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A NOTE ON DIFFUSIONS IN COMPRESSIBLE ENVIRONMENTS

BY

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Abstract. We study the equation of a motion of a passive tracer in a time-independent turbulent flow in a medium with a positive molecular diffusivity. In [6] the authors have shown the existence of an invariant probability measure for the Lagrangian velocity process. This measure is absolutely continuous with respect to the underlying physical probability for the Eulerian flow. As a result the existence of the Stokes drift has been proved. The results of [6] were derived under some technical condition on the statistics of the Eulerian velocity field. This condition was crucial in the proof in [6]. However, in applications it is difficult to check whether the velocity field satisfies this condition.

In this note we show that the main result of [6] can be stated also without the above-mentioned technical assumption. A somewhat similar result, but for time-dependent flows with different statistical properties, has been shown in [5].

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1. INTRODUCTION

Consider the following Itô stochastic equation describing the motion of a passive tracer in a turbulent flow

$$(1.1)\qquad \begin{aligned} dx(t) &= u\big(x(t)\big)\,dt+\sqrt{2\kappa}\,dw(t),\\ x(0) &= \mathbf{0}, \end{aligned}$$

where $\boldsymbol{u}\colon \boldsymbol{R}\times \boldsymbol{R}^d\times \Omega \to \boldsymbol{R}^d$, the so-called *Eulerian velocity*, is a stationary, strongly mixing, d-dimensional random field given over a certain probability triple

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$(\Omega, \mathscr{V}, \boldsymbol{P})$ and $\boldsymbol{w}(t)$, $t \geqslant 0$, is a standard d-dimensional Brownian motion independent of $\boldsymbol{u}$, given over $\mathscr{T}_1 := (\Sigma, \mathscr{A}, W)$. The parameter $\kappa > 0$ characterizes the strength of the intrinsic molecular diffusivity of the medium.

We are interested in the long run behaviour of the tracer. One of the questions is whether the trajectory process $\boldsymbol{x}(\cdot)$ obeys the law of large numbers, i.e. if there exists $\boldsymbol{v}_*$ (called the *Stokes drift*) such that

$$\lim_{t\to\infty} \frac{\boldsymbol{x}(t)}{t} = \boldsymbol{v}_* \text{ almost surely.}$$

It has been proved in [6] that for sufficiently regular velocity fields $\boldsymbol{u}(\cdot)$ with a finite-dependency range (see condition (FDR) below) and satisfying a certain regularity condition in the measure theoretic sense (see (RAC) below) there exists an absolutely continuous change of measure such that the Lagrangian process is stationary and ergodic with respect to the new measure. The proof of this result is based on an application of the Lasota–York theorem, which provides the existence of invariant densities for Markov operators satisfying a certain lower bound condition, see Theorem 5.6.2 of [7].

A certain result on σ-algebra factorization due to Skorokhod (see [9]) has been applied in the proof of the main result in [6]. In order to be able to use the factorization result the authors needed some technical condition on the velocity field. This condition is presented below (see (RAC)). In many applications this assumption becomes cumbersome. Here we show that the result of [6] is valid also without this assumption. The main idea is similar to the one used in [5]. We approximate velocity fields that do not satisfy (RAC) with the fields which do satisfy this condition.

2. NOTATION AND FORMULATION OF THE MAIN RESULT

We will assume that $\kappa = 1$ in (1.1).

For any $L > 0$ we denote by $\mathfrak{X}_L := C([0, L]; \boldsymbol{R}^d)$ and $\mathfrak{X} := C([0, +\infty); \boldsymbol{R}^d)$. These spaces are equipped with the standard topology of uniform convergence on compact sets. For any $t \geqslant 0$ we denote by $\Pi(t)\colon \mathfrak{X} \to \boldsymbol{R}^d$ the canonical projection $\Pi(t)(\pi) := \pi(t)$, $\pi \in \mathfrak{X}$. Let $\mathscr{M}_t := \sigma\{\Pi(s)\colon s \leqslant t\}$, $t \geqslant 0$, be the canonical filtration on $\mathfrak{X}$. We let $\mathscr{M} := \bigvee_{t\geqslant 0} \mathscr{M}_t$. By $\mathscr{P}$ and $\mathscr{P}_L$ we denote the spaces of all Borel probability measures on $\mathfrak{X}$ and $\mathfrak{X}_L$, respectively. By W and W_L we denote the standard Wiener measure on $(\mathfrak{X}, \mathscr{M})$ and its restriction to $\mathscr{M}_L$, respectively. For any $h \geqslant 0$ we have the shift operator $\theta_h\colon \mathfrak{X} \to \mathfrak{X}$ given by $\theta_h(\pi)(t) := \pi(t+h)$ for all $t \geqslant 0$, $\pi \in \mathfrak{X}$.

Let (Ω, d) be a Polish space with a Borel probability measure $\boldsymbol{P}$ on it. We denote by $\mathscr{B}(\Omega)$ the σ-algebra of Borel sets on Ω and by $\boldsymbol{E}[\cdot]$ the corresponding mathematical expectation. Let $\mathscr{N}$ be the σ-ring of $\boldsymbol{P}$-null sets in $\overline{\mathscr{B}(\Omega)}$, the completion of $\mathscr{B}(\Omega)$. Unless otherwise stated, we will assume that any sub-σ-algebra of $\overline{\mathscr{B}(\Omega)}$ contains $\mathscr{N}$. For abbreviation sake we write $L^p := L^p(\mathscr{T}_0)$, where $\mathscr{T}_0 := (\Omega, \overline{\mathscr{B}(\Omega)}, \boldsymbol{P})$.

Let $\boldsymbol{w}(\cdot)$ be a standard d-dimensional Brownian motion given over a certain probability space $\mathscr{T}_1 := (\Sigma, \mathscr{A}, W)$.

We suppose that $\boldsymbol{u}: \Omega \to \boldsymbol{R}^d$ is a random vector over $\mathscr{T}$ satisfying the following conditions:

(A) *Existence of a drift*. We assume that $|\boldsymbol{v}| > \|\tilde{\boldsymbol{u}}\|_{L^\infty}$, where $\boldsymbol{v} := \boldsymbol{E}\boldsymbol{u}$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{v}$.

This assumption guarantees that the mean drift dominates fluctuations, i.e. there exists $\delta > 0$ such that

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \hat{\boldsymbol{v}} \geqslant 2\delta > 0 \quad \boldsymbol{P}\text{-a.s.} \tag{2.1}$$

for all $\boldsymbol{x} \in \boldsymbol{R}^d$. Here $\hat{\boldsymbol{v}} = \boldsymbol{v}/|\boldsymbol{v}|$.

(S) $\boldsymbol{u}$ *is stationary*, i.e. for any finite collection $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in \boldsymbol{R}^d$ and any $\boldsymbol{x} \in \boldsymbol{R}^d$ the laws of $(\boldsymbol{u}(\boldsymbol{x}_1), \ldots, \boldsymbol{u}(\boldsymbol{x}_N))$ and $(\boldsymbol{u}(\boldsymbol{x}_1 + \boldsymbol{x}), \ldots, \boldsymbol{u}(\boldsymbol{x}_N + \boldsymbol{x}))$ coincide.

(FDR) *Finite dependence range*. For any $r > 0$ we denote by $\mathscr{F}_r^i$ and $\mathscr{F}_r^e$ the σ-algebras generated by $\boldsymbol{u}(\boldsymbol{x})$, $|\boldsymbol{x}| \leqslant r$ and $\boldsymbol{u}(\boldsymbol{x})$, $|\boldsymbol{x}| \geqslant r$, respectively. We assume that there exists $r_0 > 0$ such that for any $r > 0$ the σ-algebras $\mathscr{F}_r^i$ and $\mathscr{F}_{r+r_0}^e$ are independent.

(RH) *Regularity*. For any $\omega \in \Omega$ the field $\boldsymbol{u}(\cdot; \omega)$ is of class C^1 and there exists a deterministic positive constant U such that $\|\tilde{\boldsymbol{u}}(\cdot; \omega)\|_{W^{1,\infty}(\boldsymbol{R}^d)} \leqslant U$. Here $\|\cdot\|_{W^{1,\infty}(\boldsymbol{R}^d)}$ denotes the norm in the Sobolev space $W^{1,\infty}(\boldsymbol{R}^d)$.

In [6], in addition to the condition (RH), the following regularity (in the measure theoretic sense) condition has been imposed:

(RAC) All distributions of vectors $(\boldsymbol{u}(\boldsymbol{x}_1), \ldots, \boldsymbol{u}(\boldsymbol{x}_N))$, where $N \geqslant 1$, $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, $i \neq j \in \{1, \ldots, N\}$, are absolutely continuous with respect to the $N \cdot d$-dimensional Lebesgue measure.

