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DEPENDENCE STRUCTURE OF STABLE R-GARCH PROCESSES

BY

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Dedicated to Professor CZESŁAW RYLL-NARDZEWSKI on the Occasion of His 75th Birthday

Abstract. In this paper we investigate properties of R-GARCH processes with positive strictly stable innovations. We derive the unconditional distributions and analyze the dependence structure. This analysis is carried out by means of the measure of dependence – the codifference – which extends the behavior of the covariance function to situations where the covariance function is no longer defined. In the case of R-GARCH (1, 1, 0) process we determine the exact asymptotic behavior.

1. Introduction. Linear processes do not capture the structure of financial data. Therefore, GARCH-type models have become popular in the past few years as they provide a good description of financial time series.

The class of Generalized Autoregressive Conditionally Heteroskedastic processes of order (p, q) (GARCH (p, q)) was introduced to allow the conditional variance of a time series process to depend on past information. GARCH processes are non-linear stochastic processes, their distributions are heavy-tailed with time-dependent conditional variance and they model clustering of volatility. The details about the properties and applications of GARCH models can be found in [1] and [2].

However, the way that GARCH models are built imposes limits on the heaviness of the tails of their unconditional distribution. Given that a wide range of financial data exhibit remarkably fat tails, this assumption represents a major shortcoming of GARCH models in financial time series analysis. Some attempts were made to develop GARCH-type models, which can describe time

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series with remarkably fat tails (see, for example, [7], [5]). The class of Randomized Generalized Autoregressive Conditionally Heteroskedastic (R-GARCH) processes was proposed in [4] and applied in [8]. Such processes, with conditional variance dependent on the past, can have very heavy tails.

We start Section 2 with the definition of R-GARCH processes, discuss their conditional distributions and we note that this class of processes is very flexible as it includes ARCH and GARCH processes, de Vries processes and discrete versions of subordinated processes as special cases. Results concerning the R-GARCH$(r, p, 0)$ process with positive strictly stable innovations are presented in Section 3. For example, we derive the unconditional distribution of the process, and then we show that this process is stationary symmetric α-stable. As the covariance function does not exist for such processes, in Section 4 we investigate the dependence structure using another measure of dependence – the codifference. We show that the codifference tends to zero at least exponentially, and in a special case – the R-GARCH$(1, 1, 0)$ process – we determine the exact asymptotic behavior of the codifference.

2. R-GARCH processes

DEFINITION 2.1. The *Randomized Generalized Autoregressive Conditionally Heteroskedastic process of order* (r, p, q) (*R-GARCH* (r, p, q)), where $r, p, q \in N$, is defined by the equations

$$X_n = \sqrt{h_n}\,\varepsilon_n, \quad n = 0, \pm 1, \pm 2, \ldots,$$

$$h_n = \sum_{j=1}^{r} \theta_j \eta_{n-j} + \sum_{j=1}^{p} \phi_j h_{n-j} + \sum_{j=1}^{q} \psi_j X_{n-j}^2, \tag{2.1}$$

where

$$r \geqslant 1,\ \theta_r > 0, \quad \theta_j \geqslant 0,\ j = 1, \ldots, r-1,$$

$$p \geqslant 0,\ \phi_p > 0, \quad \phi_j \geqslant 0,\ j = 1, \ldots, p-1,$$

$$q \geqslant 0,\ \psi_q > 0, \quad \psi_j \geqslant 0,\ j = 1, \ldots, q-1,$$

the innovations ε_n are i.i.d. standard normal random variables ($\varepsilon_n \overset{\text{i.i.d.}}{\sim} N(0, 1)$), the innovations η_n are positive i.i.d. random variables and $\{\varepsilon_n\}$ and $\{\eta_n\}$ are independent.

