we have

$$F_{k_{n+1}}^{s,t}(a_{n+1} x + b_{n+1}) \to G_{s,t}(x),$$

and on the other hand

$$F_{k_{n+1}}^{s,t}(a_n x + b_n) = F_{k_n}^{(k_{n+1}/k_n)s, (k_{n+1}/k_n)t}(a_n x + b_n) \to G_{cs,ct}(x).$$

Hence the remaining assertions follow by the convergence of types theorem; see e.g. Proposition 0.1 in [11]. ∎

PROPOSITION 2.5. *Under the assumptions of Proposition* 2.4 *the norming constants* (a_n) *and* (b_n) *can be chosen such that either* $\alpha = 0$ *and* $\beta = \log c$ *or* $\alpha \neq 0$ *and* $\beta = 0$.

Proof. Let us consider first the case $\alpha = 0$. Suppose $\beta < 0$. Then we have $G_{0,c}(x_0 + \beta) < G_{0,c}(x_0)$ for some $x_0 \in \mathbf{R}$, which leads to a contradiction since

$$G_{0,1}(x_0) = G_{0,c}(x_0 + \beta) < G_{0,c}(x_0) = G_{0,1}(x_0) \cdot G_{1,c}(x_0) \leqslant G_{0,1}(x_0).$$

Suppose $\beta = 0$. Then we have $G_{0,1} = G_{0,c} = G_{0,1} \cdot G_{1,c}$. Choose a point of continuity x_0 of $G_{0,1}$ such that $0 < G_{0,1}(x_0) < 1$. Hence for all $m \in N$ we obtain $1 = G_{1,c}(x_0) = G_{c^{-1},1}(x_0) = G_{c^{-m},c^{-m+1}}(x_0)$, which implies

$$G_{0,1}(x_0) = \lim_{m\to\infty} G_{c^{-m},1}(x_0) = \lim_{m\to\infty} G_{c^{-m},c^{-m+1}}(x_0)\ldots G_{c^{-1},1}(x_0) = 1$$

and contradicts $G_{0,1}(x_0) < 1$. Hence β must be positive.

Now define $c_n = a_n \beta/\log c > 0$ and $H_{s,t}(x) = G_{s,t}(x\beta/\log c)$, which is the nondegenerate limiting pdf of

$$F_{k_n}^{s_n,t_n}(c_n x + b_n) = F_{k_n}^{s_n,t_n}(a_n x\beta/\log c + b_n)$$

for all sequences $0 \leqslant s_n < t_n$ with $s_n \to s$, $t_n \to t$, $s < t$. Moreover, the new norming constants fulfil

$$\frac{c_{n+1}}{c_n} = \frac{a_{n+1}}{a_n} \to c^\alpha = 1 \quad \text{and} \quad \frac{b_{n+1}-b_n}{c_n} = \frac{\log c}{\beta}\frac{b_{n+1}-b_n}{a_n} \to \log c.$$

In the remaining case $\alpha \neq 0$, define $d_n = b_n + a_n \beta/(1-c^\alpha)$ and a nondegenerate limiting pdf $H_{s,t}(x) = G_{s,t}(x+\beta/(1-c^\alpha))$ of $F_{k_n}^{s_n,t_n}(a_n x + d_n)$ for all sequences $0 \leqslant s_n < t_n$ with $s_n \to s$, $t_n \to t$, $s < t$. The new norming constants satisfy

$$\frac{d_{n+1}-d_n}{a_n} = \frac{b_{n+1}-b_n}{a_n} + \frac{a_{n+1}-a_n}{a_n}\frac{\beta}{1-c^\alpha} \to \beta\left(1+\frac{c^\alpha-1}{1-c^\alpha}\right) = 0.$$

This completes the proof. ∎

DEFINITION 2.6. In view of Propositions 2.1, 2.4 and 2.5 we call the set of pdf's $\mathscr{H} = \{G_{s,t} \mid 0 \leqslant s < t\}$ a continuous nondegenerate *max-semistable hemigroup* in analogy to continuous convolution hemigroups for summation schemes. Then the sequence (X_n) or the sequence of pdf's (F_n) is said to belong to the *domain of max-semistable attraction* of $\mathscr{H}$. Moreover, by extending the terminology given in [9], for all $t > 0$ the elements $G_{0,t}$ of $\mathscr{H}$ are *max-semiselfdecomposable* in the following sense:

$$G_{0,t}(x) = G_{0,t/c}(x) \cdot G_{t/c,t}(x) = G_{0,t}(c^\alpha x + \beta) \cdot G_{t/c,t}(x).$$

Conversely, every nondegenerate max-semiselfdecomposable pdf can be embedded into a continuous max-semistable hemigroup as we will show in the following

LEMMA 2.7. *Let G be a nondegenerate max-semiselfdecomposable pdf, i.e. for some $c > 1$, $\alpha \in R$ and a nondegenerate pdf H we have*

$$G(x) = G(c^\alpha x + \beta) \cdot H(x) \quad \textit{for all } x \in R,$$

where $\beta = 0$ if $\alpha \neq 0$ and $\beta = \log c$ if $\alpha = 0$. Without loss of generality we assume $H(x) = 0$ if $G(x) = 0$. Then clearly

$$G(x) = G(c^{n\alpha} x + n\beta) \cdot H_n(x)$$

for all $n \in N$ *and some nondegenerate pdf* H_n *with* $H_1 = H$ *and* $H_n(x) = 0$ *if* $G(x) = 0$. *The cofactors* (H_n) *fulfil* $H_n(x) \to G(x)$ *if* $\alpha \geqslant 0$ *or if* $\alpha < 0$ *and* G *has no point mass at* 0. *Moreover, the cofactors fulfil the cocycle equation*

$$H_{n+m}(x) = H_m(c^{n\alpha} x + n\beta) \cdot H_n(x) \quad \textit{for all } m,\ n \in N.$$

Proof. If $\alpha > 0$, we have

$$G(c^{n\alpha} x + n\beta) = G(c^{n\alpha} x) \to \begin{cases} 1 & \text{if } x > 0, \\ G(0) & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Obviously, $G(0) = G(0) \cdot H(0)$ such that either $G(0) = 0$ or $H(0) = 1$.

Case 1: $H(0) = 1$. Then $H(x) = 1$, and hence $G(x) = G(c^{\alpha} x)$ for all $x \geqslant 0$. Thus $G(x) = 1$ for all $x \geqslant 0$ holds true. Choose $x_0 < 0$ with $0 < G(x_0) < 1$. Then $G(x_0) = G(c^{n\alpha} x_0) \cdot H_n(x_0)$. Since $G(c^{n\alpha} x_0) \to 0$, we must have $H_n(x_0) \to \infty$, a contradiction.

Case 2: $G(0) = 0$. Since $G(c^{n\alpha} x) \to 1$ for all $x > 0$, we have $H_n(x) \to G(x)$ for all $x > 0$. Moreover, $H_n(x) = 0 = G(x)$ for all $x \leqslant 0$.

If $\alpha = 0$, then $G(c^{n\alpha} x + n\beta) = G(x + n \log c) \to 1$ for all $x \in R$ such that $H_n(x) \to G(x)$.

If $\alpha < 0$, then $G(c^{n\alpha} x + n\beta) = G(c^{-n|\alpha|} x) \to G(0)$. As in the case $\alpha > 0$, either $G(0) = 0$ or $H(0) = 1$ holds.

Case 1: $G(0) = 0$. Choose $x_0 > 0$ with $0 < G(x_0) < 1$. Then $G(x_0) = G(c^{n\alpha} x_0) \cdot H_n(x_0)$, and since $G(c^{n\alpha} x_0) \to G(0) = 0$, we must have $H_n(x_0) \to \infty$, a contradiction.

Case 2: $H(0) = 1$. Then $G(x) = 1$ for all $x \geqslant 0$ as above. Thus $H_n(x) \to G(x)$ for all $x \in R$ since $G(c^{n\alpha} x) \to G(0) = 1$ by continuity of G in $x = 0$.

Since on the one hand

$$G(x) = G\big(c^{(n+m)\alpha} x + (n+m)\beta\big) \cdot H_{n+m}(x),$$

and on the other hand

$$\begin{aligned} G(x) &= G(c^{n\alpha} x + n\beta) \cdot H_n(x) \\ &= G\big(c^{(n+m)\alpha} x + (n+m)\beta\big) \cdot H_m(c^{n\alpha} x + n\beta) \cdot H_n(x), \end{aligned}$$

we obtain the cocycle equation. ∎

THEOREM 2.8. *Every nondegenerate max-semiselfdecomposable pdf* G, *which has no point mass at* 0 *if* $\alpha < 0$, *can be embedded into a continuous nondegenerate max-semistable hemigroup* $\mathscr{H} = \{G_{s,t} \mid 0 \leqslant s < t\}$ *with* $G = G_{0,1}$.

