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SUMS OF DUFRESNE RANDOM VARIABLES

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Abstract. We study particular cases of sums of independent Dufresne random variables (which are essentially products of beta and gamma variates) such that the distribution of the sum is again a Dufresne random variable.

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1. INTRODUCTION

The *gamma* distribution with scale parameter 1 and shape parameter $c > 0$ is the probability on $(0, \infty)$ defined by

$$\gamma_c(dx) = \frac{1}{\Gamma(c)} x^{c-1} e^{-x} \mathbf{1}_{(0,\infty)}(x)\, dx.$$

The *beta* distribution with parameters $a, b > 0$ is the probability on $(0, 1)$ defined by

$$\beta_{a,b}(dx) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{(0,1)}(x)\, dx,$$

where $B(a, b) = \big(\Gamma(a)\Gamma(b)\big)/\Gamma(a+b)$.

Let us denote by $\odot$ and $*$ the multiplicative and additive convolutions, respectively, of probability distributions on $\boldsymbol{R}$; this means that if X and Y are independent variables with distributions α and β, then the distributions of XY and $X+Y$ are denoted by $\alpha \odot \beta$ and $\alpha * \beta$. Recall that the Laplace transform of a probability distribution α on $\boldsymbol{R}$ is

$$L(\alpha)(s) = \int_{\boldsymbol{R}} e^{sx} \alpha(dx).$$

It satisfies $L(\alpha * \beta)(s) = L(\alpha)(s)\, L(\beta)(s)$. Similarly, the Mellin transform of a probability distribution α defined on $[0, \infty)$ is

$$M(\alpha)(s) = \int_{[0,\infty)} x^s \alpha(dx)$$

and satisfies $M(\alpha \odot \beta)(s) = M(\alpha)(s)\, M(\beta)(s)$.

The Pochhammer symbol is defined for $a, s > 0$ by $(a)_s = \Gamma(a+s)/\Gamma(a)$. We extend this notation to a sequence $\boldsymbol{a} = (a_1, \ldots, a_p)$ of positive numbers by $(\boldsymbol{a})_s = \prod_{j=1}^{p} (a_j)_s$ with the convention $(\boldsymbol{a})_s = 1$ if $p = 0$. For two sequences $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$ of positive numbers the Dufresne distributions* $D(\boldsymbol{a};\, \boldsymbol{b})$ are introduced in Chamayou and Letac (1999) (see also Dufresne (1996), (1998)) as the distribution on $(0, \infty)$ such that for all $s \geqslant 0$ we have

$$\int_0^\infty x^s D(\boldsymbol{a};\, \boldsymbol{b})(dx) = \frac{(\boldsymbol{a})_s}{(\boldsymbol{b})_s}. \tag{1.1}$$

The later paper contains comments on existence and computation rules for these Dufresne distributions. If $p = 1$, i.e. if the sequence $\boldsymbol{a}$ is reduced to a single number a, we write of course $\boldsymbol{a} = a$, and if $p = 0$, it is convenient to write the empty sequence $\boldsymbol{a} = -$. With these notations we can write $D(a;\, a+b) = \beta_{a,b}$ and $D(c;\, -) = \gamma_c$.

The aim of this paper is to give some unexpected properties of *additive* and *multiplicative* convolutions of the $D(\boldsymbol{a}, \boldsymbol{b})$ laws. We shall use the following classical notation for the hypergeometric functions defined for $-1 < s < 1$:

$$_pF_q(\boldsymbol{a};\, \boldsymbol{b};\, s) = \sum_{n=0}^{\infty} \frac{(\boldsymbol{a})_n s^n}{(\boldsymbol{b})_n\, n!}. \tag{1.2}$$

These hypergeometric functions arise naturally as Laplace transforms of the Dufresne laws: for $-1 < s < 1$ we obtain

$$L(D(\boldsymbol{a};\, \boldsymbol{b}))(s) = {}_pF_q(\boldsymbol{a};\, \boldsymbol{b};\, s). \tag{1.3}$$

The *double gamma* distribution λ_c with parameter $c > 0$ is the probability on $\boldsymbol{R}$ with Laplace transform $L(\lambda_c)(s) = (1-s^2)^{-c}$ defined for $s \in (-1, 1)$. If $\tilde{\gamma}_c$ is the image of γ_c by $x \mapsto -x$, we clearly have

$$\lambda_c = \gamma_c * \tilde{\gamma}_c, \tag{1.4}$$

and this proves the existence of λ_c. It is easily seen that $\lambda_1(dx) = e^{-|x|}\, dx/2$, which has various names in the literature: double exponential, bilateral exponential, Laplace distribution of first kind. If c is an integer, the density of

* When writing $D(\boldsymbol{a};\, \boldsymbol{b})$ instead of $_pD_q(\boldsymbol{a};\, \boldsymbol{b})$ we depart from the tradition of hypergeometric functions, which furthermore displays, respectively, by p and q the lengths of the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$. Although there is some redundancy in the traditional notation, we still keep it for conveniency of the reader when dealing with the hypergeometric functions $_pF_q$ or the Kampé de Fériet functions $F^{k,C,D}_{k',C',D'}$ in Section 4.