In [4], conditions (RH) and (RAC) were denoted together by (R). The absolute continuity assumption (RAC) is somewhat restrictive. The purpose of this note is to show that the main result of [6] holds also without this assumption. Before we state the main result, define a stochastic process V over $(\Omega \times \mathscr{X}, \overline{\mathscr{B}(\Omega)} \otimes \mathscr{M}, P_0)$

$$V(t; \omega, \pi) := \boldsymbol{u}(\pi(t), \omega), \qquad t \geqslant 0. \tag{2.2}$$

Let $Q_{\boldsymbol{x},\omega}$ be the martingale solution of (1.1) for a fixed realization of $\omega \in \Omega$ and subject to the initial condition $\boldsymbol{x}(0) = \boldsymbol{0}$. Denote by $M_{\boldsymbol{x},\omega}$ the respective mathematical expectation.

Let $\{T_{\boldsymbol{x}}: \Omega \to \Omega, \boldsymbol{x} \in \boldsymbol{R}^d\}$ be an additive group of transformations, satisfying the following conditions:

(M) *Measurability*. The mapping $(\boldsymbol{x}, \omega) \mapsto \mathbf{1}(T_{\boldsymbol{x}}\omega)$, $(\boldsymbol{x}, \omega) \in \boldsymbol{R}^d \times \Omega$, is jointly measurable for any $A \in \mathscr{B}(\Omega)$.

(MP) *$\boldsymbol{P}$-preserving*, i.e. $\boldsymbol{P}(T_{\boldsymbol{x}} A) = \boldsymbol{P}(A)$ for all $\boldsymbol{x} \in \boldsymbol{R}^d$, $A \in \overline{\mathscr{B}(\Omega)}$.

(SC) *Stochastic continuity*. We have

$$\lim_{|\boldsymbol{x}| \to 0} \boldsymbol{P}[\mathbf{1}_A(T_{\boldsymbol{x}}\omega) - \mathbf{1}_A(\omega) \geqslant \eta] = 0 \qquad \forall \eta > 0,\ A \in \overline{\mathscr{B}(\Omega)}.$$

The main result of the present paper is the following theorem.

THEOREM 2.1. *Suppose that $\boldsymbol{u}(\cdot)$ is a stationary velocity field satisfying* (A), (FDR), (RH), *and the trajectory $\boldsymbol{x}(t)$, $t \geqslant 0$, is given by* (1.1) *with $\kappa > 0$. Then there exists a probability measure μ on $\big(\Omega \times \mathscr{X}, \overline{\mathscr{B}(\Omega) \otimes \mathscr{M}}\big)$ for the Lagrangian velocity process $\boldsymbol{U}(t) := \boldsymbol{u}(\boldsymbol{x}(t))$, $t \geqslant 0$. This process is stationary and ergodic with respect to μ.*

Ergodicity of the above measure is understood in the following sense. Any set $A \in \mathscr{B}(\mathscr{X})$ such that

$$\text{(E)} \qquad \int \big|\mathbf{1}_{\theta_h(A)}\big(V(\cdot)\big) - \mathbf{1}_A\big(V(\cdot)\big)\big|\, d\mu = 0 \qquad \forall h \geqslant 0$$

must be μ-trivial, i.e.

$$\text{(2.3)} \qquad \mu[(\omega, \pi)\colon V(\cdot; \omega, \pi) \in A] = 0 \text{ or } 1.$$

Obviously, the above theorem yields the existence of the Stokes drift under the assumptions of Theorem 2.1, i.e.

$$\text{(2.4)} \qquad \lim_{t \to \infty} \frac{\pi(t)}{t} = \int \boldsymbol{u}(\mathbf{0})\, d\mu \quad P_0\text{-a.s.}$$

Remark 2.2. It should be noted that establishing the above theorem in the present setting is more difficult than in the case of time-dependent flows considered in [5] and [6]. In these two results the velocity field is supposed to be time-space stationary. Moreover, the field is supposed to decorrelate as the time passes. This means that it is enough to wait sufficiently long fixed time and the velocity field will be independent of its history. In the present situation we have the condition (FDR), but it alone does not imply that the moving particle will see new realizations of the environment independent of what it has seen in the past. We should also impose some condition which guarantees that the particle will see new parts of the environment. This is achieved with the condition (A). Due to diffusive character of the flow, the tracer has to travel for some random time in order to see new realizations of the environment, independent of the past.

3. APPROXIMATION OF GENERAL FLOWS BY VELOCITY FIELDS SATISFYING (RAC)

In this section we will construct a sequence of velocity fields $(\boldsymbol{u}_n)$ satisfying the condition (RAC), which approximates the velocity field $\boldsymbol{u}$.

Without loss of generality we may assume that the probability space $\mathcal{T}_0$ is sufficiently rich to support a stationary Gaussian random field $\boldsymbol{g}\colon \boldsymbol{R}^d \times \Omega \to \boldsymbol{R}^d$, which satisfies the following conditions:

(G0) $\boldsymbol{g}$ and $\boldsymbol{u}$ are independent.

(G1) $\boldsymbol{g}$ is centered, i.e. $\boldsymbol{E}\boldsymbol{g}(\boldsymbol{0}) = 0$.

(G2) $\boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{y})$ are uncorrelated if $|\boldsymbol{x}-\boldsymbol{y}| > r_0$ (r_0 appeared in condition (FDR)).

This condition implies (see [8]) that the field $\boldsymbol{g}(\cdot)$ satisfies (FDR) with $\mathcal{G}_r^i$ and $\mathcal{G}_{r+r_0}^e$ in place of $\mathcal{F}_r^i$ and $\mathcal{F}_{r+r_0}^e$, respectively. Here $\mathcal{G}_r^i$ and $\mathcal{G}_r^e$ are defined analogously as $\mathcal{F}_r^i$ and $\mathcal{F}_r^e$. Namely, for any $r > 0$, $\mathcal{G}_r^i$ and $\mathcal{G}_r^e$ denote the σ-algebras generated by $\boldsymbol{g}(\boldsymbol{x})$ with $|\boldsymbol{x}| < r$ and $|\boldsymbol{x}| > r$, respectively.

(G3) Realizations of $\boldsymbol{g}$ are C^∞-regular $\boldsymbol{P}$-a.s.

(G4) $\boldsymbol{g}$ satisfies (RAC).

Let us show the construction of such a field. Let $W(dx)$ be the $\boldsymbol{R}$-valued white noise over $\mathcal{T}_0$, which is independent of $\boldsymbol{u}$. Let $g\colon \boldsymbol{R}^d \to \boldsymbol{R}$ be a compactly supported function of class C^∞. Its support is contained in some ball $B_R(\boldsymbol{0})$ of radius $R > 0$ centered at $\boldsymbol{0}$. Define

$$g_1(\boldsymbol{x}, \omega) := \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} g(\boldsymbol{x}-\boldsymbol{y})\, W(d\boldsymbol{y}),$$

where $\boldsymbol{x} = (x_1, \dots, x_d)$. Let $g_2, \dots, g_d$ be independent copies of g_1. The field $\boldsymbol{g} = (g_1, \dots, g_d)$ satisfies (G0)–(G4).

Let us now define the velocity field $\boldsymbol{u}_n$ by

$$\boldsymbol{u}_n(\boldsymbol{x}) := \boldsymbol{u}(\boldsymbol{x}) + \frac{1}{n+n_0}\, \phi\big(\boldsymbol{g}(\boldsymbol{x})\big),$$

where $\phi\colon \boldsymbol{R}^d \to \boldsymbol{R}^d$ is given by

$$\phi(\boldsymbol{x}) := \frac{\boldsymbol{x}}{(1+\|\boldsymbol{x}\|^2)^{1/2}} \quad \text{and} \quad \frac{1}{n_0} < \delta.$$

By the conditions (G0)–(G4) it is easy to check that the field $\boldsymbol{u}_n$ satisfies all the conditions (S), (RH), (FDR), (RAC). Moreover, recalling (2.1), we see that the mean drift dominates its fluctuations:

$$\boldsymbol{u}_n(\boldsymbol{x}) \cdot \hat{\boldsymbol{v}} \geqslant \delta > 0 \quad \boldsymbol{P}\text{-a.s.} \tag{3.1}$$

for all $x \in R^d$ and $n \in N$. It is also easy to check that u_n, $\nabla_x u_n$ converge to u, $\nabla_x u$, respectively, as $n \to +\infty$, uniformly on compact sets, P-a.s.

Analogously as $Q_{x,\omega}$, we define $Q_{x,\omega}^{(n)}$ as the martingale solution of (1.1) (with the obvious substitution of u_n for u) for a fixed realization of $\omega \in \Omega$ and subject to the initial condition $x(0) = 0$. By $M_{x,\omega}^{(n)}$ we denote the respective mathematical expectation.

Since the fields u and g are homogeneous, without any loss of generality we may assume that there exists a group $T_x: \Omega \to \Omega$, $x \in R^d$, satisfying the assumptions (MP), (M), (SC) of Section 2, such that

(3.2) $$u(x; \omega) = u(0; T_x \omega),\ g(x; \omega) = g(0; T_x \omega) \qquad \forall x \in R^d.$$

Hence, in particular,

(3.3) $$u_n(x; \omega) = u_n(0; T_x \omega) \qquad \forall n \geqslant 1 \text{ and } \forall x \in R^d.$$

4. NON-RETRACTION TIMES

Let us introduce some more notation after [6]. We want to introduce a family of random variables, called *non-retraction times*. They describe times after which no retraction of the diffusion can occur in the direction pointed out by the mean velocity. These are not stopping times. The notion appeared first, in a discrete setting, for random walks on a random lattice in [11].