Proof. In view of the cofactors (H_n) of Lemma 2.7 we define nondegenerate pdf's $G_{s,t}$ for all $0 \leqslant s < t$ as follows. For $t > 0$ write $t = c^{n_t} r_t$ with $n_t \in Z$

and $r_t \in [1, c)$, and define $L_t x = c^{-n_t\alpha} x - n_t \beta$. Then we have

$$L_t x = c^\alpha L_{ct} x + \beta = L_{ct}(c^\alpha x + \beta),$$

and we further define

$$G_{s,t}(x) = \begin{cases} H(L_{cs}x)^{\log_c(c/r_s)} H_{n_t-(n_s+1)}(L_t x) H(L_{ct}x)^{\log_c r_t}, & 0 < s < c^{n_s+1} \leqslant t, \\ H(L_{ct}x)^{\log_c(r_t/r_s)}, & 0 < s < t < c^{n_s+1}, \\ G(L_t x) H(L_{ct}x)^{\log_c r_t}, & 0 = s < t, \end{cases}$$

where we set $H_0 \equiv 1$. It is easy to verify that $G_{0,1} = G$.

1. We show first the hemigroup property: $G_{s,v} \cdot G_{v,t} = G_{s,t}$.

If $0 = s < v < t < c^{n_v+1}$, we have $n_v = n_t$, and hence

$$G_{s,v}(x) G_{v,t}(x) = G(L_v x) H(L_{cv}x)^{\log_c r_t} = G(L_t x) H(L_{ct}x)^{\log_c r_t} = G_{s,t}(x).$$

If $0 = s < v < c^{n_v+1} \leqslant t$, by Lemma 2.7 we have

$$\begin{aligned} G_{s,v}(x) G_{v,t}(x) &= G(L_v x) H(L_{cv}x) H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= G(L_{cv}x) H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= G\big(c^{(n_t-(n_v+1))\alpha} L_t x + (n_t - (n_v+1))\beta\big) H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= G(L_t x) H(L_{ct}x)^{\log_c r_t} = G_{s,t}(x). \end{aligned}$$

If $0 < s < v < t < c^{n_s+1}$, we have $n_s = n_v = n_t$, and hence

$$G_{s,v}(x) G_{v,t}(x) = H(L_{cv}x)^{\log_c(r_v/r_s)} H(L_{ct}x)^{\log_c(r_t/r_s)} = H(L_{ct}x)^{\log_c(r_t/r_s)} = G_{s,t}(x).$$

The cases $0 < s < v < c^{n_s+1} \leqslant t$ and $0 < s < c^{n_s+1} \leqslant v < t < c^{n_v+1}$ can be treated similarly. We prove the remaining case $0 < s < c^{n_s+1} \leqslant v < c^{n_v+1} \leqslant t$ by applying the cocycle equation of Lemma 2.7:

$$\begin{aligned} &G_{s,v}(x) G_{v,t}(x) \\ &= H(L_{cs}x)^{\log_c(c/r_s)} H_{n_v-(n_s+1)}(L_v x) H(L_{cv}x)^{\log_c r_v} \\ &\quad \times H(L_{cv}x)^{\log_c(c/r_v)} H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= H(L_{cs}x)^{\log_c(c/r_s)} H_{n_v-(n_s+1)}(L_v x) H(L_{cv}x) H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= H(L_{cs}x)^{\log_c(c/r_s)} H_{(n_v+1)-(n_s+1)}(L_{cv}x) H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= H(L_{cs}x)^{\log_c(c/r_s)} H_{(n_v+1)-(n_s+1)}\big(c^{n_t-(n_v+1)} L_t x + (n_t-(n_v+1))\beta\big) \\ &\quad \times H_{n_t-(n_v+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} \\ &= H(L_{cs}x)^{\log_c(c/r_s)} H_{n_t-(n_s+1)}(L_t x) H(L_{ct}x)^{\log_c r_t} = G_{s,t}(x). \end{aligned}$$

2. Since $ct = c^{n_t+1} r_t$ for all $t > 0$ and $L_t x = L_{ct}(c^\alpha x + \beta)$, we obtain the max-semistability $G_{cs,ct}(c^\alpha x + \beta) = G_{s,t}(x)$ for all $0 \leqslant s < t$.

3. For continuity of the hemigroup we first consider $r_s \to c$ and $r_t \to c$. If $0 = s < t$ and $r_t \to c$, we have

$$G_{0,t}(x) \to G(L_t x) H(L_{ct} x) = G(L_{ct} x) = G(L_{ct} x) H(L_{c^2 t} x)^{\log_c 1} = G_{0,c^{n_t+1}}.$$

If $0 < s < t < c^{n_s+1}$ and $r_t \to c$, we have $n_s = n_t$ and

$$G_{s,t}(x) \to H(L_{ct} x)^{\log_c(c/r_s)}$$
$$= H(L_{cs} x)^{\log_c(c/r_s)} H_{(n_t+1)-(n_s+1)}(L_{ct} x) H(L_{c^2 t} x)^{\log_c 1} = G_{s,c^{n_t+1}}.$$

All other cases can be proved similarly. Finally let us consider continuity if $s \to 0$. Then we have $n_s \to -\infty$, and thus

$$L_{cs} x \to \begin{cases} \infty & \text{if } \alpha = 0 \text{ or } \alpha > 0 \text{ and } x > 0, \\ 0 & \text{if } \alpha < 0 \text{ or } \alpha > 0 \text{ and } x = 0, \\ -\infty & \text{if } \alpha > 0 \text{ and } x < 0. \end{cases}$$

We consider these cases for

$$G_{s,t}(x) = H(L_{cs} x)^{\log_c(c/r_s)} H_{n_t-(n_s+1)}(L_t x) H(L_{ct} x)^{\log_c r_t}.$$

From Lemma 2.7 we obtain $H_{n_t-(n_s+1)}(L_t x) \to G(L_t x)$.

Case 1: $L_{cs} x \to \infty$. Then we have

$$G_{s,t}(x) \to G(L_t x) H(L_{ct} x)^{\log_c r_t} = G_{0,t}(x).$$

Case 2: $L_{cs} x \to -\infty$. Then $\alpha > 0$ and $x < 0$, in which case the proof of Lemma 2.7 implies $G(L_t x) = 0$ since $L_t x < 0$. Thus

$$G_{s,t}(x) \to 0 = G(L_t x) H(L_{ct} x)^{\log_c r_t} = G_{0,t}(x).$$

Case 3: $L_{cs} x \to 0$. If $\alpha < 0$, we have $H(0) = 1$ as in the proof of Lemma 2.7. Thus $G_{s,t}(x) \to G_{0,t}(x)$ as in the case 1.

If $\alpha > 0$ and $x = 0$, we have $G(0) = 0$ as in the proof of Lemma 2.7. Thus $G_{s,t}(0) \to 0 = G_{0,t}(0)$ as in the case 2. ∎

We now prove some results on the norming sequences (a_n) and (b_n) in (2.1). Namely, these sequences can be chosen such that they can be embedded into sequences which have properties as in (2.2). The embedding sequences in turn lead us to fully understand the limiting behaviour; see also [7] for the i.i.d. case. The method of proof is similar to the one given for domains of attraction of semistable laws in the scheme of summation; see [13].

THEOREM 2.9. *Let the norming constants (a_n) and (b_n) be chosen according to Proposition 2.5. Then there exist embedding sequences (c_n) and (d_n) with $c_n > 0$, $c_{k_n} = a_n$, $d_{k_n} = b_n$ and*

$$\frac{c_{\lfloor \lambda n \rfloor}}{c_n} \to \lambda^\alpha, \quad \frac{d_{\lfloor \lambda n \rfloor} - d_n}{c_n} \to \begin{cases} \log \lambda & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha \neq 0, \end{cases}$$

uniformly on compact subsets of $\{\lambda > 0\}$. Especially, the sequence (a_n) can be embedded into a regularly varying sequence.

Proof. Write $n = r_n k_{p_n}$ with $p_n \in N$ and $k_{p_n} \leqslant n < k_{p_n+1}$. Thus (r_n) is relatively compact in $[1, c]$. Define

$$c_n = r_n^\alpha a_{p_n} \quad \text{and} \quad d_n = \begin{cases} b_{p_n} + c_n \log r_n & \text{if } \alpha = 0, \\ b_{p_n} & \text{if } \alpha \neq 0. \end{cases}$$

Obviously, we have $c_n > 0$, $c_{k_n} = a_n$ and $d_{k_n} = b_n$.

Let (λ_n) be a positive sequence with $\lambda_n \to \lambda \in [1, c]$.

Case 1: Suppose $p_{\lfloor \lambda_n n \rfloor} \leqslant p_n - 2$ along a subsequence (n'). Then we have

$$\lambda = \limsup_{n' \to \infty} \frac{\lfloor \lambda_n n \rfloor}{n} \leqslant \limsup_{n' \to \infty} \frac{r_{\lfloor \lambda_n n \rfloor} k_{p_n - 2}}{r_n k_{p_n}} \leqslant c \limsup_{n' \to \infty} \frac{k_{p_n - 2}}{k_{p_n}} = c^{-1}$$

in contradiction to $\lambda \geqslant 1$.