The equation (2.1) can be written symbolically in the more compact form

$$h_n = \Theta(L)\eta_n + \Phi(L)h_n + \Psi(L)X_n^2, \quad n = 0, \pm 1, \pm 2, \ldots,$$

where Θ, Φ and Ψ are the r^{th}, p^{th} and q^{th} degree polynomials

$$\Theta(z) = \theta_1 z + \theta_2 z^2 + \ldots + \theta_r z^r,$$

$$\Phi(z) = \phi_1 z + \phi_2 z^2 + \ldots + \phi_p z^p, \quad \Psi(z) = \psi_1 z + \psi_2 z^2 + \ldots + \psi_q z^q,$$

and L is the backward shift operator defined by

$$L^j X_n = X_{n-j}, \quad L^j h_n = h_{n-j}, \quad L^j \eta_n = \eta_{n-j}, \quad j = 0, \pm 1, \pm 2, \ldots$$

Basic properties of R-GARCH processes are discussed in [4]. Let us recall that the conditional on $\mathscr{F}_{n-1} = \sigma\{\varepsilon_s, \eta_s: s \leqslant n-1, s \in \mathbf{Z}\}$, $n = 0, \pm 1, \pm 2, \ldots$, distribution of X_n is Gaussian:

$$X_n \mid \mathscr{F}_{n-1} \sim N(0, h_n).$$

This means that the conditional expectation $\mathrm{E}(X_n \mid \mathscr{F}_{n-1})$ is constant and it is equal to zero. The conditional variance is, in turn, not constant but it depends on the past:

$$\mathrm{Var}(X_n \mid \mathscr{F}_{n-1}) = \mathrm{E}(X_n^2 \mid \mathscr{F}_{n-1}) = h_n.$$

In the R-GARCH process the conditional variance h_n depends on the past as it is specified as a linear function of past innovations $\eta_{n-1}, \ldots, \eta_{n-r}$, lagged conditional variances $h_{n-1}, \ldots, h_{n-p}$, and squared past observations of the process $X_{n-1}^2, \ldots, X_{n-q}^2$.

The class of R-GARCH processes is a relatively large class which includes ARCH and GARCH processes, discrete versions of subordinated processes and de Vries processes as special cases. However, not every process with conditional variance dependent on the past belongs to the class of R-GARCH processes. For example, the HARCH(k) (Heterogeneous interval, Autoregressive, Conditionally Heteroskedastic) process with $k > 1$, proposed in [3], differs from all R-GARCH processes.

3. R-GARCH processes with stable innovations. If the η_n's are stable, then, on the one hand, methods usually used when the second-order moments exist cannot be applied but, on the other hand, it is possible to obtain some interesting results about unconditional distributions of R-GARCH processes.

All relevant properties of stable random variables and processes can be found in [6]. Let us recall only a few facts that are important for the rest of the paper:

- The characteristic function of a stable random variable $Z \sim S_\alpha(\sigma, \beta, \mu)$ is given by

$$\mathrm{E}\exp\{i\xi Z\} = \begin{cases} \exp\left\{-\sigma^\alpha |\xi|^\alpha \left(1 - i\beta (\operatorname{sign}\xi) \operatorname{tg}\dfrac{\pi\alpha}{2}\right) + i\mu\xi\right\} & \text{if } \alpha \neq 1, \\[2ex] \exp\left\{-\sigma |\xi| \left(1 + i\beta \dfrac{2}{\pi} (\operatorname{sign}\xi) \ln|\xi|\right) + i\mu\xi\right\} & \text{if } \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2]$ denotes index of stability, $\sigma \geqslant 0$ – scale parameter, $\beta \in [-1, 1]$ – skewness parameter, and $\mu \in \mathbf{R}$ – shift parameter.

- $Z \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric if and only if $\beta = 0$ and $\mu = 0$.