λ_c can be obtained by expansion into partial fractions of $s \mapsto (1-s^2)^{-c}$. It is easily checked that if n is a non-negative integer, we have

$$\lambda_{n+1}(dx) = e^{-|x|} \sum_{k=0}^{n} \frac{(2n-k)!}{n!\,k!\,(n-k)!\,2^{2n-2k+1}} |x|^k\, dx.$$

If c is not an integer, the density is obtained by using the Mac Donald functions (modified Bessel functions of second kind). The lambda law $\lambda_c(\boldsymbol{a};\,\boldsymbol{b}) = \lambda_c \odot D(\boldsymbol{a};\,\boldsymbol{b})$ or bilateral Dufresne laws are introduced in Chamayou (2004). However, in the present paper we shall mainly use the laws $\lambda_c(a;\,b)$ $= \lambda_c \odot \beta_{a,b-a}$ which are the multiplicative convolutions of double gammas with parameter c by betas with parameters $(a,\, b-a)$.

The symmetrical *beta* distribution of third kind with parameter $a = b = \nu + 1/2 > 0$ is defined by

$$\beta_{a,b}^{(3)}(dx) = \frac{1}{2^{a+b-1} B(a,\, b)} (1-x)^{a-1} (1+x)^{b-1} \mathbf{1}_{(-1,1)}(x)\, dx.$$

The Fourier transform is given in terms of Bessel functions (see Prudnikov et al. (1992), Vol. 3, formula 7.13.1, p. 594): ${}_0F_1(\nu+1;\, -s^2/4)$.

2. SUMS OF SEVERAL DUFRESNE RANDOM VARIABLES

We first present a property of additive infinite divisibility for some Dufresne laws.

PROPOSITION 2.1. *If* $t > 0$, *the Laplace transform of* $D(t/2,\, t/2+1/2;\, t+1)$ *is as follows*:

$${}_2F_1(t/2,\, t/2+1/2;\, t+1;\, s) = \big(2/(1+\sqrt{1-s}\,)\big)^t.$$

This is an infinitely divisible law associated with the Lévy process $t \mapsto X\big(P(t)\big)$, *where* P *and* X *are independent Lévy processes whose respective laws at time* t *are* $\gamma_{t,2}$ *and the inverse Gaussian law*

$$\nu_{1/2,t}(dx) = \frac{t}{2\sqrt{\pi}} x^{-3/2} \exp\left(-\frac{(x-t)^2}{x}\right) \mathbf{1}_{(0,\infty)}(x)\, dx,$$

respectively.

Proof. The Laplace transform of a Dufresne law is given by the Gauss hypergeometric function: ${}_2F_1(a,\, b;\, c;\, s)$. Then the Laplace transform of $D(t/2,$ $t/2+1/2;\, t+1)$ is

$${}_2F_1(t/2,\, t/2+1/2;\, t+1;\, s) \;=\; \big(2/(1+\sqrt{1-s}\,)\big)^t$$

according to Abramovitz and Stegun (1970), p. 556, formula 15-1-13. Moreover, by Seshadri (1993), p. 40, formula 2-9, we know that if for $\alpha > 0$ and $p > 0$

we define

$$\nu_{\alpha,p}(dx) = \frac{p\sqrt{\alpha}}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{(x-p\alpha)^2}{2\alpha x}\right) \mathbf{1}_{(0,\infty)}(x)\, dx, \tag{2.1}$$

then for $s < 1/(2\alpha)$ we get

$$\int_0^\infty e^{sx} \nu_\alpha(dx) = \exp\left(p\left(1-\sqrt{1-2\alpha s}\right)\right). \tag{2.2}$$