Case 2: Suppose $p_{\lfloor \lambda_n n \rfloor} = p_n - 1$ along a subsequence (n'). Since (r_n) is relatively compact, every subsequence of (n') contains a further subsequence (n'') with $r_n \to r \in [1, c]$ along (n''), and thus along (n'') we have

$$r_{\lfloor \lambda_n n \rfloor} = \frac{\lfloor \lambda_n n \rfloor k_{p_n}}{n k_{p_n - 1}} r_n \to \lambda c r \quad \text{and} \quad \frac{c_{\lfloor \lambda_n n \rfloor}}{c_n} = \frac{r^\alpha_{\lfloor \lambda_n n \rfloor} a_{p_n - 1}}{r_n^\alpha a_{p_n}} \to \frac{(\lambda c r)^\alpha}{r^\alpha c^\alpha} = \lambda^\alpha.$$

In the case $\alpha = 0$ we further obtain

$$\frac{d_{\lfloor \lambda_n n \rfloor} - d_n}{c_n} = \frac{b_{p_n - 1} - b_{p_n}}{a_{p_n}} + \frac{c_{\lfloor \lambda_n n \rfloor}}{c_n} \log r_{\lfloor \lambda_n n \rfloor} - \log r_n$$
$$\to -\log c + \log(\lambda c r) - \log r = \log \lambda$$

along the subsequence (n''), and in the case $\alpha \neq 0$ we have along (n'')

$$\frac{d_{\lfloor \lambda_n n \rfloor} - d_n}{c_n} = \frac{b_{p_n - 1} - b_{p_n}}{r_n^\alpha a_{p_n}} \to 0.$$

Since every subsequence of (n') contains a further subsequence (n'') with these properties, we get the asserted convergence results along the whole subsequence.

Case 3: The assumption of $p_{\lfloor \lambda_n n \rfloor} = p_n$, $p_{\lfloor \lambda_n n \rfloor} = p_n + 1$ or $p_{\lfloor \lambda_n n \rfloor} = p_n + 2$ along a subsequence (n') can be treated similarly to the case 2 to obtain the same convergence results.

Case 4: Suppose $p_{\lfloor \lambda_n n \rfloor} \geqslant p_n + 3$ along a subsequence (n'). Then we have

$$\lambda^{-1} = \limsup_{n' \to \infty} \frac{n}{\lfloor \lambda_n n \rfloor} \leqslant \limsup_{n' \to \infty} \frac{r_n k_{p_n}}{r_{\lfloor \lambda_n n \rfloor} k_{p_n + 3}} \leqslant c \limsup_{n' \to \infty} \frac{k_{p_n}}{k_{p_n + 3}} = c^{-2}$$

in contradiction to $\lambda \leqslant c$.

Now let (λ_n) be a positive sequence with $\lambda_n \to \lambda > c$. Write $\lambda = \gamma c^p$ with $\gamma \in [1, c)$ and $p \in N$ as well as $\lambda_n = \gamma_n c^p$ with $\gamma_n \to \gamma$. Then according to the above cases we have

$$\frac{c_{\lfloor\lambda_n n\rfloor}}{c_n}=\frac{c_{\lfloor\gamma_n c^{p_n}\rfloor}}{c_{\lfloor c^{p}n\rfloor}}\frac{c_{\lfloor c^{p}n\rfloor}}{c_{\lfloor c^{p-1}n\rfloor}}\cdots\frac{c_{\lfloor cn\rfloor}}{c_n}\to\gamma^{\alpha}c^{p\alpha}=\lambda^{\alpha}$$

and

$$\frac{d_{\lfloor\lambda_n n\rfloor}-d_n}{c_n}=\frac{d_{\lfloor\gamma_n c^{p_n}\rfloor}-d_{\lfloor c^{p_n}\rfloor}}{c_{\lfloor c^{p_n}\rfloor}}\frac{c_{\lfloor c^{p_n}\rfloor}}{c_n}+\ldots+\frac{d_{\lfloor cn\rfloor}-d_n}{c_n}$$

$$\to\begin{cases}\log\gamma+p\log c=\log\lambda & \text{if } \alpha=0,\\ 0 & \text{if } \alpha\neq 0.\end{cases}$$

Finally, let (λ_n) be a positive sequence with $\lambda_n\to\lambda\in(0,1)$. Thus $\lambda^{-1}>1$ and we have

$$\frac{c_{\lfloor\lambda_n n\rfloor}}{c_n}=\left(\frac{c_{\lfloor\lambda_n^{-1}\lambda_n n\rfloor}}{c_{\lfloor\lambda_n n\rfloor}}\right)^{-1}\to(\lambda^{-1})^{-\alpha}=\lambda^{\alpha}$$

and

$$\frac{d_{\lfloor\lambda_n n\rfloor}-d_n}{c_n}=-\frac{c_{\lfloor\lambda_n n\rfloor}}{c_n}\frac{d_{\lfloor\lambda_n^{-1}\lambda_n n\rfloor}-d_{\lfloor\lambda_n n\rfloor}}{c_{\lfloor\lambda_n n\rfloor}}\to\begin{cases}-\log(\lambda^{-1})=\log\lambda & \text{if } \alpha=0,\\ 0 & \text{if } \alpha\neq 0.\end{cases}$$

This proves the theorem. ■

THEOREM 2.10. *Under the conditions of Theorem 2.9 for all* $0\leqslant s_n<t_n$ *with* $s_n\to s$, $t_n\to t$, $s<t$, *the sequence* $\left(c_n^{-1}(M_n^{s_n,t_n}-d_n)\right)$ *is stochastically compact, i.e. the distributions are weakly relatively compact and all limit points are nondegenerate. Moreover, each pdf of the limit points belongs to*

$$\{x\mapsto G_{\lambda s,\lambda t}(\lambda^{\alpha}x+p_{\alpha}(\lambda))\mid\lambda\in[1,c)\},$$

where $p_{\alpha}(\lambda)=0$ *if* $\alpha\neq 0$ *and* $p_0(\lambda)=\log\lambda$. *In particular, if* (r_n) *is a positive sequence with* $r_n\to r>0$, *we have*

$$F^{s_n,t_n}_{\lfloor k_n r_n\rfloor}(c_{\lfloor k_n r_n\rfloor}x+d_{\lfloor k_n r_n\rfloor})\to G_{rs,rt}(r^{\alpha}x+p_{\alpha}(r)),\qquad F^{s_n,t_n}_{\lfloor k_n r_n\rfloor}(a_n x+b_n)\to G_{rs,rt}(x).$$

Proof. The last assertions follow by Theorem 2.9:

$$F^{s_n,t_n}_{\lfloor k_n r_n\rfloor}(c_{\lfloor k_n r_n\rfloor}x+d_{\lfloor k_n r_n\rfloor})$$

$$=F_{k_n}^{(\lfloor k_n r_n\rfloor/k_n)s_n,(\lfloor k_n r_n\rfloor/k_n)t_n}\left(a_n\left(\frac{c_{\lfloor k_n r_n\rfloor}}{c_{k_n}}x+\frac{d_{\lfloor k_n r_n\rfloor}-d_{k_n}}{c_{k_n}}\right)+b_n\right)$$

$$\to G_{rs,rt}(r^{\alpha}x+p_{\alpha}(r)),$$

and thus

$$F^{s_n,t_n}_{\lfloor k_n r_n\rfloor}(a_n x+b_n)=F_{k_n}^{(\lfloor k_n r_n\rfloor/k_n)s_n,(\lfloor k_n r_n\rfloor/k_n)t_n}(c_{k_n}x+b_{k_n})\to G_{rs,rt}(x).$$

To prove stochastic compactness, write again $n = r_n k_{p_n}$ with $p_n \in N$ and $k_{p_n} \leqslant n < k_{p_n+1}$. Thus (r_n) is relatively compact in $[1, c]$, and for every limit point $r \in [1, c]$ we get, by the previous arguments, along a subsequence (n')

$$P\{c_n^{-1}(M_n^{s_n,t_n} - d_n) \leqslant x\} = F^{s_n,t_n}_{\lfloor k_{p_n} r_n \rfloor}(c_{\lfloor k_{p_n} r_n \rfloor} x + d_{\lfloor k_{p_n} r_n \rfloor}) \to G_{rs,rt}(r^{\alpha} x + p_{\alpha}(r)).$$

For $r = 1$ and $r = c$ the limit points coincide according to Propositions 2.4 and 2.5. ∎

3. RANDOMIZED LIMITS

In this section we present randomization results for the domain of attraction of max-semistable hemigroups including the proofs. Theorem 3.6, being the most essential of these results, discusses the limit of the randomized sequence $(a_n^{-1}(M_{T_n}^{s,t} - b_n))$, where (T_n) is a sequence of positive integer valued random variables such that $T_n/k_n \to D$ in probability for some positive random variable D. Note that we do not assume (T_n) and (X_n) to be independent, neither require information on the dependence structure between the two sequences. As stated in the Introduction, the proof follows corresponding results on the central limit theorem. Therefore we give preparatory results relying on characteristics of mixing sequences of random variables developed by Rényi, as well as on Anscombe's condition.

PROPOSITION 3.1. *Under the conditions and with the notation of Theorem* 2.10 *for every* $\lambda > 0$ *and* $0 \leqslant s < t$ *the sequence* $(a_n^{-1}(M_{\lfloor k_n \lambda \rfloor}^{s,t} - b_n))$ *is a mixing sequence of random variables in the sense of Rényi, i.e. for any event* A *with positive probability we have*

$$P(a_n^{-1}(M_{\lfloor k_n \lambda \rfloor}^{s,t} - b_n) \leqslant x \mid A) \to G_{\lambda s, \lambda t}(x).$$

Proof. Let $A_n = \{M_{\lfloor k_n \lambda \rfloor}^{s,t} < a_n x + b_n\}$. Then by Lemma 6.2.1 of [4] it is sufficient to prove for every fixed $m \in N$

$$P(A_n \cap A_m) \to P(A_m) \cdot G_{\lambda s, \lambda t}(x). \tag{3.1}$$

Case 1: $s > 0$. Hence $\lfloor \lfloor k_n \lambda \rfloor s \rfloor > \lfloor \lfloor k_m \lambda \rfloor t \rfloor$ for sufficiently large $n \in N$, and thus (3.1) holds true since then A_n and A_m are independent events.