- If Z_1 and Z_2 are independent random variables, $Z_i \sim S_\alpha(\sigma_i, 1, 0)$, $i = 1, 2$, $0 < \alpha < 1$, and $a_1, a_2 \geqslant 0$, then

$$a_1 Z_1 + a_2 Z_2 \sim S_\alpha\left(\left((a_1\sigma_1)^\alpha + (a_2\sigma_2)^\alpha\right)^{1/\alpha}, 1, 0\right).$$

• The Laplace transform $Ee^{-\gamma Z}, \gamma \geqslant 0$, of the random variable $Z \sim S_\alpha(\sigma, 1, 0)$, $0 < \alpha < 1$, equals

$$Ee^{-\gamma Z} = \exp\left\{-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}\gamma^\alpha\right\}.$$

In this section we consider the R-GARCH$(r, p, 0)$ process with η_n's strictly stable random variables totally skewed to the right, i.e. we assume that

$$\eta_n \sim S_{\alpha/2}\left(2\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right) \tag{3.1}$$

with $0 < \alpha < 2$. This means that the index of stability of η_n is smaller than one, and thus the first moment of η_n does not exist. The choice of the scale parameter provides results in a compact form.

Moreover, we assume that the polynomials $\Theta(z)$ and $1-\Phi(z)$ do not have common roots and the polynomial $1-\Phi(z)$ has no roots in the closed unit disk $\{z: |z| \leqslant 1\}$. In Proposition 3.1 we show that it is possible to represent the R-GARCH$(r, p, 0)$ process as the R-GARCH$(\infty, 0, 0)$ process with positive coefficients δ_j. Then we derive the unconditional distribution of X_n and of sums of X_n's (Proposition 3.2). It turns out that all finite-dimensional distributions of $\{X_n\}$ are symmetric α-stable ($S\alpha S$) and they are invariant under a shift of the time index. Thus the process $\{X_n\}$ is stationary symmetric α-stable, $0 < \alpha < 2$ (Corollary 3.1).

PROPOSITION 3.1. *If the polynomials $\Theta(z)$ and $1-\Phi(z)$ do not have common roots, then the conditional variance h_n for R-GARCH$(r, p, 0)$ with the innovations given by* (3.1) *can be represented in the form*

$$h_n = \sum_{j=1}^{\infty} \delta_j \eta_{n-j} \quad a.s. \tag{3.2}$$

with positive[1] *δ_j's satisfying $\delta_j < Q^{-j}$ eventually*[2], *$Q > 1$, if and only if the polynomial $1-\Phi(z)$ has no roots in the closed unit disk $\{z: |z| \leqslant 1\}$. The sequence $\{h_n\}$ is then stationary, strictly stable with index of stability $\alpha/2$, and*

$$h_n \sim S_{\alpha/2}\left(2\left(\sum_{j=1}^{\infty} \delta_j^{\alpha/2}\right)^{2/\alpha}\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right). \tag{3.3}$$

The δ_j's are the coefficients in the series expansion of $\Theta(z)/(1-\Phi(z))$, $|z| < 1$.

Proof. This proposition may be proved in the analogous way as Theorem 7.12.2 in [6] but here the coefficients δ_j are determined by the following

[1] More precisely, the δ_j's are non-negative and at least one of them is positive.

[2] "$a_j < b_j$ eventually" means that there is a j_0 such that $a_j < b_j$ for all $j > j_0$.

system of equations:

$$
(3.4)\qquad \begin{cases} \delta_1 = \theta_1, \\ \delta_2 - \delta_1 \phi_1 = \theta_2, \\ \delta_3 - \delta_2 \phi_1 - \delta_1 \phi_2 = \theta_3, \\ \dots\dots\dots\dots\dots \\ \delta_r - \delta_{r-1}\phi_1 - \delta_{r-2}\phi_2 - \dots - \delta_1 \phi_{r-1} = \theta_r, \\ \delta_s - \delta_{s-1}\phi_1 - \delta_{s-2}\phi_s - \dots - \delta_1 \phi_{s-1} = 0, \quad s > r, \end{cases}
$$

with the understanding that $\phi_i = 0$ if $i > p$. It follows from (3.4) and Definition 2.1 that the δ_j's are real, non-negative and at least one of them is positive. In order to get $\delta_1 > 0$ we assume that $\theta_1 > 0$.