Moreover, for every $\beta < 2$ we have

$$\int_0^\infty e^{\beta p} \gamma_{t,2}(dp) = (2-\beta)^{-t}. \tag{2.3}$$

Using (2.3) for $\beta = 1-\sqrt{1-2\alpha s}$ we see that for $s < 1$

$$\left(2/(1+\sqrt{1-s})\right)^t = \int_0^\infty \int_0^\infty e^{sx} \nu_{1/2,p}(dx)\, \gamma_{t,2}(dp),$$

which completes the proof. ∎

Note. This law is considered by Imhof (1986). The corresponding density of probability $\int_0^\infty \nu_{1/2,p}(dx)\gamma_{t,2}(dp)$ can be expressed by using the parabolic cylinder functions:

$$f(t, x) = \frac{t 2^{(3/2)t}}{\sqrt{2\pi}} e^{-x/2} x^{t/2-1} D_{-(t+1)}(\sqrt{2x})\, \mathbf{1}_{(0,\infty)}(x)\, dx. \tag{2.4}$$

For this point see Prudnikov et al. (1992), Tome V, p. 45, formula 2.1.9-1. Note that the law on integers $\sum_{m=0}^{\infty} f(m, x)\delta_m$ is considered in Chamayou (1984).

We now apply the previous Laplace transform computation to the bilateral Dufresne laws:

Corollary 2.1. *If $t > 0$, the Laplace transform of the law $\lambda_{t+1/2}(t;\, 2t+1)$ is as follows:*

$$ {}_2F_1(t/2,\, t/2+1/2;\, t+1;\, s^2) = \left(2/(1+\sqrt{1-s^2})\right)^t.$$

This is an infinitely divisible law associated with the Lévy process $t \mapsto X(P(t))$, where P and X are independent Lévy processes whose respective laws at time t are $\gamma_{t,2}$ and the Bessel law

$$\nu_p(dx) = \frac{p e^p}{\pi} \frac{K_1(\sqrt{x^2+p^2})}{\sqrt{x^2+p^2}} \mathbf{1}_{(-\infty,\infty)}(x)\, dx, \tag{2.5}$$

where K_1 is the order-one modified Bessel function of second kind.

Proof. The Laplace transform of the density of a bilateral Dufresne variable $\lambda_d(a; c)$ is given by the generalized hypergeometric function:

$$ {}_3F_2\left(d, a/2, (a+1)/2; c/2, (c+1)/2; s^2\right) \tag{2.6} $$

(see Chamayou (2004)). If we set $d = a+1/2 = c/2$, the function (2.6) takes the form ${}_2F_1\left(a/2, (a+1)/2; a+1; s^2\right)$, which allows us to use Proposition 2.1 to establish the result, where

$$ {}_2F_1\left(a/2, a/2+1/2; a+1; s^2\right) = \left(2/(1+\sqrt{1-s^2})\right)^a. \tag{2.7} $$

We get the previous Laplace transform by integration of the expression

$$ \frac{1}{\Gamma(a)}\int_0^\infty \exp\left(-p(1+\sqrt{1-s^2})/2\right)p^{a-1}\,dp. \tag{2.8} $$

Then we get

$$ \int_{-\infty}^{\infty} e^{sx}\,\nu_p(dx) = \exp\left(p(1-\sqrt{1-s^2})\right). \tag{2.9} $$

For that point, see Prudnikov et al. (1992), Tome 2, p. 357, formula 2-16-12-4, and Abramowitz and Stegun (1970) for the representation of $K_{1/2}$. Thus, if $P(a)$ is a random variable following a gamma law $\gamma_a(ds)$ and if $\left(X(p)\right)_{p\geq 0}$ is a Bessel process independent of $P(a)$ such that $X(p)$ has the law ν_p, then $\left(2/(1+\sqrt{1-s^2})\right)^a$ is the Laplace transform of the law $X\left(P(a)/2\right)$, i.e. follows the above Dufresne law. ∎

Note. The probabilistic interpretation of the law of $\lambda_{t+1/2}(t; 2t+1)$ from the law of $D(t/2, t/2+1/2; t+1)$ is standard: if X and Z are independent random variables with respective laws $N(0, 1)$ and $D(t/2, t/2+1/2; t+1)$, then the consideration of the Laplace transform of $Z\sqrt{2|X|}$ shows that it is a $\lambda_{t+1/2}(t; 2t+1)$ law. We also remark, using the asymptotical representation of K_1 (see Abramovitz and Stegun (1970)), that for $p\to\infty$ the law of $X/\sqrt{p}$ tends to an $N(0, 1)$ law.