Case 2: $s = 0$. Since

$$P(A_n) = P\{M_{\lfloor k_n \lambda \rfloor}^{0,(\lfloor k_m \lambda \rfloor / \lfloor k_n \lambda \rfloor)t} < a_n x + b_n\} \cdot P\{M_{\lfloor k_n \lambda \rfloor}^{(\lfloor k_m \lambda \rfloor / \lfloor k_n \lambda \rfloor)t, t} < a_n x + b_n\}$$

and both $P(A_n)$ and the latter probability on the right-hand side converge to $G_{0,\lambda t}(x)$ for continuity points, we have

$$P\{M_{\lfloor k_n \lambda \rfloor}^{0,(\lfloor k_m \lambda \rfloor / \lfloor k_n \lambda \rfloor)t} < a_n x + b_n\} \to 1$$

if $G_{0,\lambda t}(x) > 0$. Thus

$$P(A_n \cap A_m) = P(\{M^{0,(\lfloor k_m\lambda\rfloor/\lfloor k_n\lambda\rfloor)t}_{\lfloor k_n\lambda\rfloor} < a_n x + b_n\} \cap A_m)$$
$$\times P\{M^{(\lfloor k_m\lambda\rfloor/\lfloor k_n\lambda\rfloor)t,t}_{\lfloor k_n\lambda\rfloor} < a_n x + b_n\} \to P(A_m)\cdot G_{0,\lambda t}(x).$$

Hence (3.1) is fulfilled. ∎

LEMMA 3.2. *Let (Y_n) and (Z_n) be sequences of positive integer valued random variables with $Y_n \to \infty$ and $Y_n/Z_n \to L$ in probability for some positive random variable L. Then under the conditions and with the notation of Theorems* 2.9 *and* 2.10 *we have*

$$\frac{c_{Y_n}}{c_{Z_n}} \to L^\alpha \quad \textit{and} \quad \frac{d_{Y_n}-d_{Z_n}}{c_{Z_n}} \to p_\alpha(L) \quad \textit{in probability.}$$

Proof. Every subsequence (n') contains a further subsequence (n'') such that the event $\Gamma = \{Y_n/Z_n \to L$ and $Y_n \to \infty$ along $(n'')\}$ has probability one. For $\omega \in \Gamma$ let us write $Y_n(\omega) = \lambda_n(\omega) Z_n(\omega) = \lfloor \lambda_n(\omega) Z_n(\omega) \rfloor$, and thus we obtain $\lambda_n(\omega) \to L(\omega)$ and $Z_n(\omega) \to \infty$ along (n''). Hence for all $\omega \in \Gamma$ it follows by Theorem 2.9 that

$$\frac{c_{Y_n(\omega)}}{c_{Z_n(\omega)}} \to L(\omega)^\alpha \quad \text{and} \quad \frac{d_{Y_n(\omega)}-d_{Z_n(\omega)}}{c_{Z_n(\omega)}} \to p_\alpha\big(L(\omega)\big)$$

along (n''). Since Γ has probability one, the assertions hold true. ∎

THEOREM 3.3. *Under the conditions and with the notation of Theorem* 2.10 *for all $0 \leqslant s < t$ the following Anscombe condition is fulfilled*:

$$\lim_{a\downarrow 0}\limsup_{n\to\infty} P\{\max_{|m-n|\leqslant an} |c_n^{-1}(M_m^{s,t}-M_n^{s,t})| > \varepsilon\} = 0 \quad \textit{for any } \varepsilon > 0.$$

Proof. Choose $a_1 \in (0, 1)$ with $(1+a_1)s < (1-a_1)t$ such that for all $0 < a \leqslant a_1$ and $|m-n| \leqslant an$ we have $M_n^{(1+a)s,(1-a)t} \leqslant M_m^{s,t} \leqslant M_n^{(1-a)s,(1+a)t}$, and thus we obtain

$$\text{(3.2)}\quad \limsup_{n\to\infty} P\{\max_{|m-n|\leqslant an} |c_n^{-1}(M_m^{s,t}-M_n^{s,t})| > \varepsilon\}$$
$$\leqslant \limsup_{n\to\infty} P\{c_n^{-1}\max(|M_n^{(1-a)s,(1+a)t}-M_n^{s,t}|,\ |M_n^{(1+a)s,(1-a)t}-M_n^{s,t}|) > \varepsilon\}$$
$$\leqslant \limsup_{n\to\infty} P\{c_n^{-1}(M_n^{(1-a)s,(1+a)t}-M_n^{s,t}) > \varepsilon\}$$
$$+\limsup_{n\to\infty} P\{c_n^{-1}(M_n^{s,t}-M_n^{(1+a)s,(1-a)t}) > \varepsilon\}$$
$$\leqslant \limsup_{n\to\infty} P\{c_n^{-1}\max(M_n^{(1-a)s,s},\ M_n^{t,(1+a)t}) > \varepsilon + c_n^{-1} M_n^{s,t}\}$$

$$+\limsup_{n\to\infty} P\{c_n^{-1}\max(M_n^{s,(1+a)s},\ M_n^{(1-a)t,t}) > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\}$$

$$\leqslant \limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)s,s} > \varepsilon + c_n^{-1} M_n^{s,t}\}$$

$$+\limsup_{n\to\infty} P\{c_n^{-1} M_n^{t,(1+a)t} > \varepsilon + c_n^{-1} M_n^{s,t}\}$$

$$+\limsup_{n\to\infty} P\{c_n^{-1} M_n^{s,(1+a)s} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\}$$

$$+\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)t,t} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\}$$

$$\leqslant 2\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)s,(1+a)s} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\}$$

$$+2\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)t,(1+a)t} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\}.$$

Let us now consider the first expression of the last inequality on the right-hand side of (3.2). Define probability measures $\mu_{\lambda(1-a)s,\lambda(1+a)s}$ and $\mu_{\lambda(1+a)s,\lambda(1-a)t}$ by

$$\mu_{\lambda(1-a)s,\lambda(1+a)s}(-\infty,\ x] = G_{\lambda(1-a)s,\lambda(1+a)s}(\lambda^\alpha x + p_\alpha(\lambda)),$$
$$\mu_{\lambda(1+a)s,\lambda(1-a)t}(-\infty,\ x] = G_{\lambda(1+a)s,\lambda(1-a)t}(\lambda^\alpha x + p_\alpha(\lambda)),$$

respectively. Hence it follows by Theorem 2.10 that the distributions of the random vectors $(c_n^{-1}(M_n^{(1-a)s,(1+a)s} - d_n),\ c_n^{-1}(M_n^{(1+a)s,(1-a)t} - d_n))$ are stochastically compact and every weak limit point is contained in

$$\{\mu_{\lambda(1-a)s,\lambda(1+a)s} \otimes \mu_{\lambda(1+a)s,\lambda(1-a)t} \mid \lambda \in [1,\ c]\}.$$

Consequently, by the portmanteau theorem we obtain

$$\text{(3.3)}\quad \limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)s,(1+a)s} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\}$$

$$\leqslant \limsup_{n\to\infty} P\{c_n^{-1}(M_n^{(1-a)s,(1+a)s} - d_n) \geqslant \varepsilon + c_n^{-1}(M_n^{(1+a)s,(1-a)t} - d_n)\}$$

$$\leqslant \sup_{\lambda\in[1,c]} \int_{\{x \geqslant \varepsilon + y\}} d\mu_{\lambda(1-a)s,\lambda(1+a)s} \otimes \mu_{\lambda(1+a)s,\lambda(1-a)t}(x,\ y)$$

$$= \int_{\{x \geqslant \varepsilon + y\}} d\mu_{\lambda_0(1-a)s,\lambda_0(1+a)s} \otimes \mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(x,\ y)$$

for some $\lambda_0 \in [1,\ c]$, since $\lambda \mapsto \mu_{\lambda(1-a)s,\lambda(1+a)s} \otimes \mu_{\lambda(1+a)s,\lambda(1-a)t}$ is continuous on the positive half line. For $a \to 0$ we have

$$G_{\lambda_0(1-a)s,\lambda_0(1-a)t}(\lambda_0^\alpha x + p_\alpha(\lambda_0)) \to G_{\lambda_0 s,\lambda_0 t}(\lambda_0^\alpha x + p_\alpha(\lambda_0)),$$
$$G_{\lambda_0(1+a)s,\lambda_0(1-a)t}(\lambda_0^\alpha x + p_\alpha(\lambda_0)) \to G_{\lambda_0 s,\lambda_0 t}(\lambda_0^\alpha x + p_\alpha(\lambda_0)).$$

Observe that for all $y \in R$

$$G_{\lambda_0(1-a)s,\lambda_0(1-a)t}(y) = G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(y) \cdot G_{\lambda_0(1+a)s,\lambda_0(1-a)t}(y)$$

so that for every point of continuity $\lambda_0^\alpha x + p_\alpha(\lambda_0)$ of $G_{\lambda_0 s,\lambda_0 t}$ with $G_{\lambda_0 s,\lambda_0 t}(\lambda_0^\alpha x + p_\alpha(\lambda_0)) > 0$ we obtain

$$G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha x + p_\alpha(\lambda_0)) \to 1.$$

Let $y_0 = \inf\{x \in R \mid G_{\lambda_0 s,\lambda_0 t}(\lambda_0^\alpha x + p_\alpha(\lambda_0)) > 0\} \in R \cup \{-\infty\}$.