The distribution of h_n follows immediately from (3.2) and (3.1). ■

PROPOSITION 3.2. *The unconditional distribution of the sum* $\sum_{k=0}^{m-1} X_{n-k}$, $m \geqslant 1$, *in the R-GARCH* $(r, p, 0)$ *model with innovations* η_n *given by formula* (3.1) *is symmetric stable with index of stability* α:

$$
(3.5)\qquad \sum_{k=0}^{m-1} X_{n-k} \sim S_\alpha\Big(\Big[\sum_{j=1}^{m-1}\Big(\sum_{i=1}^{j}\delta_i\Big)^{\alpha/2} + \sum_{j=m}^{\infty}\Big(\sum_{i=j-m+1}^{j}\delta_i\Big)^{\alpha/2}\Big]^{1/\alpha}, 0, 0\Big).
$$

Proof. As $\{\varepsilon_n\}$ and $\{\eta_n\}$ are independent, the characteristic function of $\sum_{k=0}^{m-1} X_{n-k}$ can be expressed by the Laplace transform of $\sum_{k=0}^{m-1} h_{n-k}$. Indeed,

$$
\begin{aligned}
(3.6)\qquad \mathrm{E}\Big(\exp\Big\{i\xi\sum_{k=0}^{m-1} X_{n-k}\Big\}\Big) &= \mathrm{E}\Big(\exp\Big\{i\xi\sum_{k=0}^{m-1}\sqrt{h_{n-k}}\,\varepsilon_{n-k}\Big\}\Big) \\
&= \mathrm{E}\Big(\mathrm{E}\Big[\exp\Big\{i\xi\sum_{k=0}^{m-1}\sqrt{h_{n-k}}\,\varepsilon_{n-k}\Big\}\,\Big|\,\eta_{n-1}, \eta_{n-2}, \dots\Big]\Big) \\
&= \mathrm{E}\left(\exp\left\{-\frac{\xi^2\sum_{k=0}^{m-1} h_{n-k}}{2}\right\}\right).
\end{aligned}
$$

In order to get the Laplace transform of $\sum_{k=0}^{m-1} h_{n-k}$, we write

$$
\sum_{k=0}^{m-1} h_{n-k} = \sum_{k=0}^{m-1}\Big(\sum_{j=1}^{\infty}\delta_j\eta_{n-k+j}\Big) = \sum_{j=1}^{m-1}\Big(\sum_{i=1}^{j}\delta_i\Big)\eta_{n-j} + \sum_{j=m}^{\infty}\Big(\sum_{i=j-m+1}^{j}\delta_i\Big)\eta_{n-j},
$$

and thus $\sum_{k=0}^{m-1} h_{n-k}$ is positive strictly stable with index of stability $\alpha/2$:

$$
\sum_{k=0}^{m-1} h_{n-k} \sim S_{\alpha/2}\left(2\Big[\sum_{j=1}^{m-1}\Big(\sum_{i=1}^{j}\delta_i\Big)^{\alpha/2} + \sum_{j=m}^{\infty}\Big(\sum_{i=j-m+1}^{j}\delta_i\Big)^{\alpha/2}\Big]^{2/\alpha}\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right).
$$

Applying the formula for the Laplace transform of $\sum_{k=0}^{m-1} h_{n-k}$ in (3.6), we obtain

$$
\mathrm{E}\Big(\exp\Big\{i\xi\sum_{k=0}^{m-1} X_{n-k}\Big\}\Big) = \exp\Big\{-|\xi|^\alpha\Big[\sum_{j=1}^{m-1}\Big(\sum_{i=1}^{j}\delta_i\Big)^{\alpha/2} + \sum_{j=m}^{\infty}\Big(\sum_{i=j-m+1}^{j}\delta_i\Big)^{\alpha/2}\Big]\Big\}.
$$