For any $\pi \in \mathfrak{X}$, $l \in [0, +\infty)$ we let

(4.1) $$D(l; \pi) := \min[t \geqslant 0:\ \hat{v} \cdot \pi(t) \leqslant -1 + l].$$

For abbreviation sake we write $D := D(\hat{v} \cdot \pi(0))$,

$$U_n(\pi) := \min[t \geqslant 0:\ \hat{v} \cdot \pi(t) \geqslant u], \qquad \tilde{U}_u(\pi) := \min[t \geqslant 0:\ \hat{v} \cdot \pi(t) \leqslant u]$$

and

(4.2) $$M_*(\pi) := \sup[\hat{v} \cdot (\pi(t) - \pi(0)):\ 0 \leqslant t \leqslant D(\pi)].$$

The last random variable is defined for those π for which $D(\pi) < +\infty$.

For any $t \geqslant 0$ we define also

(4.3) $$A(t) := [\pi:\ \inf_{s \in [0,t]} (\pi(s) \cdot \hat{v} - \pi(0) \cdot \hat{v}) \geqslant -1].$$

We introduce the sequence of $(\mathcal{M}_t)$-stopping times $(S_k)_{k \geqslant 0}$, $(R_k)_{k \geqslant 0}$ and the sequence of maxima $(M_k)_{k \geqslant 0}$ letting

(4.4) $$\begin{gathered} S_0 := 0, \quad R_0 := 0, \quad M_0 := \hat{v} \cdot \pi(0), \\ S_1 := U_{M_0 + r_0 + 1} \leqslant +\infty, \quad R_1 := D \circ \theta_{S_1} + S_1 \leqslant +\infty, \\ M_1 := \max[\hat{v} \cdot \pi(t),\ 0 \leqslant t \leqslant R_1] \leqslant +\infty, \end{gathered}$$

where $r_0 > 0$ is as in (FDR).

By induction we set for any $k \geqslant 1$

$$S_{k+1} := U_{M_k + r_0 + 1}, \qquad R_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}, \tag{4.5}$$
$$M_{k+1} := \max[\hat{v}\cdot\pi(t),\ 0 \leqslant t \leqslant R_{k+1}].$$

The following summarizes the properties of the above-defined stopping times and random variables (see Lemmas 3.1 and 3.2 of [6]).

LEMMA 4.1. *There exists a deterministic constant γ such that for all $n \in N$*

$$Q^{(n)}_{x,\omega}[D = \infty] \geqslant \gamma \quad P\text{-a.s.}, \tag{4.6}$$

$$Q^{(n)}_{x,\omega}[R_k < \infty] \leqslant (1-\gamma)^k \qquad \forall k \geqslant 1,\ P\text{-a.s.} \tag{4.7}$$

for all $x \in R^d$.

Remark 4.2. In fact, in Lemmas 3.1 and 3.2 of [6], the path measures $Q^{(n)}_{x,\omega}$ do not depend on n. The careful inspection of the proof of the mentioned results in [6] shows that the constants appearing in Lemma 4.1 are uniform for all $n \in N$.

Define, after [6], $K := \inf[k \geqslant 1:\ R_k = +\infty]$, with the convention that $K = \infty$ if the set over which the infimum is taken is empty.

We have (see Corollary 3.3 in [6]) the following lemma which is a consequence of Lemma 4.1.

LEMMA 4.3. *For all $n \in N$ and $x \in R^d$, we have*

$$Q^{(n)}_{x,\omega}[K < \infty] = 1 \quad P\text{-a.s.}, \qquad Q^{(n)}_{x,\omega}[S_K < \infty] = 1 \quad P\text{-a.s.}$$

The above corollary allows us to define the first non-retraction time as $\tau_1 := S_K < \infty$, $P^{(n)}_0$-a.s. The subsequent times are defined recursively: $\tau_m := \tau_m + \tau_1 \circ \theta_{\tau_m}$ for $m > 1$.

The following lemma summarizes important properties of the non-retraction time τ_1.

LEMMA 4.4. *There exist constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5 (independent of ω) such that for all $x \in R^d$ and $n \in N$*

$$Q^{(n)}_{x,\omega}[2^m \leqslant M_* < 2^{m+1},\ D < \infty] \leqslant \gamma_4 \exp(-\gamma_5 2^m), \tag{4.8}$$

$$M^n_{x,\omega}[M_*^4,\ D < \infty] \leqslant \gamma_1, \tag{4.9}$$

$$M^n_{x,\omega}[\hat{v}\cdot\pi(\tau_1)]^4 \leqslant \gamma_2, \tag{4.10}$$

$$Q^{(n)}_{x,\omega}[\tau_1 > u] \leqslant \frac{\gamma^3}{1+u^4} \quad \text{for } u > 0 \tag{4.11}$$

and, in consequence,

$$E^{(n)}_0[\tau_1^2,\ D = \infty] < \infty. \tag{4.12}$$

The proof of this lemma will be postponed till the Appendix.

5. THE CONSTRUCTION OF THE INVARIANT MEASURE

For any $a, b \in \mathbf{R} \cup \{-\infty, +\infty\}$, $a \leqslant b$, we let $\mathscr{V}_{a,b}$ be the σ-algebra generated by $\boldsymbol{u}(x)$, $\boldsymbol{g}(x)$, where $a \leqslant x \cdot \hat{v} \leqslant b$. We write $\mathscr{V}_a$ for $\mathscr{V}_{-\infty,a}$. Let $\mathscr{T}_2 := (\Omega, \mathscr{V}_0, \boldsymbol{P})$,

$$\boldsymbol{P}_D^{(n)}(d\omega) := \frac{Q_\omega^{(n)}[D = +\infty]}{P_0^{(n)}[D = +\infty]} \boldsymbol{P}(d\omega),$$

$$P_D^{(n)}(d\omega, d\pi) := \frac{\mathbf{1}_{[D(\pi) = +\infty]}}{P_0^{(n)}[D = +\infty]} P_0^{(n)}(d\omega, d\pi)$$

and $\mathscr{T}_D := (\Omega, \mathscr{V}_0, \boldsymbol{P}_D)$. Note that in the light of (4.6) of Lemma 4.1, $\boldsymbol{P}_D^{(n)}$ is equivalent to $\boldsymbol{P}$. Analogously we define

$$\boldsymbol{P}_D(d\omega) := \frac{Q_\omega[D = +\infty]}{P_0[D = +\infty]} \boldsymbol{P}(d\omega), \qquad P_D(d\omega, d\pi) := \frac{\mathbf{1}_{[D(\pi) = +\infty]}}{P_0[D = +\infty]} P_0(d\omega, d\pi).$$

Also, for any probability triple $\mathscr{T}$ the symbol $\mathscr{D}(\mathscr{T})$ denotes the set of all probability densities with respect to the relevant probability measure, i.e. the non-negative elements of $L^1(\mathscr{T})$ whose integral equals 1.

An important role in the proof of the main result of [6] is played by a certain transport operator (see Section 4.2 in [6]). For any $n \in \mathbf{N}$, the field $\boldsymbol{u}_n(\cdot)$ has absolutely continuous finite-dimensional distributions (see condition (RAC)), i.e. for any $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$, $m \geqslant 1$ ($\boldsymbol{x}_i \neq \boldsymbol{x}_j$ for $i \neq j$), the random vector $(\boldsymbol{u}_n(\boldsymbol{x}_1), \ldots, \boldsymbol{u}_n(\boldsymbol{x}_m))$ is absolutely continuous with respect to the $m \cdot d$-dimensional Lebesgue measure. According to Section 4.2 of [6], there exists a density preserving operator $\mathscr{Q}_n$: $L^1(\mathscr{T}_D^n) \to L^1(\mathscr{T}_D^n)$ satisfying the condition

$$\text{(5.1)} \qquad \int M_\omega^{(n)}[G(T_{\pi(\tau_1)}(\omega)), D = \infty] F(\omega) \boldsymbol{P}(d\omega) = \int G(\omega) \mathscr{Q}_n F(\omega) Q_\omega^{(n)}[D = \infty] \boldsymbol{P}(d\omega)$$

for any F and G that are correspondingly $\mathscr{V}_0$- and $\mathscr{V}_{0,\infty}$-measurable. We call this operator a *transport operator*.

Let us recall the construction of the operator $\mathscr{Q}_n$ in more detail. Since $\boldsymbol{u}_n(\cdot)$ satisfies the condition (RAC) in addition to (RH), the filtration $(\mathscr{V}_t)_{t \geqslant 0}$ admits a *factorization* with respect to $\mathscr{V}_0$, i.e. for any $t \geqslant 0$ there exists a σ-algebra $\mathscr{R}^t$ such that $\mathscr{V}_0$ and $\mathscr{R}^t$ are $\boldsymbol{P}$-independent and $\mathscr{V}_t$ is generated by $\mathscr{V}_0$ and $\mathscr{R}^t$. Let $\mathscr{R} := \bigvee_{t \geqslant 0} \mathscr{R}^t$.