3. SUMS OF TWO DISTINCT DUFRESNE VARIABLES WITH THE SAME LAWS

PROPOSITION 3.1. *Let a, b, c be such that $0 < a, b < 1$,*

$$ \max(a, b) < c < \min(1+a, 1+b), $$

and let D_1, D_2, D_1', D_2' be independent random variables with respective laws

$$ D(a, b; c),\ D(1-a, 1-b; 2-c),\ D(c-a, c-b; c),\ D(a+1-c, b+1-c; 2-c). $$

Then $D_1+D_2 \sim D_1'+D_2'$.

Proof. The Laplace transform of the two sums is written as the product of two Gauss hypergeometrical functions, i.e.

$$ {}_2F_1(a, b; c; s)\, {}_2F_1(1-a, 1-b; 2-c; s) $$

and

$$ {}_2F_1(c-a, c-b; c; s)\, {}_2F_1(a+1-c, b+1-c; 2-c; s). $$

The result is obtained by application of the "Darling products" identity (see Darling (1932)). ■

COROLLARY 3.1. *Let a, c be such that $0 < a < 1/2$, $a+1/2 < c < a+1$, and let $\Lambda_1, \Lambda_2, \Lambda'_1, \Lambda'_2$ be four independent random variables with respective laws*

$$ \lambda_{c+1/2}(2a; 2c),\ \lambda_{3/2-c}(1-2a; 3-2c),\ \lambda_{c+1/2}\big(2(c-a)-1; 2-c\big), $$
$$ \lambda_{3/2-c}\big(2(1+a-c); 3-2c\big). $$

Then $\Lambda_1+\Lambda_2 \sim \Lambda'_1+\Lambda'_2$.

Proof. The proof is identical to the previous one for the Laplace transforms of the lambda laws, i.e.

$$ {}_3F_2\big(d, a/2, (a+1)/2; c/2, (c+1)/2; s^2\big) \tag{3.1} $$

by choosing $d = c+1/2$ and $d = 3/2-c$, respectively. ■

PROPOSITION 3.2. *Let a, b be such that $a > 1/2$, $0 < b < a-1/2$, and let D_0, Γ_1, Δ_1 be independent random variables with respective laws*

$$ D(2a, 2b, a-1/2; a+b+1/2, 2a-1),\ D(b; -),\ D(b, a-b-1/2; a+b+1/2). $$

Then $D_0 \sim \Gamma_1+\Delta_1$.

Proof. The Laplace transform of D_0 and $\Gamma_1+\Delta_1$ are written by using the hypergeometrical functions

$$ {}_3F_2(2a, 2b, a-1/2; a+b+1/2, 2a-1; s) $$

and

$$ \frac{1}{(1-s)^b}\big({}_2F_1(b, a-b-1/2; a+b+1/2; s)\big), $$

respectively. The result is obtained by application of formula 14, p. 498, in Prudnikov et al. (1992), Vol. 3. ■

COROLLARY 3.2. *Let $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda'_0, \Lambda'_1, \Lambda'_2$ be independent random variables with respective laws*

$$ \lambda_4(3; 6),\ \lambda_1,\ \lambda_3(1; 6),\ \lambda_8(7; 13),\ \lambda_2,\ \lambda_6(3; 12). $$

Then $\Lambda_0 \sim \Lambda_1+\Lambda_2$ and $\Lambda'_0 \sim \Lambda'_1+\Lambda'_2$.

Proof. We use again formula 14, p. 498, in Prudnikov et al. (1992), Vol. 3, with argument s^2:

$$ {}_3F_2(2a, 2b, a-1/2; a+b+1/2, 2a-1; s^2) = \frac{1}{(1-s^2)^b}\left({}_2F_1(b, a-b-1/2; a+b+1/2; s^2)\right) $$

as an identity for the Laplace transforms of the lambda laws of $\Lambda_0^{(\cdot)}$ and $\Lambda_1^{(\cdot)}+\Lambda_2^{(\cdot)}$. By choosing $a = b+1$, $b = 1$ and $a = b+2$, $b = 2$, respectively, we obtain the assertion. ∎

PROPOSITION 3.3. *Let a, b be such that $a > 0$, $1/2 < b < a+1/2$ and let D_0, Γ_1, Δ_1 be independent random variables with respective laws*

$D(2a+1, b+1/2; 2a-b+3/2)$, $D(1; -)$, $D(2a, a+1, b-1/2; a, 2a-b+3/2)$.

Then $D_0 \sim \Gamma_1+\Delta_1$.