Case 1: $y_0 > -\infty$. Choose $a_2 \in (0, a_1]$ such that for all $0 < a \leqslant a_2$ we have

$$\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(-\infty, y_0 - \varepsilon/4] \leqslant \delta/8.$$

Further choose $y_1 \in (y_0, y_0 + \varepsilon/4)$ such that $\lambda_0^\alpha y_1 + p_\alpha(\lambda_0)$ is a point of continuity of $G_{\lambda_0 s,\lambda_0 t}$. Then there exists $a_3 \in (0, a_2]$ such that

$$G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha y_1 + p_\alpha(\lambda_0)) \geqslant 1 - \delta/8 \qquad \text{for all } 0 < a \leqslant a_3.$$

By (3.3) we obtain for all $0 < a \leqslant a_3$

$$\begin{aligned}
&\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)s,(1+a)s} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\} \\
&\leqslant \int_{\{x > \varepsilon/2 + y\}} d\mu_{\lambda_0(1-a)s,\lambda_0(1+a)s} \otimes \mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(x, y) \\
&= \int\int 1_{\{x > \varepsilon/2 + y\}}\, d\mu_{\lambda_0(1-a)s,\lambda_0(1+a)s}(x)\, d\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(y) \\
&= \int 1 - G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha(y + \varepsilon/2) + p_\alpha(\lambda_0))\, d\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(y) \\
&\leqslant \delta/8 + \int_{\{y > y_0 - \varepsilon/4\}} 1 - G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha(y + \varepsilon/2) + p_\alpha(\lambda_0))\, d\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(y) \\
&\leqslant \delta/8 + 1 - G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha y_1 + p_\alpha(\lambda_0)) \leqslant \delta/4.
\end{aligned}$$

Case 2: $y_0 = -\infty$. Since $(\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t})_{0 < a \leqslant a_1}$ is uniformly tight, choose $y_1 \in R$ such that

$$\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(-\infty, y_1) \leqslant \delta/8 \qquad \text{for all } 0 < a \leqslant a_1.$$

Further choose $y_2 \in (y_1, y_1 + \varepsilon/2]$ such that $\lambda_0^\alpha y_2 + p_\alpha(\lambda_0)$ is a point of continuity of $G_{\lambda_0 s,\lambda_0 t}$. Then there exists $a_3 \in (0, a_1]$ such that

$$G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha y_2 + p_\alpha(\lambda_0)) \geqslant 1 - \delta/8 \qquad \text{for all } 0 < a \leqslant a_3$$

as in the case 1. By (3.3) we obtain for all $0 < a \leqslant a_3$, as in the case 1,

$$\begin{aligned}
&\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)s,(1+a)s} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\} \\
&\leqslant \delta/8 + \int_{\{y > y_1\}} 1 - G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha(y + \varepsilon/2) + p_\alpha(\lambda_0))\, d\mu_{\lambda_0(1+a)s,\lambda_0(1-a)t}(y) \\
&\leqslant \delta/8 + 1 - G_{\lambda_0(1-a)s,\lambda_0(1+a)s}(\lambda_0^\alpha y_2 + p_\alpha(\lambda_0)) \leqslant \delta/4.
\end{aligned}$$

Finally, in both cases, for all $0 < a \leqslant a_3$ we obtain

$$\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)s,(1+a)s} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\} \leqslant \delta/4. \tag{3.4}$$

Analogously we can see that for the latter expression on the right-hand side of (3.2) there exists $a_0 \in (0, a_1]$ such that for all $0 < a \leqslant a_0$ we have

$$\limsup_{n\to\infty} P\{c_n^{-1} M_n^{(1-a)t,(1+a)t} > \varepsilon + c_n^{-1} M_n^{(1+a)s,(1-a)t}\} \leqslant \delta/4,$$

which together with (3.2) and (3.4) gives the asserted result. ∎

The following lemma is proved in [3] and will be helpful for further results. The proof given in [3] is based on arguments of Hilbert space techniques.

LEMMA 3.4. *Let (Y_n) be a sequence of independent random variables and (k_n), (m_n) be two sampling sequences with $k_n \leqslant m_n$, $k_n \to \infty$. If A_n is an event depending only on $Y_{k_n}, \ldots, Y_{m_n}$, then for any event A we have*

$$\limsup_{n\to\infty} P(A_n \mid A) = \limsup_{n\to\infty} P(A_n),$$

where we set $P(A_n \mid A) = P(A_n)$ if $P(A) = 0$.

The next lemma is crucial for the proof of randomized limit theorems. Recall the three steps of proof for the randomized central limit theorem given at the end of the Introduction. Whereas in the first step $T_n = \lfloor nD \rfloor$ mixing properties are sufficient, in the last two steps approximations of T_n by $\lfloor nD \rfloor$ with discrete D, and of arbitrary $D > 0$ by discrete random variables are applied, respectively. Anscombe's condition defines the quality of these approximations for randomized maxima as follows:

LEMMA 3.5. *Under the conditions and with the notation of Theorem* 2.10 *let (U_n) be a sequence of positive integer valued random variables with $U_n/n \to U$ in probability for some positive random variable U.*

(a) *If U is discrete, for all $0 \leqslant s < t$ we have*

$$c_{\lfloor nU \rfloor}^{-1} (M_{U_n}^{s,t} - M_{\lfloor nU \rfloor}^{s,t}) \to 0 \text{ in probability.} \tag{3.5}$$

(b) *For $m \in N$ define positive discrete random variables V_m by*

$$V_m = k \cdot 2^{-m} \quad \text{if } (k-1)2^{-m} < U \leqslant k \cdot 2^{-m}.$$

Further, for $m, n \in N$ let $U_{m,n}$ be positive integer valued random variables given by $U_{m,n} = U_n + \lfloor n(V_m - U) \rfloor$. Then for every $\varepsilon > 0$ and $0 \leqslant s < t$ we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\{|c_{\lfloor nV_m \rfloor}^{-1} (M_{U_n}^{s,t} - M_{\lfloor nV_m \rfloor}^{s,t})| > \varepsilon\} = 0, \tag{3.6}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\{|c_{\lfloor nV_m \rfloor}^{-1} (M_{U_{m,n}}^{s,t} - M_{\lfloor nV_m \rfloor}^{s,t})| > \varepsilon\} = 0. \tag{3.7}$$

Proof. (a) Let (u_k) be a (countable) sequence with $p_k = P\{U = u_k\} > 0$ and $\sum_k p_k = 1$. For $\delta > 0$ choose $N \in N$ such that $\sum_{k>N} p_k \leqslant \delta/2$. For any $a > 0$ we have

$$\limsup_{n\to\infty} P\{|U_n - \lfloor nU\rfloor| > a\lfloor nU\rfloor\} = \limsup_{n\to\infty} P\left\{\left|\frac{U_n}{\lfloor nU\rfloor} - 1\right| > a\right\} = 0.$$

Thus with the events $E_{n,k} = \{|U_n - \lfloor nu_k\rfloor| \leqslant a\lfloor nu_k\rfloor\}$ we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} P\{|c^{-1}_{\lfloor nU\rfloor}(M^{s,t}_{U_n} - M^{s,t}_{\lfloor nU\rfloor})| > \varepsilon\}\\
&\quad\leqslant \limsup_{n\to\infty} P\{|c^{-1}_{\lfloor nU\rfloor}(M^{s,t}_{U_n} - M^{s,t}_{\lfloor nU\rfloor})| > \varepsilon,\ |U_n - \lfloor nU\rfloor| \leqslant a\lfloor nU\rfloor\}\\
&\qquad + \limsup_{n\to\infty} P\{|U_n - \lfloor nU\rfloor| > a\lfloor nU\rfloor\}\\
&\quad\leqslant \delta/2 + \sum_{k\leqslant N} p_k \limsup_{n\to\infty} P(\{|c^{-1}_{\lfloor nu_k\rfloor}(M^{s,t}_{U_n} - M^{s,t}_{\lfloor nu_k\rfloor})| > \varepsilon\} \cap E_{n,k} \mid U = u_k)\\
&\quad\leqslant \delta/2 + \sum_{k\leqslant N} \limsup_{n\to\infty} P\{\max_{|m-\lfloor nu_k\rfloor| \leqslant a\lfloor nu_k\rfloor} |c^{-1}_{\lfloor nu_k\rfloor}(M^{s,t}_m - M^{s,t}_{\lfloor nu_k\rfloor})| > \varepsilon\}.
\end{aligned}$$

Now choose $a \in (0, 1)$ such that, by Theorem 3.3, for all $k \leqslant N$ we have

$$\limsup_{n\to\infty} P\{\max_{|m-\lfloor nu_k\rfloor| \leqslant a\lfloor nu_k\rfloor} |c^{-1}_{\lfloor nu_k\rfloor}(M^{s,t}_m - M^{s,t}_{\lfloor nu_k\rfloor})| > \varepsilon\} \leqslant \frac{\delta}{2N}.$$

Hence we obtain

$$\limsup_{n\to\infty} P\{|c^{-1}_{\lfloor nU\rfloor}(M^{s,t}_{U_n} - M^{s,t}_{\lfloor nU\rfloor})| > \varepsilon\} \leqslant \frac{\delta}{2} + N\frac{\delta}{2N} = \delta,$$

which proves (3.5) since $\varepsilon > 0$ and $\delta > 0$ are arbitrary.