The factoring property has an important consequence. Let us put $\mathscr{T}_3 := (\Omega, \mathscr{R}, \boldsymbol{P})$ and let $\mathscr{T}_2 \otimes \mathscr{T}_3 := (\Omega \times \Omega, \mathscr{V}_0 \otimes \mathscr{R}, \boldsymbol{P} \otimes \boldsymbol{P})$. The condition (RAC) implies (see Section 4.1 in [6]; see also *ibidem*, Appendix B) the existence of an isometric isomorphism

$$\mathscr{X}: L^p(\mathscr{T}_1) \to L^p(\mathscr{T}_2 \otimes \mathscr{T}_3) \qquad (p \in [1, \infty])$$

such that

(Z1) $\mathscr{Z}F \geqslant 0$ for $F \geqslant 0$ and $\mathscr{Z}\mathbf{1} = \mathbf{1}$;

(Z2) for any $F_1, \ldots, F_N \in L^p(\mathscr{T}_1)$ and $\Phi\colon \boldsymbol{R}^N \to \boldsymbol{R}$ bounded and continuous we have

$$\mathscr{Z}\big(\Phi(F_1, \ldots, F_N)\big) = \Phi(\mathscr{Z}F_1, \ldots, \mathscr{Z}F_N);$$

(Z3) $\mathscr{Z}F(\omega, \omega') = F(\omega)$ for all $F \in L^p(\mathscr{T}_2)$, $\mathscr{Z}G(\omega, \omega') = G(\omega')$ for all $G \in L^p(\mathscr{T}_3)$;

(Z4) $\mathscr{Z}F$ is $\mathscr{V}_0 \otimes \mathscr{R}^t$-measurable if F is $\mathscr{V}_t$-measurable for any $t \geqslant 0$.

Let $\mathscr{T}_W := (\mathscr{X}, \mathscr{M}, W)$, where W denotes the standard Wiener measure. Then

$$v_L^{(n)}(\pi, \omega) := \exp\left\{\int_0^L \boldsymbol{u}_n\big(\pi(s), \omega\big)\, d\pi(s) - \frac{1}{2}\int_0^L \big|\boldsymbol{u}_n\big(\pi(s), \omega\big)\big|^2\, ds\right\}$$

is the Radon–Nikodym derivative of $Q_{\omega,L}^{(n)}$ with respect to W_L, the restriction of W to $\mathscr{X}_L$. Here $Q_{\omega,L}^{(n)}$ denotes the restriction of $Q_\omega^{(n)}$ to $\mathscr{M}_L$ for $L > 0$. Let $\int_0^L \boldsymbol{u}_n\big(\pi(s), \omega\big)\, d\pi(s)$ denote the stochastic integral with respect to the Wiener process $\pi(\cdot)$ over the probability space $\mathscr{T}_W$. The Radon–Nikodym derivative of $Q_{\omega,L}$ with respect to W_L, denoted by v_L, can be defined analogously. Let $\boldsymbol{F}_k$ denote the law in $\boldsymbol{R}^d$ of the random vector $\pi(S_k)$ over $\mathscr{T}_W$.

One can show that

$$\mathscr{Z}\big(v_L^{(n)}(\pi; \bullet)\big)(\omega, \omega') = \bar{v}_L^{(n)}(\pi; \omega, \omega'), \tag{5.2}$$

where

$$\bar{v}_L^{(n)}(\pi; \omega, \omega') := \exp\left\{\int_0^L \boldsymbol{U}_n\big(\pi(s); \omega, \omega'\big)\, d\pi(s) - \frac{1}{2}\int_0^L \big|\boldsymbol{U}_n\big(\pi(s); \omega, \omega'\big)\big|^2\, ds\right\} \quad \forall L > 0.$$

Here

$$\boldsymbol{U}_n(\boldsymbol{x}) = \mathscr{Z}\, \boldsymbol{u}_n(\boldsymbol{x}) \in L^\infty(\mathscr{T}_2 \otimes \mathscr{T}_3).$$

Remark 5.1. In fact, we can find a modification of $\boldsymbol{U}_n$ defined over $\mathscr{T}_2 \otimes \mathscr{T}_3$ that is of C^1-class of regularity and such that $\|\tilde{\boldsymbol{U}}_n(\cdot; \omega, \omega')\|_{W^{1,\infty}(\boldsymbol{R}^d)} \leqslant U + 1$ for all (ω, ω'), where $\tilde{\boldsymbol{U}}_n(\cdot; \omega, \omega') := \boldsymbol{U}_n(\cdot; \omega, \omega') - \boldsymbol{v}$ and U is as in (RH).

The linear operators $\mathscr{Q}_n$ satisfying (5.1), whose existence has been announced above, are defined in the following way. For any bounded and $\mathscr{V}_0$-measurable function F define

$$\mathscr{Q}_n F(\omega') := \int \mathscr{K}^{(n)}(\omega, \omega')\, F(\omega)\, \boldsymbol{P}(d\omega), \tag{5.3}$$

where

$$\mathscr{K}^{(n)}(\omega, \omega') := \sum_{k,L=1}^{\infty} \int_{\boldsymbol{R}^d} \mathscr{M}_{k,L,\boldsymbol{x}}^{(n)}(\omega, T_{-\boldsymbol{x}}\omega')\, \boldsymbol{F}_k(d\boldsymbol{x}), \tag{5.4}$$

and

$$\mathcal{M}^{(n)}_{k,L,x}(\omega, \omega') := M_{k,L,x}[\bar{v}^{(n)}_{S_k}(\pi, \omega, \omega'), A(S_k), L-1 \leqslant S_k < L]. \tag{5.5}$$

Due to Proposition 4.4 of [6] this operator can be extended to a density preserving operator $\mathcal{Q}_n$: $L^1(\mathcal{T}^n_D) \to L^1(\mathcal{T}^n_D)$.

We will show that $\mathcal{Q}_n$ has an invariant density. Precisely we will show the following lemma:

LEMMA 5.2. *For any $n \in N$, there exists a $\mathcal{V}_0$-measurable, strictly positive element $H^{(n)}_* \in \mathcal{D}(\mathcal{T}^n_D) \cap L^2(\mathcal{T}^n_D)$ such that $\mathcal{Q}_n H^{(n)}_* = H^{(n)}_*$. In addition, for any $F \in \mathcal{D}(\mathcal{T}^n_D)$*

$$\lim_{m\to\infty} \int |(\mathcal{Q}_n)^m F(\omega) - H^{(n)}_*(\omega)| P^{(n)}_D(d\omega) = 0. \tag{5.6}$$

We will begin with the following lemma:

LEMMA 5.3. *Let $p \in (1, 2)$. There exists a constant $C > 0$, independent of n, such that*

$$\|\mathcal{Q}_n G\|_{L^2(P^{(n)}_D)} \leqslant C\|G\|_{L^p(P^{(n)}_D)} \qquad \forall G \in L^p(P^{(n)}_D). \tag{5.7}$$

Proof. Let q be the coefficient adjoint to p, i.e. $1/p+1/q = 1$. By the definition of the operator $\mathcal{Q}_n$, the square of the left-hand side of (5.7) can be majorized by

$$\frac{1}{\gamma P^{(n)}_0[D=\infty]} \int \Big(\int \mathcal{K}^{(n)}(\omega, \omega') Q^{(n)}_{\omega'}[D=\infty]\, G(\omega) P(d\omega)\Big)^2 P(d\omega') \tag{5.8}$$
$$\leqslant \frac{\|G\|^2_{L^p(P)}}{\gamma P^{(n)}_0[D=\infty]} \int \Big(\int \big(\mathcal{K}^{(n)}(\omega, \omega') Q^{(n)}_{\omega'}[D=\infty]\big)^q P(d\omega)\Big)^{2/q} P(d\omega').$$