This means that $\sum_{k=0}^{m-1} X_{n-k}$ is $S\alpha S$ and

$$\sum_{k=0}^{m-1} X_{n-k} \sim S_\alpha\Big(\Big[\sum_{j=1}^{m-1} \big(\sum_{i=1}^{j} \delta_i\big)^{\alpha/2} + \sum_{j=m}^{\infty} \big(\sum_{i=j-m+1}^{j} \delta_i\big)^{\alpha/2}\Big]^{1/\alpha}, 0, 0\Big). \blacksquare$$

Remark 3.1. *If the process $\{X_n\}$ is used for modeling the behavior of logarithmic returns, then Proposition 3.2 is of great importance. It gives information about unconditional distributions of the returns, i.e.*

$$X_n \sim S_\alpha\Big(\big(\sum_{j=1}^{\infty} \delta_j^{\alpha/2}\big)^{1/\alpha}, 0, 0\Big), \tag{3.7}$$

and also about unconditional distribution of aggregated returns.

For example, if the R-GARCH$(r, p, 0)$ model with η_n's given by (3.1) describes daily returns, then formula (3.5) determines the unconditional distributions of weekly, monthly, etc. returns, where m denotes the number of trading days within the given interval. Note that in this model all kinds of returns are symmetric stable with the same characteristic exponent α but with different scale parameters. Moreover, by choosing the index of stability of the η_n's appropriately, any unconditionally stable distributed X_n except the normal ones can be constructed. This implies, in turn, that the unconditional distribution of X_n is heavy-tailed.

Corollary 3.1. *The R-GARCH$(r, p, 0)$ process with innovations η_n given by* (3.1) *is stationary symmetric α-stable,* $0 < \alpha < 2$.

Proof. A similar argument as in Proposition 3.2 shows that all finite linear combinations of the X_n's are symmetric stable with the same index of stability α. This, in turn, implies that $\{X_n\}$ is a symmetric α-stable stochastic process and all its finite-dimensional distributions are symmetric α-stable. Moreover, the process $\{X_n\}$ is stationary. Indeed, as $\{\varepsilon_n\}$ and $\{\eta_n\}$ are independent, the characteristic function of the random vector $(X_{n_1+k}, \ldots, X_{n_d+k})$ can be calculated in the following way:

$$\begin{aligned} \mathrm{E}\big(\exp\{i \sum_{j=1}^{d} \xi_j X_{n_j+k}\}\big) &= \mathrm{E}\big(\exp\{i \sum_{j=1}^{d} \xi_j \sqrt{h_{n_j+k}}\,\varepsilon_{n_j+k}\}\big) \\ &= \mathrm{E}\big(\mathrm{E}\big[\exp\{i \sum_{j=1}^{d} \xi_j \sqrt{h_{n_j+k}}\,\varepsilon_{n_j+k}\} \,|\, \eta_{m-1}, \eta_{m-2}, \ldots\big]\big) \\ &= \mathrm{E}\Big(\exp\Big\{-\frac{\sum_{j=1}^{d} \xi_j^2 h_{n_j+k}}{2}\Big\}\Big), \end{aligned}$$

where $m = \max(n_1+k,\ n_2+k, \ldots, n_d+k)$.

By the stationarity of $\{h_n\}$ (see Proposition 3.1),

$$\mathrm{E}\Big(\exp\Big\{-\frac{\sum_{j=1}^{d} \xi_j^2 h_{n_j+k}}{2}\Big\}\Big) = \mathrm{E}\Big(\exp\Big\{-\frac{\sum_{j=1}^{d} \xi_j^2 h_{n_j}}{2}\Big\}\Big),$$

and then

$$\mathrm{E}\big(\exp\{i\sum_{j=1}^{d}\xi_j X_{n_j+k}\}\big) = \mathrm{E}\big(\exp\{i\sum_{j=1}^{d}\xi_j X_{n_j}\}\big).$$

Therefore, for any $d \geqslant 1$ and $n_1, \ldots, n_d$, $k \in \boldsymbol{Z}$,

$$(X_{n_1+k}, \ldots, X_{n_d+k}) \stackrel{d}{=} (X_{n_1}, \ldots, X_{n_d}),$$

and thus the process $\{X_n\}$ is stationary. ■

4. Dependence structure. The covariance function is an extremely powerful tool in the study of stationary Gaussian time series, but it is not defined for stationary symmetric α-stable, $\alpha < 2$, time series. Therefore, let us consider another measure of dependence – the codifference – which is defined for stable time series and reduces to the covariance when $\alpha = 2$.