(b) We have $0 \leqslant V_m - U < 2^{-m}$. Thus $V_m \to U$ in probability. Further, for any fixed $m \in N$ we obtain

$$\frac{U_{m,n}}{n} = \frac{U_n}{n} + \frac{\lfloor n(V_m - U)\rfloor}{n} \to V_m \text{ in probability.}$$

Choose $m_0 \in N$ with $P\{U > m_0\} \leqslant \delta/3$ and $m_1 \geqslant m_0$ such that for all $m \geqslant m_1$ we have $P\{U \leqslant m\cdot 2^{-m}\} \leqslant \delta/3$. By Theorem 3.3 further choose $m_2 \geqslant m_1$ such that for all $m \geqslant m_2$ we have

$$\limsup_{n\to\infty} P\{\max_{|l-n|\leqslant m^{-1}n} |c_n^{-1}(M^{s,t}_l - M^{s,t}_n)| > \varepsilon\} \leqslant \delta/3. \tag{3.8}$$

Let $p_k(n) = \lfloor nk\cdot 2^{-m}\rfloor$ and define events

$$E_k = \{(k-1)2^{-m} < U \leqslant k\cdot 2^{-m}\} = \{V_m = k\cdot 2^{-m}\},$$

$$G_{k,n} = \{|U_n - p_k(n)| \leqslant n\cdot 2^{-m}\}, \qquad \tilde{G}_{k,n} = \{|U_n - p_k(n)| \leqslant m^{-1} p_k(n)\},$$

$$A_{k,n} = \{\max_{|l-p_k(n)|\leqslant m^{-1}p_k(n)} |c^{-1}_{p_k(n)}(M^{s,t}_l - M^{s,t}_{p_k(n)})| > \varepsilon\}.$$

By Lemma 3.4 and (3.8), for all $k \in N$ and $s > 0$ we have

$$\limsup_{n\to\infty} P(A_{k,n} \mid E_k) \leqslant \delta/3. \tag{3.9}$$

Further, for sufficiently large $n \in N$ and $k \geqslant m+1$ we obtain

$$n \cdot 2^{-m} \leqslant \frac{1}{k-1} p_k(n) \leqslant m^{-1} p_k(n),$$

and hence $G_{k,n}$ is contained in $\tilde{G}_{k,n}$. Using the relation

$$\limsup_{n\to\infty} P\{|U_n - \lfloor nV_m \rfloor| > n \cdot 2^{-m}\} \leqslant P\{|U - V_m| \geqslant 2^{-m}\} = 0$$

we imply (3.6) in the case $s > 0$ by (3.9), since for $m \geqslant m_2$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} P\{|c_{\lfloor nV_m \rfloor}^{-1} (M_{U_n}^{s,t} - M_{\lfloor nV_m \rfloor}^{s,t})| > \varepsilon\} \\
&\quad \leqslant \limsup_{n\to\infty} P\{|c_{\lfloor nV_m \rfloor}^{-1} (M_{U_n}^{s,t} - M_{\lfloor nV_m \rfloor}^{s,t})| > \varepsilon, |U_n - \lfloor nV_m \rfloor| \leqslant n \cdot 2^{-m}\} \\
&\qquad + \limsup_{n\to\infty} P\{|U_n - \lfloor nV_m \rfloor| > n \cdot 2^{-m}\} \\
&\quad \leqslant \limsup_{n\to\infty} \sum_{k=m+1}^{m_0 \cdot 2^m} P(\{|c_{p_k(n)}^{-1} (M_{U_n}^{s,t} - M_{p_k(n)}^{s,t})| > \varepsilon\} \cap G_{k,n} \cap E_k) \\
&\qquad + P\{U > m_0\} + P\{U \leqslant m \cdot 2^{-m}\} \\
&\quad \leqslant \frac{2\delta}{3} + \limsup_{n\to\infty} \sum_{k=m+1}^{m_0 \cdot 2^m} P(\{|c_{p_k(n)}^{-1} (M_{U_n}^{s,t} - M_{p_k(n)}^{s,t})| > \varepsilon\} \cap \tilde{G}_{k,n} \cap E_k) \\
&\quad \leqslant \frac{2\delta}{3} + \sum_{k=m+1}^{m_0 \cdot 2^m} P(E_k) \limsup_{n\to\infty} P(A_{k,n} \mid E_k) \leqslant \frac{2\delta}{3} + \frac{\delta}{3} \sum_{k=m+1}^{m_0 \cdot 2^m} P(E_k) \leqslant \delta.
\end{aligned}$$

In the case $s = 0$ we further have to discuss the application of Lemma 3.4 in (3.9). As in the proof of Theorem 3.3, analogously to (3.2) we get

$$\begin{aligned}
&\limsup_{n\to\infty} P\{\max_{|l - p_k(n)| \leqslant m^{-1} p_k(n)} |c_{p_k(n)}^{-1} (M_l^{0,t} - M_{p_k(n)}^{0,t})| > \varepsilon\} \\
&\quad \leqslant 2 \limsup_{n\to\infty} P\{c_{p_k(n)}^{-1} M_{p_k(n)}^{(1-m^{-1})t,(1+m^{-1})t} > \varepsilon + c_{p_k(n)}^{-1} M_{p_k(n)}^{0,(1-m^{-1})t}\} \\
&\quad \leqslant 2 \limsup_{n\to\infty} P\{c_{p_k(n)}^{-1} M_{p_k(n)}^{(1-m^{-1})t,(1+m^{-1})t} > \varepsilon + c_{p_k(n)}^{-1} M_{p_k(n)}^{p_k(n)^{-1/2},(1-m^{-1})t}\}.
\end{aligned}$$

Since the distributions of the random variables

$$c_{p_k(n)}^{-1} (M_{p_k(n)}^{0,(1-m^{-1})t} - d_{p_k(n)}) \quad \text{and} \quad c_{p_k(n)}^{-1} (M_{p_k(n)}^{p_k(n)^{-1/2},(1-m^{-1})t} - d_{p_k(n)})$$

have the same weak limiting behaviour, we further obtain as in the proof of Theorem 3.3, analogously to (3.4),

$$\limsup_{n\to\infty} P\{c_{p_k(n)}^{-1} M_{p_k(n)}^{(1-m^{-1})t,(1+m^{-1})t} > \varepsilon + c_{p_k(n)}^{-1} M_{p_k(n)}^{p_k(n)^{-1/2},(1-m^{-1})t}\} \leqslant \delta/6$$

for arbitrary $\varepsilon > 0$, $\delta > 0$ and sufficiently large $m \geqslant m_2$. The advantage of the new situation is that now the events

$$B_{k,n} = \{c_{p_k(n)}^{-1} M_{p_k(n)}^{(1-m^{-1})t,(1+m^{-1})t} > \varepsilon + c_{p_k(n)}^{-1} M_{p_k(n)}^{p_k(n)^{-1/2},(1-m^{-1})t}\}$$

depend only on $X_{\lfloor p_k(n)^{1/2}\rfloor}, \ldots, X_{\lfloor p_k(n)(1+m^{-1})t\rfloor}$ such that Lemma 3.4 can be applied and we get also (3.9) in the case $s = 0$ since

$$\limsup_{n\to\infty} P(A_{k,n} \mid E_k) \leqslant 2 \limsup_{n\to\infty} P(B_{k,n} \mid E_k).$$

This proves (3.6) and analogously we get (3.7) since for all $m \in N$

$$\limsup_{n\to\infty} P\{|U_{m,n} - \lfloor nV_m \rfloor| > m^{-1} \lfloor nV_m \rfloor\} = 0.$$

Moreover, with the events $H_{k,n} = \{|U_{m,n} - p_k(n)| \leqslant m^{-1} p_k(n)\}$ we obtain for $m \geqslant m_2$, as before,

$$\begin{aligned}
&\limsup_{n\to\infty} P\{|c_{\lfloor nV_m\rfloor}^{-1} (M_{U_{m,n}}^{s,t} - M_{\lfloor nV_m\rfloor}^{s,t})| > \varepsilon\} \\
&\leqslant \frac{2\delta}{3} + \limsup_{n\to\infty} \sum_{k=m+1}^{m_0\cdot 2^m} P(\{|c_{p_k(n)}^{-1} (M_{U_{m,n}}^{s,t} - M_{p_k(n)}^{s,t})| > \varepsilon\} \cap H_{k,n} \cap E_k) \\
&\leqslant \frac{2\delta}{3} + \sum_{k=m+1}^{m_0\cdot 2^m} P(E_k) \limsup_{n\to\infty} P(A_{k,n} \mid E_k) \leqslant \frac{2\delta}{3} + \frac{\delta}{3} \sum_{k=m+1}^{m_0\cdot 2^m} P(E_k) \leqslant \delta.
\end{aligned}$$