Applying the definition of $\mathcal{K}^{(n)}(\omega, \omega')$, we see that the right-hand side of the inequality in (5.8) equals

$$\frac{\|G\|^2_{L^p(P)}}{\gamma P^{(n)}_0[D=\infty]} \int \Big(\int \Big(\int \sum_{k,L=1}^{\infty} \bar{v}_{S_k}(\pi, \omega, \omega')$$
$$\times \mathbf{1}_{A(S_k)} \mathbf{1}_{[L-1,L)}(S_k)\, Q^{(n)}_{T_{\pi(S_k)}\omega'}[D=\infty]\, W(d\pi)\Big)^q P(d\omega)\Big)^{2/q} P(d\omega'),$$

$$\frac{\|G\|^2_{L^p(P)}}{\gamma P^{(n)}_0[D=\infty]} \tag{5.9}$$
$$\times \int \Big(\int \Big(\sum_{k=1}^{\infty} M_{\omega,\omega'}\big[Q^{(n)}_{T_{\pi(S_k)}\omega'}[D=\infty], A(S_k), S_k < \infty\big]\Big)^q P(d\omega)\Big)^{2/q} P(d\omega'),$$

$$\frac{\|G\|^2_{L^p(P)}}{\gamma P^{(n)}_0[D=\infty]} \tag{5.10}$$
$$\times \Big(\int\int \Big(\sum_{k=1}^{\infty} M_{\omega,\omega'}\big[Q^{(n)}_{T_{\pi(S_k)}\omega'}[D=\infty], A(S_k), S_k < \infty\big]\Big)^q P(d\omega) P(d\omega')\Big)^{2/q},$$

where the passage from (5.9) to (5.10) follows from Jensen's inequality. Using the properties (Z3) and (Z4) of the operator $\mathscr{Z}$, and recalling the definition of the stopping time τ_1, we conclude that (5.10) equals

$$(5.11)\qquad \frac{\|G\|^2_{L^p(P)}}{\gamma P_0^{(n)}[D=\infty]}\Big(\int \big(Q_\omega^{(n)}[D=\infty,\ \tau_1<\infty]\big)^q P(d\omega)\Big)^{2/q} \leqslant \frac{\|G\|^2_{L^p(P)}}{\gamma P_0^{(n)}[D=\infty]} \leqslant \frac{\|G\|^2_{L^p(P_D^{(n)})}}{\gamma^{1+2/p}},$$

where the last inequality follows from the definition of $P_D^{(n)}$ and the lower bound for $Q_\omega^{(n)}[D=\infty]$. ∎

Before we proceed with the proof of Lemma 5.2 we observe the following.

LEMMA 5.4. *For every $n \in N$, there exists an $L^1(P_D^{(n)})$-weakly compact set $\mathfrak{K}_n \subset \mathscr{D}(\mathscr{T}_D^n)$ such that*

$$(5.12)\qquad \lim_{m\to\infty}\inf_{K\in\mathfrak{K}_n} \|(\mathscr{Q}_n)^m G - K\|_{L^1(P_D^{(n)})} = 0 \qquad \forall G \in \mathscr{D}(\mathscr{T}_D^n).$$

Proof. Observe that there exists $C_1 > 0$ such that, for all $G \in \mathscr{D}(\mathscr{T}_D^n) \cap L^2(P_D^{(n)})$,

$$(5.13)\qquad \limsup_{m\to\infty} \|(\mathscr{Q}_n)^m G\|_{L^2(P_D^{(n)})} \leqslant C_1.$$

Indeed, the application of Lemma 5.3 yields that, for $\theta \in (0, 1)$ (such that $\theta + (1-\theta)/2 = 1/p$),

$$(5.14)\qquad \|(\mathscr{Q}_n)^{m+1} G\|_{L^2(P_D^{(n)})} \leqslant C\,\|(\mathscr{Q}_n)^m G\|_{L^p(P_D^{(n)})} \leqslant C\,\|(\mathscr{Q}_n)^m G\|^{\theta}_{L^1(P_D^{(n)})} \|(\mathscr{Q}_n)^m G\|^{1-\theta}_{L^2(P_D^{(n)})} = C\,\|(\mathscr{Q}_n)^m G\|^{1-\theta}_{L^2(P_D^{(n)})},$$

where the last equality follows from Proposition 4.4 of [6]. Iterating (5.14) we get

$$(5.15)\qquad \|(\mathscr{Q}_n)^{m+1} G\|_{L^2(P_D^{(n)})} \leqslant C^{\sum_{i=0}^{m}(1-\theta)^m} \|G\|^{(1-\theta)^{m-1}}_{L^2(P_D^{(n)})}.$$

Hence (5.13) follows. Notice that C_1 does not depend on n.

The set whose existence has been proclaimed in Lemma 5.4 can be defined as $\mathfrak{K}_n := \{\|G\|_{L^2(P_D^{(n)})} \leqslant C_1\}$. ∎

Proof of Lemma 5.2. The existence of $H_*^{(n)} \in \mathscr{D}(\mathscr{T}_D^n)$ satisfying (5.6) has been shown in [6], Theorem 4.7. It is a consequence of Theorem 5.6.2 of [7] and the existence of the uniform (with respect to ω) lower bound for $\mathscr{Q}_n F$ (see Lemma 4.8 in [6]).

It remains to prove that $H_*^{(n)}$ is an element of $L^2(P_D^{(n)})$. Due to (5.13), for any $G \in \mathscr{D}(\mathscr{T}_D^n) \cap L^2(P_D^{(n)})$ and sufficiently large $m \in N$, the sequence $\big((\mathscr{Q}_n)^m G\big)$ is bounded in $L^2(P_D^{(n)})$. Hence we can extract an L^2-weakly convergent subsequence

$((\mathcal{Q}_n)^{m_k} G)$. On the other hand, due to (5.13) and by virtue of the Komlós theorem, $k^{-1}\sum_{i=1}^{k}((\mathcal{Q}_n)^{m_i} G)$ a.s. converges to $H_*^{(n)}$ (see [3]). Fatou's lemma and (5.13) yield that

$$\|H_*^{(n)}\|_{L^2(\boldsymbol{P}_D^{(n)})} \leqslant \liminf_{k\to\infty} \left\| \frac{1}{k} \sum_{i=1}^{k} (\mathcal{Q}_n)^{m_k} G \right\|_{L^2(\boldsymbol{P}_D^{(n)})} \leqslant C_1. \tag{5.16}$$ ■

After [6] (see Theorem 2.2) we define the measures

$$\mu_n(d\omega, d\pi) := \frac{1}{Z_n} \sum_{m=1}^{\infty} \Big[\int_0^{\infty} \int_{\boldsymbol{R}^d} \mathcal{H}_m^{(n)}(\boldsymbol{x}, s, \omega, \pi)\, ds\, d\boldsymbol{x}\Big] P_0^{(n)}(d\omega, d\pi), \tag{5.17}$$

where

$$\begin{aligned}(5.18)\qquad & \mathcal{H}_m^{(n)}(\boldsymbol{x}, s, \omega, \pi) \\ & := \mathbb{1}_{[D(\boldsymbol{x}\cdot\dot{v}) = +\infty]}(\pi)\, p_n^{\omega}(s, \boldsymbol{x}, \boldsymbol{0})\, Q_{\omega,s}^{n,\boldsymbol{x},\boldsymbol{0}}[A(s), S_m \leqslant s < S_{m+1}]\, H_*^{(n)}(T_{\boldsymbol{x}}\omega),\end{aligned}$$

and Z_n is the corresponding normalizing constant, i.e.

$$Z_n := \iint \sum_{m=1}^{\infty} \Big[\int_0^{\infty} \int_{\boldsymbol{R}^d} \mathcal{H}_m^{(n)}(\boldsymbol{x}, s, \omega, \pi)\, ds\, d\boldsymbol{x}\Big] P_0^{(n)}(d\omega, d\pi). \tag{5.19}$$

For the sake of convenience write

$$h^{(n)}(\pi, \omega) := \sum_{m=1}^{\infty} \Big[\int_0^{\infty} \int_{\boldsymbol{R}^d} \mathcal{H}_m^{(n)}(\boldsymbol{x}, s, \omega, \pi)\, ds\, d\boldsymbol{x}\Big]. \tag{5.20}$$

Due to Lemma 5.5 of [6] we have

$$Z_n = \int_{\Omega} M_{\omega}^n \Big[\int_0^{\tau_1} F(s)\, ds, D = \infty\Big] H_*^{(n)}(\omega)\, \boldsymbol{P}(d\omega). \tag{5.21}$$

According to Theorem 2.2 of [6] these measures are regular measures for processes $V_n(\cdot)$ (where $V_n(\cdot)$ are defined by (2.2), with an obvious substitution of $\boldsymbol{u}_n$ for $\boldsymbol{u}$).

The main idea of the proof of the main theorem relies on a subtraction of a suitable subsequence (μ_n) which weakly converges to some measure μ, being the desired measure, whose existence has been announced in Theorem 2.1. We will define the measure μ by analogy with μ_n (see (5.17) and 5.18)). To do this we will identify a suitable limit H_* of the sequence $(H_*^{(n)})$ of invariant densities of the transport operators $(\mathcal{Q}_n)$.

LEMMA 5.5. *There exists a square integrable function $H_*\colon \Omega \to [0, \infty)$ and a subsequence (n_i) such that for any further subsequence (n_{i_j}) of (n_i)*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} H_*^{(n_{i_j})}(\omega) = H_*(\omega) \quad \boldsymbol{P}\text{-a.s.} \tag{5.22}$$

The set of ω's, on which the above convergence takes place, does not depend on the choice of the subsequence (n_{i_j}).