DEFINITION 4.1. Let $\{X_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ be a symmetric α-stable stationary time series with $0 < \alpha \leqslant 2$. The *codifference*, $\mathrm{CD}(n)$, is then defined by the equation

$$\mathrm{CD}(n) = \mathrm{CD}(X_n, X_0) = \ln \mathrm{E}\exp\{i(X_n - X_0)\} - 2\ln \mathrm{E}\exp\{iX_0\}. \tag{4.1}$$

Properties of the codifference are described in [6].

As we have shown in Corollary 3.1, the R-GARCH$(r, p, 0)$ process with innovations η_n given by (3.1) is stationary symmetric α-stable, $0 < \alpha < 2$. Therefore, the dependence structure of this process cannot be characterized by the covariance and we shall use the codifference. In Theorem 4.1 we show that this measure of dependence is positive and bounded by an exponentially decaying function.

THEOREM 4.1. *Let Q be as in Proposition* 3.1. *Then for the R-GARCH$(r, p, 0)$ model with the innovations η_n given by* (3.1) *and for any $0 < \alpha < 2$ there is a constant K depending only on α and Q such that*

$$0 \leqslant \limsup_{n\to\infty} Q^{(n\alpha)/2}\,\mathrm{CD}(n) \leqslant K. \tag{4.2}$$

Proof. First we show that the codifference for the R-GARCH$(r, p, 0)$ model with innovations η_n given by (3.1) can be described in terms of δ_j's by the formula

$$\mathrm{CD}(n) = \begin{cases} 2\sum_{j=1}^{\infty}\delta_j^{\alpha/2} & \text{for } n = 0, \\ \sum_{j=1}^{\infty}[\delta_{j+n}^{\alpha/2} + \delta_j^{\alpha/2} - (\delta_{j+n}+\delta_j)^{\alpha/2}] & \text{for } n \leqslant 0. \end{cases} \tag{4.3}$$

Let us note that, by Proposition 3.2,

$$\mathrm{E}\exp\{iX_0\} = \exp\{-\sum_{j=1}^{\infty}\delta_j^{\alpha/2}\},$$

and therefore, if $n = 0$, then the formula for $\mathrm{CD}(0)$ reduces to

$$\mathrm{CD}(0) = -2\ln \mathrm{E}\exp\{iX_0\} = 2\sum_{j=1}^{\infty} \delta_j^{\alpha/2},$$

giving the first part of (4.3).

We now focus on the case $n \geqslant 1$. As the sequences $\{\varepsilon_n\}$ and $\{\eta_n\}$ are independent, then conditioning on the η_n's we obtain

$$\text{(4.4)} \quad \mathrm{E}(\exp\{i(X_n - X_0)\}) = \mathrm{E}(\exp\{i(\sqrt{h_n}\,\varepsilon_n - \sqrt{h_0}\,\varepsilon_0)\})$$
$$= \mathrm{E}(\mathrm{E}[\exp\{i(\sqrt{h_n}\,\varepsilon_n - \sqrt{h_0}\,\varepsilon_0)\} \mid \eta_{n-1}, \eta_{n-2}, \ldots]) = \mathrm{E}\left(\exp\left\{-\frac{h_n + h_0}{2}\right\}\right).$$

Since h_n is given by (3.2) for every n, we have

$$h_n + h_0 = \sum_{j=1}^{\infty} \delta_j \eta_{n-j} + \sum_{j=1}^{\infty} \delta_j \eta_{-j} = \sum_{j=1}^{n} \delta_j \eta_{n-j} + \sum_{j=1}^{\infty} (\delta_{j+n} + \delta_j)\eta_{-j},$$

and thus the distribution of $h_n + h_0$ is positive strictly stable with index of stability $\alpha/2$, i.e.