This completes the proof. ∎

Theorem 3.6. *Under the conditions and with the notation of Theorem 2.10 let (T_n) be a sequence of positive integer valued random variables such that $T_n/k_n \to D$ in probability for some positive random variable D with distribution ϱ. Then for all $0 \leqslant s < t$ we have the randomized limit*

$$P\{a_n^{-1}(M_{T_n}^{s,t} - b_n) \leqslant x\} \to \int_0^\infty G_{rs,rt}(x)\, d\varrho(r).$$

It is easy to see that the limiting integral mixture of pdf's is again a pdf. As described in the Introduction, the proof of the theorem is divided into three consecutive steps. The first two steps are formulated separately by the following lemmas:

Lemma 3.7. *Theorem 3.6 holds in the special case of $T_n = \lfloor k_n D \rfloor$ with discrete $D > 0$.*

Proof. Let (d_k) be a (countable) sequence with $p_k = P\{D = d_k\} > 0$ and $\sum_k p_k = 1$. For $\delta > 0$ choose $N \in \mathbf{N}$ such that $\sum_{k>N} p_k \leqslant \delta/4$. Since the sequence $(a_n^{-1}(M^{s,t}_{\lfloor k_n d_k \rfloor} - b_n))$ is mixing by Proposition 3.1, for every $k \leqslant N$ and sufficiently large $n \in \mathbf{N}$ in view of Theorem 2.10 we have

$$\left|P(a_n^{-1}(M^{s,t}_{\lfloor k_n d_k \rfloor} - b_n) \leqslant x \mid D = d_k) - G_{d_k s, d_k t}(x)\right| \leqslant \delta/2$$

for each (fixed) point of continuity. Consequently,

$$\left|P\{a_n^{-1}(M^{s,t}_{T_n} - b_n) \leqslant x\} - \int_0^\infty G_{rs,rt}(x)\, d\varrho(r)\right|$$

$$= \left|\sum_k p_k P(a_n^{-1}(M^{s,t}_{\lfloor k_n d_k \rfloor} - b_n) \leqslant x \mid D = d_k) - \sum_k p_k G_{d_k s, d_k t}(x)\right|$$

$$\leqslant \frac{\delta}{2} + \sum_{k \leqslant N} p_k \left|P(a_n^{-1}(M^{s,t}_{\lfloor k_n d_k \rfloor} - b_n) \leqslant x \mid D = d_k) - G_{d_k s, d_k t}(x)\right| \leqslant \frac{\delta}{2} + \frac{\delta}{2} \sum_{k \leqslant N} p_k \leqslant \delta.$$

Since $\delta > 0$ is arbitrary, this proves the lemma. ∎

Lemma 3.8. *Theorem 3.6 holds in the special case of $T_n/k_n \to D$ in probability with discrete $D > 0$.*

Proof. Let us write

$$\frac{M^{s,t}_{T_n} - b_n}{a_n} = \frac{M^{s,t}_{\lfloor k_n D \rfloor} - b_n}{a_n} + \frac{c_{\lfloor k_n D \rfloor}}{c_{k_n}} \frac{M^{s,t}_{T_n} - M^{s,t}_{\lfloor k_n D \rfloor}}{c_{\lfloor k_n D \rfloor}}.$$

Since by Lemma 3.2 we have $c_{\lfloor k_n D \rfloor} c_{k_n}^{-1} \to D^\alpha$ in probability, and by Lemma 3.7 we obtain

$$P\{a_n^{-1}(M^{s,t}_{\lfloor k_n D \rfloor} - b_n) \leqslant x\} \to \int_0^\infty G_{rs,rt}(x)\, d\varrho(r),$$

in view of Cramer's theorem it is sufficient to prove for any $\varepsilon > 0$

$$\lim_{n \to \infty} P\{|c^{-1}_{\lfloor k_n D \rfloor}(M^{s,t}_{T_n} - M^{s,t}_{\lfloor k_n D \rfloor})| > \varepsilon\} = 0. \tag{3.10}$$

Let us write $n = \alpha_n k_{p_n}$ with $p_n \in \mathbf{N}$, $k_{p_n} \leqslant n < k_{p_n+1}$ and define $U_n = \alpha_n T_{p_n}$. Then $U_{k_n} = T_n$ and $U_n/n \to D$ in probability. It follows by Lemma 3.5 (a) that

$$\lim_{n \to \infty} P\{|c^{-1}_{\lfloor nD \rfloor}(M^{s,t}_{U_n} - M^{s,t}_{\lfloor nD \rfloor})| > \varepsilon\} = 0 \quad \text{for any } \varepsilon > 0.$$

In particular, we get (3.10) along the subsequence (k_n). ∎

Proof of Theorem 3.6. For $m \in \mathbf{N}$ assume that random variables D_m with distributions ϱ_m are determined as

$$D_m = k \cdot 2^{-m} \quad \text{if } (k-1)2^{-m} < D \leqslant k \cdot 2^{-m}.$$

D_m is positive and discrete and $0 \leqslant D_m - D < 2^{-m}$. Thus $D_m \to D$ in probability. Further, let us define positive integer valued random variables $D_{m,n} = T_n + \lfloor k_n(D_m - D) \rfloor$. Then for fixed $m \in N$ we have

$$\frac{D_{m,n}}{k_n} = \frac{T_n}{k_n} + \frac{\lfloor k_n(D_m - D) \rfloor}{k_n} \to D_m \text{ in probability.}$$

For all $0 \leqslant s < t$ let us write

$$\frac{M_{T_n}^{s,t} - b_n}{a_n} = \frac{M_{D_{m,n}}^{s,t} - b_n}{a_n} + \frac{c_{\lfloor k_n D_m \rfloor}}{c_{k_n}} \frac{M_{T_n}^{s,t} - M_{D_{m,n}}^{s,t}}{c_{\lfloor k_n D_m \rfloor}}.$$

Since by Lemma 3.2 we have $c_{\lfloor k_n D_m \rfloor} c_{k_n}^{-1} \to D_m^\alpha \to D^\alpha$ subsequently as $n \to \infty$ and $m \to \infty$, and by Lemma 3.8 we obtain

$$P\{a_n^{-1}(M_{D_{m,n}}^{s,t} - b_n) \leqslant x\} \to \int_0^\infty G_{rs,rt}(x)\, d\varrho_m(r) \to \int_0^\infty G_{rs,rt}(x)\, d\varrho(r),$$

again subsequently as $n \to \infty$ and $m \to \infty$, in view of the extension of Cramer's theorem for doubly indexed sequences (see Lemma 2 in [3]) it is sufficient to prove for all $\varepsilon > 0$

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\{|c_{\lfloor k_n D_m \rfloor}^{-1}(M_{T_n}^{s,t} - M_{D_{m,n}}^{s,t})| > \varepsilon\} = 0. \tag{3.11}$$

For (3.11) it is sufficient to prove for all $\varepsilon > 0$ and $m \to \infty$ the following convergences:

$$\limsup_{n \to \infty} P\{|c_{\lfloor k_n D_m \rfloor}^{-1}(M_{T_n}^{s,t} - M_{\lfloor k_n D_m \rfloor}^{s,t})| > \varepsilon\} \to 0, \tag{3.12}$$

$$\limsup_{n \to \infty} P\{|c_{\lfloor k_n D_m \rfloor}^{-1}(M_{D_{m,n}}^{s,t} - M_{\lfloor k_n D_m \rfloor}^{s,t})| > \varepsilon\} \to 0. \tag{3.13}$$

Let us write $n = \alpha_n k_{p_n}$ with $p_n \in N$, $k_{p_n} \leqslant n < k_{p_n+1}$ and define $U_n = \alpha_n T_{p_n}$. Then $U_{k_n} = T_n$ and $U_n/n \to D$ in probability. By (3.6) we obtain for any $\varepsilon > 0$

$$\lim_{m \to \infty} \lim_{n \to \infty} P\{|c_{\lfloor n D_m \rfloor}^{-1}(M_{U_n}^{s,t} - M_{\lfloor n D_m \rfloor}^{s,t})| > \varepsilon\} = 0,$$

which in particular gives (3.12) along the subsequence (k_n).