Proof. According to the definition of the measure $P_D^{(n)}$ we have

$$(5.23)\qquad \int H_*^{(n)}(\omega)\, P(d\omega) \leqslant P_0^{(n)}[D=\infty] \int \frac{H_*^{(n)}(\omega)}{Q_\omega^n [D=\infty]} P_D^{(n)}(d\omega).$$

Applying (4.6) of Lemma 4.1 we can write

$$(5.24)\qquad \int H_*^{(n)}(\omega)\, P(d\omega) \leqslant \frac{1}{\gamma} \int H_*^{(n)}(\omega)\, P_D^{(n)}(d\omega) = \frac{1}{\gamma} \qquad \forall n \in N,$$

where the last equality follows from the fact that $H_*^{(n)}$ are $P_D^{(n)}$-densities. Application of the Komlós theorem (see [3]) yields the existence of an integrable function H_* and a subsequence (n_i) such that (5.22) holds. Square integrability of H_* is a consequence of the Fatou lemma, (4.6) and (5.16):

$$(5.25)\qquad \int H_*^2(\omega)\, P(d\omega) \leqslant \liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \int_\Omega (H_*^{(n_i)}(\omega))^2\, P(d\omega)$$

$$\leqslant \frac{1}{\gamma} \liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \int_\Omega (H_*^{(n_i)}(\omega))^2\, P_D^{(n_i)}(d\omega) \leqslant \frac{C_1}{\gamma} < \infty. \quad ■$$

By analogy with $\mathscr{H}_m^{(n)}(\cdot,\cdot,\cdot,\cdot)$, given H_*, we can define $\mathscr{H}_m(\cdot,\cdot,\cdot,\cdot)$ (see (5.18)). Similarly, we can define $h(\pi, \omega)$ (see (5.20)). Now, let us define the measure μ:

$$(5.26)\qquad \mu(d\omega, d\pi) := Z^{-1} h(\omega, \pi)\, Q_\omega(d\pi)\, P(d\omega),$$

where, by analogy with (5.21), we define

$$(5.27)\qquad Z := \int_\Omega M_\omega \Big[\int_0^{\tau_1} F(s)\, ds,\ D=\infty\Big] H_*(\omega)\, P(d\omega).$$

Let n be a positive integer, $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$ and $F_1, \ldots, F_n \in C_b(\boldsymbol{R}^d)$. For any $m \in N$, define

$$(5.28)\qquad F(\omega, \pi) := \prod_{p=1}^{n} F_p\big(\boldsymbol{u}(\pi(t_p))\big), \qquad F_m(\omega, \pi) := \prod_{p=1}^{n} F_p\big(\boldsymbol{u}_m(\pi(t_p))\big).$$

LEMMA 5.6. *There exists a subsequence (n_i) such that*

$$(5.29)\qquad \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \int\int F_{n_i}(\omega, \pi)\, h^{(n_i)}(\omega, \pi)\, P_0^{(n_i)}(d\omega, d\pi)$$

$$= \int\int F(\omega, \pi)\, h(\omega, \pi)\, P_0(d\omega, d\pi).$$

Proof. Without loss of generality we may assume that $F_1, \ldots, F_n$ are non-negative. For any $m \in N$, set

$$F_m(s) := \prod_{p=1}^{n} F_p\big(\boldsymbol{u}_m(\theta_s(\pi)(t_p))\big) \quad \text{and} \quad F(s) := \prod_{p=1}^{n} F_p\big(\boldsymbol{u}(\theta_s(\pi)(t_p))\big).$$

In view of Lemma 5.5 of [6], we have to show the existence of a sequence (n_i) such that

$$(5.30)\quad \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\int_{\Omega} M_{\omega}^{n_i}\Big[\int_0^{\tau_1} F_{n_i}(s)\,ds,\ D=\infty\Big] H_*^{(n_i)}(\omega)\,\boldsymbol{P}(d\omega) = \int_{\Omega} M_{\omega}\Big[\int_0^{\tau_1} F(s)\,ds,\ D=\infty\Big] H_*(\omega)\,\boldsymbol{P}(d\omega).$$

We have

$$(5.31)\quad \lim_{k\to\infty}\Big|\frac{1}{k}\sum_{i=1}^{k}\int_{\Omega} M_{\omega}^{n_i}\Big[\int_0^{\tau_1} F_{n_i}(s)\,ds,\ D=\infty\Big] H_*^{(n_i)}(\omega)\,\boldsymbol{P}(d\omega) - \int_{\Omega} M_{\omega}\Big[\int_0^{\tau_1} F(s)\,ds,\ D=\infty\Big] H_*(\omega)\,\boldsymbol{P}(d\omega)\Big|$$
$$\leqslant \lim_{k\to\infty}\Big|\frac{1}{k}\sum_{i=1}^{k}\int_{\Omega}\Big(M_{\omega}^{n_i}\Big[\int_0^{\tau_1} F_{n_i}(s)\,ds,\ D=\infty\Big] - M_{\omega}\Big[\int_0^{\tau_1} F(s)\,ds,\ D=\infty\Big]\Big) H_*^{(n_i)}(\omega)\Big|\,\boldsymbol{P}(d\omega)$$
$$+\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\int_{\Omega} M_{\omega}^{n_i}\Big[\int_0^{\tau_1} F_{n_i}(s)\,ds,\ D=\infty\Big]\,|H_*^{(n_i)}(\omega)-H_*(\omega)|\,\boldsymbol{P}(d\omega).$$

Let us denote the first and the second expression on the right-hand side of (5.31) by I_1 and I_2, respectively. First let us consider I_1:

$$(5.32)\quad I_1 = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\int_{\Omega}\Big|\int\int_0^{\infty} \mathbf{1}_{[0,\tau_1]}(s)\,\mathbf{1}_{[D=\infty]}(\pi)\prod_{p=1}^{m} F_p\big(\boldsymbol{u}_{n_i}(\pi(t_p+s))\big)\,ds\,Q_{\omega}^{(n_i)}(d\pi) - \int\int_0^{\infty} \mathbf{1}_{[0,\tau_1]}(s)\,\mathbf{1}_{[D=\infty]}(\pi)\prod_{p=1}^{m} F_p\big(\boldsymbol{u}(\pi(t_p+s))\big)\,ds\,Q_{\omega}(d\pi)\Big|H_*^{(n_i)}(\omega)\,\boldsymbol{P}(d\omega).$$

Due to Theorem 11.1.4 of [10], $Q_{\omega}^{(n)}$ converges weakly to Q_{ω}. Moreover, this convergence is uniform for all $\omega\in\Omega$. This is a consequence of the uniform (with respect to ω) convergence of $(\boldsymbol{u}_n(\cdot))$ to $\boldsymbol{u}(\cdot)$. Since $H_*^{(n_i)}$ are integrable (see (5.24)), we conclude that $I_1 = 0$.

Now consider I_2. We have

$$(5.33)\quad I_2 \leqslant \liminf_{k\to\infty}\frac{\|F\|}{k}\sum_{i=1}^{k}\int_{\Omega} M_{\omega}^{(n_i)}[\tau_1,\ D=\infty]\,|H_*^{(n_i)}(\omega)-H_*(\omega)|\,\boldsymbol{P}(d\omega).$$

To see that the right-hand side of (5.33) vanishes observe that $(H_*^{(n)})_{n\in N}$ is bounded in $L^2(\boldsymbol{P})$. Indeed, it is enough to perform calculations similar to the ones in (5.25). Thus we can choose a subsequence of $(H_*^{(n)})$ which weakly converges in $L^2(\boldsymbol{P})$. To simplify the notation, without loss of generality, we may assume that this subsequence coincides with $(H_*^{(n_i)})$ chosen in Lemma 5.5. This lemma allows us also to identify the weak L^2-limit of $(k^{-1}\sum_{i=1}^k H_*^{(n_i)})$ as H_*. Since $E_0^{(n)}[\tau_1^2, D = \infty] \in L^2$ (see (4.12)), the result follows. ■

Setting $F_n(\omega, \pi) = F(\omega, \pi) \equiv 1$ in Lemma 5.6, we get

$$(5.34)\quad Z = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} Z_{n_i} = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \int_{\Omega} M_\omega^{n_i}[\tau_1, D = \infty]\, H_*^{(n_i)}(\omega)\, \boldsymbol{P}(d\omega)$$

$$\leqslant \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} E_0^{(n_i)}[\tau_1, D = \infty] \cdot \int_{\Omega} \big(H_*^{(n_i)}(\omega)\big)^2\, \boldsymbol{P}(d\omega) < \infty.$$

The last inequality follows from the Cauchy and Jensen inequalities. The finiteness of the outmost right-hand side of the expression above is a consequence of (4.12) and the boundedness of $(H_*^{(n)})$ in $L^2(\boldsymbol{P})$. ■

6. THE PROOF OF THE MAIN THEOREM

We will show now that μ defined in (5.26) is a regular invariant measure.

Let us show first stationarity.