$$h_n + h_0 \sim S_{\alpha/2}\left(2\left[\sum_{j=1}^{n} \delta_j^{\alpha/2} + \sum_{j=1}^{\infty} (\delta_j + \delta_{j+n})^{\alpha/2}\right]^{2/\alpha} \left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right).$$

Applying the formula for the Laplace transform of $h_n + h_0$ in (4.4) we obtain

$$\text{(4.5)} \quad \mathrm{E}(\exp\{i(X_n - X_0)\}) = \exp\left\{-\left[\sum_{j=1}^{n} \delta_j^{\alpha/2} + \sum_{j=1}^{\infty} (\delta_j + \delta_{j+n})^{\alpha/2}\right]\right\}.$$

Therefore, if $n \geqslant 1$, then, by (4.5) and (4.1),

$$\text{(4.6)} \quad \mathrm{CD}(n) = -\sum_{j=1}^{n} \delta_j^{\alpha/2} - \sum_{j=1}^{\infty} (\delta_j + \delta_{j+n})^{\alpha/2} + 2\sum_{j=1}^{\infty} \delta^{\alpha/2}$$
$$= -\sum_{j=1}^{n} \delta_j^{\alpha/2} - \sum_{j=1}^{\infty} (\delta_j + \delta_{j+n})^{\alpha/2} + \sum_{j=1}^{n} \delta_j^{\alpha/2} + \sum_{j=n+1}^{\infty} \delta_j^{\alpha/2} + \sum_{j=1}^{\infty} \delta_j^{\alpha/2}$$
$$= \sum_{j=1}^{\infty} \delta_{j+n}^{\alpha/2} + \sum_{j=1}^{\infty} \delta_j^{\alpha/2} - \sum_{j=1}^{\infty} (\delta_{j+n} + \delta_j)^{\alpha/2},$$

which implies the second part of (4.3).

As we have the formula for $\mathrm{CD}(n)$, let us recall two inequalities, which hold for every $0 < \gamma \leqslant 1$ and $r, s \in \boldsymbol{R}$:

$$\text{(4.7)} \quad |r+s|^{\gamma} \leqslant |r|^{\gamma} + |s|^{\gamma}$$

and

$$\text{(4.8)} \quad \big||r+s|^{\gamma} - |r|^{\gamma} - |s|^{\gamma}\big| \leqslant 2|r|^{\gamma}.$$

Suppose $n \geqslant 1$ is such that $0 \leqslant \delta_j < Q^{-j}$ for each $j > n$. Then, after applying (4.8) to each term in (4.3) (with $r = \delta_{j+n}$ and $\gamma = \alpha/2$), we get

$$Q^{(n\alpha)/2}\,\mathrm{CD}(n) \leqslant 2Q^{(n\alpha)/2}\sum_{j=1}^{\infty}\delta_{j+n}^{\alpha/2} < 2Q^{(n\alpha)/2}\sum_{j=1}^{\infty}Q^{-(j+n)\alpha/2} \tag{4.9}$$

$$= 2\sum_{j=1}^{\infty}Q^{-j\alpha/2} = 2\frac{Q^{-\alpha/2}}{1-Q^{-\alpha/2}}.$$

Moreover, by (4.7), $\delta_j^{\alpha/2}+\delta_{j+n}^{\alpha/2}-(\delta_j+\delta_{j+n})^{\alpha/2} \geqslant 0$, and thus

$$\mathrm{CD}(n) \geqslant 0. \tag{4.10}$$

Formulae (4.9) and (4.10) imply (4.2). ∎

Theorem 4.1 shows that $\mathrm{CD}(n)$ tends to zero exponentially (or faster) for R-GARCH$(r, p, 0)$ processes with positive strictly stable innovations η_n.