Let $U_{m,n} = U_n + \lfloor n(D_m - D) \rfloor$. Then $U_{m,k_n} = T_n + \lfloor k_n(D_m - D) \rfloor = D_{m,n}$ so that for all $\varepsilon > 0$ in view of (3.7) we have

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\{|c_{\lfloor n D_m \rfloor}^{-1}(M_{U_{m,n}}^{s,t} - M_{\lfloor n D_m \rfloor}^{s,t})| > \varepsilon\} = 0.$$

In particular, we get (3.13) along the subsequence (k_n). ∎

The stochastic compactness in Theorem 2.10 enables us to observe the following limiting behaviour of randomized maxima if the random sampling sequence proportionally to n instead of the sampling sequence k_n converges in probability to a positive random variable:

THEOREM 3.9. *Under the conditions of Theorem 3.6 let (U_n) be a sequence of positive integer valued random variables such that $U_n/n \to U$ in probability for some positive random variable U with distribution η. Then the sequence $(c_n^{-1}(M_{U_n}^{s,t}-d_n))$ is stochastically compact and every weak limit point of the distributions has explicitly a pdf which for some $\lambda \in [1, c)$ can be written as*

$$\int_0^\infty G_{\lambda rs, \lambda rt}(\lambda^\alpha x + p_\alpha(\lambda))\, d\eta(r).$$

Proof. Let us write $n = \lambda_n k_{p_n}$ with $p_n \in N$ and $k_{p_n} \leqslant n < k_{p_n+1}$. Thus (λ_n) is relatively compact in $[1, c]$. Let (n') be a subsequence such that $\lambda_n \to \lambda$ along (n'), where $\lambda \in [1, c]$ is an arbitrary limit point. Let us put

$$\frac{M_{U_n}^{s,t}-d_n}{c_n} = \frac{M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}-d_n}{c_n} + \frac{c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}}{c_n} \frac{M_{U_n}^{s,t}-M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}}{c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}}.$$

Recall from the proof of Theorem 2.9 that the embedding sequences fulfil $c_n = \lambda_n^\alpha a_{p_n}$ and $d_n = b_{p_n} + a_{p_n} p_\alpha(\lambda_n)$. Since $\lfloor \lambda_n U_{k_{p_n}} \rfloor / k_{p_n} \to \lambda U$ in probability, from Theorem 3.6 we obtain

$$P\{c_n^{-1}(M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}-d_n) \leqslant x\} = P\{\lambda_n^{-\alpha} a_{p_n}^{-1}(M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}-b_{p_n}-a_{p_n}p_\alpha(\lambda_n)) \leqslant x\}$$

$$= P\{a_{p_n}^{-1}(M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}-b_{p_n}) \leqslant \lambda_n^\alpha x + p_\alpha(\lambda_n)\} \to \int_0^\infty G_{\lambda rs, \lambda rt}(\lambda^\alpha x + p_\alpha(\lambda))\, d\eta(r).$$

Since $\lfloor \lambda_n U_{k_{p_n}} \rfloor / n = \lfloor \lambda_n U_{k_{p_n}} \rfloor / \lambda_n k_{p_n} \to U$ in probability, we further infer from Lemma 3.2 that $c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor} c_n^{-1} \to U^\alpha$ in probability. Hence in view of Cramer's theorem it is sufficient to prove

$$c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{-1}(M_{U_n}^{s,t}-M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}) \to 0 \text{ in probability.} \tag{3.14}$$

Let V_m be defined as in Lemma 3.5 and write

$$\frac{M_{U_n}^{s,t}-M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}}{c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}} = \frac{c_{\lfloor nV_m \rfloor}}{c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}} \left[\frac{M_{U_n}^{s,t}-M_{\lfloor nV_m \rfloor}^{s,t}}{c_{\lfloor nV_m \rfloor}} - \frac{M_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{s,t}-M_{\lfloor nV_m \rfloor}^{s,t}}{c_{\lfloor nV_m \rfloor}}\right].$$

We have the following convergence in probability:

$$\frac{\lfloor nV_m \rfloor}{\lfloor \lambda_n U_{k_{p_n}} \rfloor} = \frac{\lfloor nV_m \rfloor}{n} \frac{\lambda_n k_{p_n}}{\lfloor \lambda_n U_{k_{p_n}} \rfloor} \to \frac{V_m}{U} \to 1$$

subsequently as $n \to \infty$ and $m \to \infty$, and hence by Lemma 3.2 we obtain

$$c_{\lfloor nV_m \rfloor} c_{\lfloor \lambda_n U_{k_{p_n}} \rfloor}^{-1} \to (V_m/U)^\alpha \to 1 \text{ in probability.}$$

Further, for every $\varepsilon > 0$ it follows by (3.6) that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\{|c_{\lfloor nV_m \rfloor}^{-1}(M_{U_n}^{s,t}-M_{\lfloor nV_m \rfloor}^{s,t})| > \varepsilon\} = 0,$$

and since $\lfloor \lambda_n U_{k_{p_n}} \rfloor / n \to U$ in probability, we obtain again by (3.6)

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\{|c^{-1}_{\lfloor nV_m \rfloor}(M^{s,t}_{\lfloor \lambda_n U_{k_{p_n}} \rfloor} - M^{s,t}_{\lfloor nV_m \rfloor})| > \varepsilon\} = 0.$$

Hence (3.14) follows by Cramer's theorem.

Finally, observe that the pdf's of limit points for $\lambda = 1$ and $\lambda = c$ coincide since Propositions 2.4 and 2.5 imply $G_{crs,crt}(c^\alpha x + p_\alpha(c)) = G_{rs,rt}(x)$. ∎

4. CONCLUDING REMARKS

The convergence condition (2.1) together with the mixing property of Proposition 3.1 imply convergence of the joint distributions of

$$\left(a_n^{-1}(M^{s,t}_{\lfloor k_n \lambda \rfloor} - b_n),\ T_n/k_n\right)$$

to the product measure $\mu_{\lambda s, \lambda t} \otimes \varrho$ for every $\lambda > 0$ and $0 \leqslant s < t$, where $\mu_{\lambda s, \lambda t}(-\infty, x] = G_{\lambda s, \lambda t}(x)$ and ϱ is the distribution of D. This follows analogously to Lemma 2 in [15] where the special case of i.i.d. random variables is proved. Convergence of the joint distributions in turn leads us to randomized limits as Silvestrov and Teugels show in Theorem 1 of [15] for more general extremal processes. In fact, Theorem 3.6 of the present paper is identical to the statement of Theorem 1 in [15] for the special case of max-semistable hemigroups. The same is true for Theorem 3.9 if we apply the stochastic compactness result of Theorem 2.10 together with an appropriate mixing property and consider Theorem 1 of [15] along certain subsequences. But the concrete form of the limits in both Theorems 3.6 and 3.9 can only be obtained by the results of Section 2 of the present paper.

As stated before, the results of [15] even hold for more general situations than the present hemigroup setting, and convergence of randomized extremal processes is also considered in the Skorokhod topology. But note that the methods of proof are different. Whereas the proofs in Section 3 of the present paper follow corresponding results on the randomized central limit theorem, containing the verification of an Anscombe condition, the proofs of comparable results in [15] avoid Anscombe's condition and make use of monotonicity arguments instead. We think that Anscombe's condition is of independent interest and gives raison d'être to the methods of Section 3 in addition to the results of Silvestrov and Teugels.

We emphasize that the methods of the last section apply also for normalized maxima where the norming constants are also randomized. Namely, under the conditions and with the notation of Theorems 3.6 and 2.10 we have

$$P\{c^{-1}_{T_n}(M^{s,t}_{T_n} - d_{T_n}) \leqslant x\} \to \int_0^\infty G_{rs,rt}(r^\alpha x + p_\alpha(r))\, d\varrho(r)$$

and the sequence $(c_{U_n}^{-1}(M_{U_n}^{s,t}-d_{U_n}))$ is stochastically compact, where every weak limit point belongs to a pdf

$$\int_0^\infty G_{\lambda rs,\lambda rt}\big((\lambda r)^\alpha x+p_\alpha(\lambda r)\big)\,d\eta(r) \quad \text{for some } \lambda\in[1, c).$$

For the proof of these assertions one has to observe that $(c_{\lfloor k_n\lambda\rfloor}^{-1}(M_{\lfloor k_n\lambda\rfloor}^{s,t}-d_{\lfloor k_n\lambda\rfloor}))$ is a mixing sequence of random variables for each $\lambda > 0$, as in the proof of Proposition 3.1. Applications of Lemma 3.5, where, as in the proofs of Lemma 3.8 and Theorems 3.6 and 3.9, $(c_{T_n}^{-1}(M_{T_n}^{s,t}-d_{T_n}))$ and $(c_{U_n}^{-1}(M_{U_n}^{s,t}-d_{U_n}))$ have to be suitably decomposed, lead us then to the desired results. The details are left to the reader.

Moreover, randomizations of the last section apply for sampling sequences $k_n = n$, i.e. in stable situations. Suppose (1.1) holds for all $t > 0$. As argued in Remark 2.2 the convergence condition of Section 2 is fulfilled. Since in this case we have $c = 1$, embedding is superfluous, and hence $c_n = a_n$ and $d_n = b_n$ in terms of Section 2. Thus Theorem 2.9 is fulfilled by (2.2) and one observes easily the following stability equation by convergence of types:

$$G_{s,t}(x) = G_{rs,rt}\big(r^\alpha x+p_\alpha(r)\big)$$

for all $0 \leqslant s < t$ and $r > 0$, where $p_\alpha(r)$ is as in Theorem 2.10. Since the proofs of Section 3 do not necessarily depend on semistability, we get randomized limit theorems by Theorem 3.6 and the above remark also in the stable case. In particular, under the conditions and with the notation of Theorem 3.6 we have for all $0 \leqslant s < t$

$$P\{a_n^{-1}(M_{T_n}^{s,t}-b_n) \leqslant x\} \to \int_0^\infty G_{rs,rt}(x)\,d\varrho(r)$$

and

$$P\{a_{T_n}^{-1}(M_{T_n}^{s,t}-b_{T_n}) \leqslant x\} \to \int_0^\infty G_{rs,rt}\big(r^\alpha x+p_\alpha(r)\big)\,d\varrho(r).$$

The last limit coincides with $G_{s,t}(x)$ in view of the above-given stability equation.

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