Let $n \geqslant 1$, $F_1, \ldots, F_m \in C_b(\boldsymbol{R}^d)$ and $0 \leqslant t_1 \leqslant \ldots \leqslant t_m$. Since $\boldsymbol{u}_n(\cdot)$ satisfies the condition (RAC), we can use Proposition 5.8 of [6], i.e. for any $h \geqslant 0$ we have

$$(6.1)\quad \frac{1}{Z_n}\iint \prod_{p=1}^{m} F_p\big(\boldsymbol{u}_n(\pi(t_p+h))\big)\, h^{(n)}(\omega, \pi)\, Q_\omega^{(n)}(d\pi)\, \boldsymbol{P}(d\omega)$$

$$= \frac{1}{Z_n}\iint \prod_{p=1}^{m} F_p\big(\boldsymbol{u}_n(\pi(t_p))\big)\, h^{(n)}(\omega, \pi)\, Q_\omega^{(n)}(d\pi)\, \boldsymbol{P}(d\omega).$$

Choosing a suitable subsequence as in Lemma 5.5 and taking the Cesaro means on both sides of the above equation we obtain

$$(6.2)\quad \frac{1}{k}\sum_{i=1}^{k} \frac{1}{Z_{n_i}}\iint \prod_{p=1}^{m} F_p\big(\boldsymbol{u}_{n_i}(\pi(t_p+h))\big)\, h^{(n_i)}(\omega, \pi)\, Q_\omega^{(n_i)}(d\pi)\, \boldsymbol{P}(d\omega)$$

$$= \frac{1}{k}\sum_{i=1}^{k} \frac{1}{Z_{n_i}}\iint \prod_{p=1}^{m} F_p\big(\boldsymbol{u}_{n_i}(\pi(t_p))\big)\, h^{(n_i)}(\omega, \pi)\, Q_\omega^{(n_i)}(d\pi)\, \boldsymbol{P}(d\omega).$$

Applying Lemma 5.6 and letting $k \to \infty$ we get

$$(6.3)\quad \frac{1}{Z}\iint \prod_{p=1}^{m} F_p\big(\boldsymbol{u}(\pi(t_p+h))\big)\, h(\omega, \pi)\, Q_\omega(d\pi)\, \boldsymbol{P}(d\omega)$$

$$= \frac{1}{Z}\iint \prod_{p=1}^{m} F_p\big(\boldsymbol{u}(\pi(t_p))\big)\, h(\omega, \pi)\, Q_\omega(d\pi)\, \boldsymbol{P}(d\omega).$$

Hence μ is stationary.

Now we proceed with the proof of ergodicity of μ. We have to show that for any bounded and Borel measurable $F\colon \mathscr{X} \to \boldsymbol{R}$ satisfying

$$F \circ \theta_t(V(\cdot)) = F(V(\cdot)) \quad \forall t \geqslant 0, \ \mu\text{-a.s.} \tag{6.4}$$

we have $F(V(\cdot))$ μ-a.s.

For any $\varepsilon > 0$ we can find $N \geqslant 1$, $0 \leqslant t_1 \leqslant \ldots \leqslant t_N$ and a bounded continuous function $F^{(N)}\colon (\boldsymbol{R}^d)^N \to \boldsymbol{R}$ approximating F in the following sense:

$$\int\int |F(V(\cdot)) - F^{(N)}(V(t_1), \ldots, V(t_N))|\, d\mu < \varepsilon. \tag{6.5}$$

This yields in turn that

$$\int\int \big|F(V(\cdot))\big(F(V(\cdot)) - F^{(N)}(V(t_1), \ldots, V(t_N))\big)\big|\, d\mu < 2\varepsilon \sup |F|. \tag{6.6}$$

Let $q \geqslant q_0$ be arbitrary integers. Set $V^{(q_0)}(t) := V(t \wedge \tau_{q_0})$, $t \geqslant 0$. Using (6.4) we conclude that

$$\begin{aligned}\int\int F(V(\cdot))\, F^{(N)}\big(V^{(q_0)}(t_1), \ldots, V^{(q_0)}(t_N)\big)\, d\mu \\ = \int\int F\big(\theta_{\tau_q}(V(\cdot))\big)\, F^{(N)}\big(V^{(q_0)}(t_1), \ldots, V^{(q_0)}(t_N)\big)\, d\mu.\end{aligned} \tag{6.7}$$

Applying (6.5) we can approximate the right-hand side of (6.7) in the following way:

$$\begin{aligned}\int\int \Big|\Big(F\big(\theta_{\tau_q}(V(\cdot))\big) - F^{(N)}\big(V(\tau_q + t_1), \ldots, V(\tau_q + t_N)\big)\Big) \\ \times F^{(N)}\big(V^{(q_0)}(t_1), \ldots, V^{(q_0)}(t_N)\big)\Big|\, d\mu \leqslant 2\varepsilon \sup |F^{(N)}|.\end{aligned} \tag{6.8}$$

Lemma 5.6 together with (5.34) yield that

$$\begin{aligned}\int\int F^{(N)}\big(V(\tau_q + t_1), \ldots, V(\tau_q + t_N)\big)\, F^{(N)}\big(V^{(q_0)}(t_1), \ldots, V^{(q_0)}(t_N)\big)\, d\mu \\ = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \int\int F^{(N)}\big(V(\tau_q + t_1), \ldots, V(\tau_q + t_N)\big) \\ \times F^{(N)}\big(V_{n_i}^{(q_0)}(t_1), \ldots, V_{n_i}^{(q_0)}(t_N)\big)\, d\mu_{n_i}.\end{aligned} \tag{6.9}$$

By virtue of Proposition 4.5 of [6], the above expression equals

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \int\int F^{(N)}\big(V(t_1), \ldots, V(t_N)\big)(\mathscr{Q}_{n_i})^{q-q_0}\, Y_{n_i}\, d\mu_{n_i}, \tag{6.10}$$

where Y_{n_i} are $\mathscr{V}_0$-measurable functions satisfying

$$\int\int Y_{n_i}\, d\mu_{n_i} = \int\int F^{(N)}\big(V_{n_i}^{(q_0)}(t_1), \ldots, V_{n_i}^{(q_0)}(t_N)\big)\, d\mu_{n_i}.$$

Due to Lemma 5.2, for any n_i there exists $\hat{q}_{n_i}$ such that for $q \geqslant \hat{q}_{n_i}$

$$\begin{aligned}\Big|\int\int F^{(N)}\big(V(t_1), \ldots, V(t_N)\big)(\mathscr{Q}_{n_i})^{q-q_0}\, Y_{n_i}\, d\mu_{n_i} \\ - \int\int F^{(N)}\big(V(t_1), \ldots, V(t_N)\big)\, d\mu_{n_i} \int\int F^{(N)}\big(V_{n_i}^{(q_0)}(t_1), \ldots, V_{n_i}^{(q_0)}(t_N)\big)\, d\mu_{n_i}\Big| < \varepsilon.\end{aligned} \tag{6.11}$$

Hence, we can choose an increasing sequence (q_{n_i}) such that

$$(6.12)\quad \frac{1}{k}\sum_{i=1}^{k}\Big|\int\int F^{(N)}(V(t_1),\ldots,V(t_N))\,d\mu_{n_i}\int\int F^{(N)}(V_{n_i}^{(q_0)}(t_1),\ldots,V_{n_i}^{(q_0)}(t_N))\,d\mu_{n_i} - \int\int F^{(N)}(V(t_1),\ldots,V(t_N))(\mathcal{Q}_{n_i})^{q_{n_i}-q_0}Y_{n_i}\,d\mu_{n_i}\Big| \leqslant \varepsilon.$$

Due to Lemma 5.6, without loss of generality we may assume that (n_i) is such that

$$(6.13)\quad \frac{1}{k}\sum_{i=1}^{k}\Big|\int\int F^{(N)}(V(t_1),\ldots,V(t_N))\,d\mu_{n_i}\int\int F^{(N)}(V_{n_i}^{(q_0)}(t_1),\ldots,V_{n_i}^{(q_0)}(t_N))\,d\mu_{n_i} - \int\int F^{(N)}(V(t_1),\ldots,V(t_N))\,d\mu\int\int F^{(N)}(V^{(q_0)}(t_1),\ldots,V^{(q_0)}(t_N))\,d\mu\Big| \leqslant \varepsilon.$$

Hence

$$(6.14)\quad \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\int\int F^{(N)}(V(t_1),\ldots,V(t_N))(\mathcal{Q}_{n_i})^{q_{n_i}-q_0}Y_{n_i}\,d\mu_{n_i} = \int\int F^{(N)}(V(t_1),\ldots,V(t_N))\,d\mu\int\int F^{(N)}(V^{(q_0)}(t_1),\ldots,V^{(q_0)}(t_N))\,d\mu.$$

Combining (6.7)–(6.14) we get

$$(6.15)\quad \Big|\int\int F^{(N)}(V(t_1),\ldots,V(t_N))F^{(N)}(V^{(q_0)}(t_1),\ldots,V^{(q_0)}(t_N))\,d\mu - \int\int F^{(N)}(V(t_1),\ldots,V(t_N))\,d\mu\int\int F^{(N)}(V^{(q_0)}(t_1),\ldots,V^{(q_0)}(t_N))\,d\mu\Big| \leqslant 2\varepsilon\sup|F^{(N)}|.$$

Letting $q_0\to\infty$, we see that (6.15) becomes

$$(6.16)\quad \Big|\int\int\big(F^{(N)}(V(t_1),\ldots,V(t_N))\big)^2\,d\mu - \Big(\int\int F^{(N)}(V(t_1),\ldots,V(t_N))\,d\mu\Big)^2\Big| \leqslant 2\varepsilon\sup|F^{(N)}|.$$

By (6.5) and (6.6), the above equality implies

$$(6.17)\quad \Big|\int\int[F(V(\cdot))]^2\,d\mu-\Big[\int\int F(V(\cdot))\,d\mu\Big]^2\Big| < 2\varepsilon(2\sup|F|+3\sup|F^{(N)}|).$$

Observe that $F^{(N)}$ in (6.5) can be chosen in such a way that, for all $N\in N$,

$$(6.18)\quad \sup|F^{(N)}|\leqslant 2\sup|F|.$$

Due to an arbitrary choice of $\varepsilon>0$, we conclude that $F(V(\cdot))=\text{const}$ μ-a.s. ∎

APPENDIX A. THE PROOF OF LEMMA 4.4

Inequality (4.8) is proved in [6] in the Appendix (see (A.11) therein). Statement (4.9) follows easily from (4.8).