Remark 4.1. *If $p = 0$, then*

$$h_n = \sum_{j=1}^{r}\theta_j\eta_{n-j} = \sum_{j=1}^{\infty}\delta_j\eta_{n-j},$$

where $\delta_j = \theta_j$ for $1 \leqslant j \leqslant r$ and $\delta_j = 0$ for $j > r$, and X_n and X_0 are independent for any $n \geqslant r$.

In the case of the R-GARCH(1, 1, 0) process with stable innovations it is possible to find an explicit formula for the codifference and to determine its exact asymptotic behavior.

THEOREM 4.2. *In the case of the R-GARCH(1, 1, 0) process with innovations η_n given by* (3.1) *and with $\theta_1 > 0$ and $0 < \phi_1 < 1$*

$$\lim_{n\to\infty}\phi_1^{(-n\alpha)/2}\,\mathrm{CD}(n) = \frac{\theta_1^{\alpha/2}}{1-\phi_1^{\alpha/2}},$$

i.e. the codifference is asymptotically proportional to $\phi_1^{(n\alpha)/2}$, and thus it tends to zero exponentially.

Proof. In the case of the R-GARCH(1, 1, 0) process with innovations η_n given by (3.1) with $\theta_1 > 0$ and $0 < \phi_1 < 1$, the conditional variance h_n depends on η_{n-1} and h_{n-1}, i.e.

$$h_n = \theta_1\eta_{n-1}+\phi_1 h_{n-1} = \Theta(L)\eta_n+\Phi(L)h_n,$$

where $\Theta(z) = \theta_1 z$ and $\Phi(z) = \phi_1 z$. As $0 < \phi_1 < 1$, the polynomial $1-\Phi(z)$ has no root in the closed unit disk $\{z\colon |z| \leqslant 1\}$. By Proposition 3.1, the δ_j's are the coefficients in the series expansion of $\theta_1 z/(1-\phi_1 z)$, $|z| < 1$. Since

$$\frac{\theta_1 z}{1-\phi_1 z} = \theta_1 z\sum_{j=0}^{\infty}\phi_1^j z^j = \sum_{j=1}^{\infty}\theta_1\phi_1^{j-1}z^j,$$

the δ_j's are given by

$$\delta_j = \theta_1\phi_1^{j-1}, \quad j = 1, 2, \ldots$$

By Corollary 3.1, $\{X_n\}$ is a stationary $S\alpha S$ process. As by (4.3), $\mathrm{CD}(n)$ is expressed in terms of the δ_j's, we get for $n \geqslant 1$

$$\begin{aligned}\mathrm{CD}(n) &= \sum_{j=1}^{\infty} [(\theta_1 \phi_1^{j-1})^{\alpha/2} + (\theta_1 \phi_1^{j+n-1})^{\alpha/2} - (\theta_1 \phi_1^{j+n-1} + \theta_1 \phi_1^{j-1})^{\alpha/2}] \\ &= \theta_1^{\alpha/2} \sum_{j=1}^{\infty} [\phi_1^{(j-1)\alpha/2} (1 + \phi_1^{(n\alpha)/2} - (\phi_1^n + 1)^{\alpha/2})] \\ &= \frac{\theta_1^{\alpha/2}}{1-\phi_1^{\alpha/2}} (1 + \phi_1^{(n\alpha)/2} - (\phi_1^n + 1)^{\alpha/2}).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \phi_1^{(-n\alpha)/2} \mathrm{CD}(n) = \frac{\theta_1^{\alpha/2}}{1-\phi_1^{\alpha/2}} \lim_{n\to\infty} \left[1 + \frac{1-(1+\phi_1^n)^{\alpha/2}}{\phi_1^{(n\alpha)/2}}\right]$$

and, in order to complete the proof, it is enough to notice that

$$\lim_{n\to\infty} \frac{1-(1+\phi_1^n)^{\alpha/2}}{\phi_1^{(n\alpha)/2}} = 0. \quad \blacksquare$$

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