A.1. Proof of (4.10). With no loss of generality we suppose that $\boldsymbol{x}=\boldsymbol{0}$. We can write

$$\hat{\boldsymbol{v}}\cdot\pi(\tau_1)\leqslant r_0+1+\sum_{k=1}^{K-1}\big(r_0+1+M_k-\hat{\boldsymbol{v}}\cdot\pi(S_k)\big),$$

with random variable K defined before the statement of Lemma 4.3. Using the Hölder inequality we get

$$\text{(A.1)} \qquad M_{\omega}^{(n)}[\hat{v}\cdot\pi(\tau_1)]^4 \leqslant 8\{(r_0+1)^4+M_{\omega}^{(n)}[K^3\sum_{k=1}^{K-1}\eta_k^4]\},$$

where $\eta_k := r_0+1+M_k-\hat{v}\cdot\pi(S_k)$. The right-hand side of (A.1) can be estimated by

$$8\{(r_0+1)^4+\sum_{1\leqslant k'<k} M_{\omega}^{(n)}[k^3\eta_{k'}^4 \mathbf{1}_{[R_{k-1}<+\infty, D\circ\theta_{S_k}=+\infty]}]\}.$$

Since $R_{k-1} = D\circ\theta_{S_{k-1}}+S_{k-1}$, upon a multiple application of the strong Markov property of $Q_{\omega}^{(n)}$ and (4.6) it follows that the expression above is less than or equal to

$$\text{(A.2)} \qquad 8\{(r_0+1)^4+\sum_{1\leqslant k'<k} k^3(1-\gamma)^{k-1-k'}M_{\omega}^{(n)}[\eta_{k'}^4\mathbf{1}_{[R_{k'}<+\infty]}]\}$$

$$\leqslant 8\{(r_0+1)^4+\sum_{1\leqslant k'<k} k^3(1-\gamma)^{k-1-k'}$$

$$\times M_{\omega}^{(n)}\big[M_{\pi(S_{k'}),\omega}^{(n)}[(r_0+1+M_*)^4, D<+\infty], S_{k'}<+\infty\big]\}.$$

By virtue of (4.9) and (4.7) we conclude that the right-hand side of (A.2) is less than or equal to

$$\text{(A.3)} \qquad 8\{(r_0+1)^4+8[(r_0+1)^4+\gamma_1^4]\sum_{1\leqslant k'<k} k^3(1-\gamma)^{k-1}\}$$

$$= 8\{(r_0+1)^4+8[(r_0+1)^4+\gamma_1^4]\sum_{k=1}^{\infty} k^4(1-\gamma)^{k-1}\} < +\infty.$$

A.2. Proof of (4.11). Again, we let $x = \mathbf{0}$. Note that

$$\text{(A.4)} \quad Q_{\omega}^{(n)}[\tau_1>u] \leqslant Q_{\omega}^{(n)}\left[\tau_1>u,\ \hat{v}\cdot\pi(\tau_1)\leqslant\frac{\delta}{2}u\right]+Q_{\omega}^{(n)}\left[\hat{v}\cdot\pi(\tau_1)\geqslant\frac{\delta}{2}u\right].$$

By virtue of (4.10) and Chebyshev's inequality the second term on the right-hand side of (A.4) is less than or equal to C/u^4 for some constant C that can be chosen independently of ω. On the other hand, the first term there can be estimated as follows:

$$\text{(A.5)} \qquad Q_{\omega}^{(n)}\left[\tau_1>u,\ \hat{v}\cdot\pi(\tau_1)\leqslant\frac{\delta}{2}u\right] \leqslant Q_{\omega}^{(n)}[T_{\delta u/2}>u]$$

$$\leqslant Q_{\omega}^{(n)}[T_{U_L}\geqslant u]+Q_{\omega}^{(n)}[T_{U_L}<u,\ \pi(T_{U_L})\notin\partial^+U_L].$$

Here $2L = \delta u$ and $U_L(x)$ is a cylinder centered at x with width $2L$ in the direction $\hat{v}$ and radius $2L(2+U)/\delta$ in the directions normal to $\hat{v}$, i.e.

$$U_L(x) := \{z \in R^d\colon\ |\hat{v}\cdot(z-x)| < L,\ |e\cdot(z-x)| < (2+U)2L/\delta \text{ for any } e \perp v,\ |e| = 1\}.$$

Here U is as in the condition (RH). The proof will be completed when we show that the right-hand side of (A.5) vanishes as $L \to \infty$.

Let $U_L := U_L(0)$. We divide $\partial U_L(x)$ into three subsets:

$$\text{(A.6)} \qquad \partial^+ U_L(x) := \{z \in \partial U_L(x)\colon\ \hat{v}\cdot(z-x) \geqslant L/2\},$$

$$\text{(A.7)} \qquad \partial^- U_L(x) := \{z \in \partial U_L(x)\colon\ \hat{v}\cdot(z-x) \leqslant -L/2\},$$

$$\text{(A.8)} \qquad \partial^0 U_L(x) := \partial U \backslash \big(\partial^+ U_L(x) \cup \partial^- U_L(x)\big).$$

The following lemma characterizes the exit times from $U_L(x)$.

LEMMA A.1. *There exist deterministic constants $c_1, c_2 > 0$ independent of L, n and ω such that for all $x \in R^d$*

$$\text{(A.9)} \qquad Q^{(n)}_{x,\omega}[T_{U_L(x)} > 2L/\delta] \leqslant c_1 \exp(-L/c_1),$$

$$\text{(A.10)} \qquad Q^{(n)}_{x,\omega}[T_{U_L(x)} \leqslant 2L/\delta,\ \pi(T_{U_L(x)}) \notin \partial^+ U_L(x)] \leqslant c_2 \exp(-L/c_2).$$

Proof. The process

$$w_\omega(t;\pi) := \pi(t) - \int_0^t u_n(\pi(s),\omega)\,ds, \qquad t \geqslant 0,$$

is a d-dimensional standard Brownian motion starting at x over $(\mathscr{X}, \mathscr{M}, Q^{(n)}_{x,\omega})$ for any ω. On the event $[T_{U_L} > 2L/\delta]$

$$|w_\omega(T_{U_L})| = \Big|\pi(T_{U_L}) - \int_0^{2L/\delta} u_n(\pi(s),\omega)\,ds\Big| \geqslant L.$$

Hence

$$Q^{(n)}_\omega[T_{U_L} > 2L/\delta] \leqslant Q^{(n)}_\omega[|w_\omega(2L/\delta)| \geqslant L] \leqslant \exp\{-\delta L/4\}.$$

On the other hand,

$$\text{(A.11)} \qquad Q^{(n)}_\omega[T_{U_L} \leqslant 2L/\delta,\ \pi(T_{U_L}) \notin \partial^+ U_L]$$

$$\leqslant Q^{(n)}_\omega[T_{U_L} \leqslant 2L/\delta,\ \pi(T_{U_L}) \in \partial^- U_L] + Q^{(n)}_\omega[T_{U_L} \leqslant 2L/\delta,\ \pi(T_{U_L}) \in \partial^0 U_L]$$

$$\leqslant Q^{(n)}_\omega[\sup_{0 \leqslant t \leqslant 2L/\delta} |w_\omega(t)| \geqslant L] + Q^{(n)}_\omega[\sup_{0 \leqslant t \leqslant 2L/\delta} |w_\omega(t)| \geqslant 2L/\delta].$$

Using elementary estimates on the law of the maximum of a Brownian motion, we bound the right-hand side of (A.11) from above by $\exp\{-\delta L/4d\}$.

Using (A.9) and (A.10) of Lemma A.1 we conclude that the right-hand side of (A.5) is less than or equal to

$$c_1 \exp\left\{-\frac{\delta u}{2c_1}\right\} + c_2 \exp\left\{-\frac{\delta u}{2c_2}\right\}$$

and (4.11) follows. ∎

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