Proof. The Laplace transform of D_0 and $\Gamma_1+\Delta_1$ are written by using the hypergeometrical functions

$$ {}_2F_1(2a+1, b+1/2; 2a-b+3/2; s) $$

and

$$ \frac{1}{1-s}\left({}_3F_2(2a, a+1, b-1/2; a, 2a-b+3/2; s)\right), $$

respectively. The result is obtained by application of formula 15, p. 498, in Prudnikov et al. (1992), Vol. 3. ∎

4. PRODUCTS OF ONE DUFRESNE VARIABLE BY SUMS OF TWO DUFRESNE VARIABLES WITH THE SAME LAW AS THE SUM OF TWO OTHER DUFRESNE VARIABLES

PROPOSITION 4.1. *Consider the following eight independent Dufresne variables with respective laws:*

- $D_3 \sim D(1, 1, 1; 2, 2)$, *i.e. multiplicative convolution of an exponential distribution by two uniform distributions;*
- $\Delta_2 \sim D(1, 1; 3/2)$;
- $\Delta_{3,0}, \Delta_{3,1}, \Delta_{3,2} \sim D(1, 1, 1; 3/2, 2)$, *i.e. multiplicative convolution of a uniform distribution by $D(1, 1; 3/2)$;*
- $D_{1,1} \sim D(2; 5/2)$, *i.e.* $\beta_{2,1/2}$;
- $D_{1,2} \sim D(2; 3)$, *i.e.* $\beta_{2,1}$;
- $D_0 \sim D(1; -)$, *i.e. an exponential distribution.*

Then

$$D_{1,1}(D_0+D_3) \sim \Delta_2+\Delta_{3,0}, \tag{4.1}$$

$$D_{1,2}(\Delta_2+\Delta_{3,0}) \sim \Delta_{3,1}+\Delta_{3,2}. \tag{4.2}$$

Proof. The Laplace transform of the product of a beta distribution by the sum of Dufresne variables can be written as the Euler integral defining the Kampé de Fériet series, i.e.

$$F^{1:C;C'}_{1:D;D'}\left[\begin{matrix} a: & (c); & (c'); \\ a+b: & (d); & (d'); \end{matrix} \; s, s\right]$$
$$= \frac{1}{B(a, b)}\int_0^1 x^{a-1}(1-x)^{b-1}\left({}_CF_D((c); (d); sx)\, {}_{C'}F_{D'}((c'); (d'); sx)\right) dx$$

(see Exton (1978), p. 39, formula 2.1.5.7). Krupnikov (2001) gives the following reduction formula:

$$F^{1:1;3}_{1:0;2}\left[\begin{matrix} 2: & 1; & 1, 1, 1; \\ 5/2: & -; & 2, 2; \end{matrix} \; s, s\right] \tag{4.3}$$
$$= \left({}_2F_1(1, 1; 3/2; s)\right)\left({}_3F_2(1, 1, 1; 3/2, 2; s)\right),$$

whence we can derive the result, iterating the previous integral relation. For that purpose, see Exton (1978), p. 184, formula A 1-2-100, we write

$$F^{k+1:C;C'}_{k+1:D;D'}\left[\begin{matrix} a, (\boldsymbol{a}): & (c); & (c'); \\ a+b, (\boldsymbol{a}+\boldsymbol{b}): & (d); & (d'); \end{matrix} \; s, s\right]$$
$$= \frac{1}{B(a, b)}\int_0^1 x^{a-1}(1-x)^{b-1}\, F^{k:C;C'}_{k:D;D'}\left[\begin{matrix} (\boldsymbol{a}): & (c); & (c'); \\ (\boldsymbol{a}+\boldsymbol{b}): & (d); & (d'); \end{matrix} \; sx, sx\right] dx.$$

Using another reduction formula given by Krupnikov (2001):

$$F^{2:1;3}_{2:0;2}\left[\begin{matrix} 2, 2: & 1; & 1, 1, 1; \\ 5/2, 3: & -; & 2, 2; \end{matrix} \; s, s\right] = \left({}_3F_2(1, 1, 1; 3/2, 2; s)\right)^2 \tag{4.4}$$

we obtain the second result. ∎

Proposition 4.2. *Let a, b, c, d be positive real numbers such that $c < a < b+c$ and consider the following five independent Dufresne random variables with respective laws:*

- $D_3 \sim D(c, d; a)$;
- $D_2 \sim D(b; -)$, *i.e.* γ_b;
- $D_1 \sim D(a; b+c)$, *i.e.* $\beta_{a,b+c-a}$;
- $D_2' \sim D(b+c+d-a, c; b+c)$;
- $D_1' \sim D(a-c; -)$, *i.e.* γ_{a-c}.

Then

$$D_1(D_2+D_3) \sim D'_1+D'_2. \tag{4.5}$$

Proof. The Laplace transform of $D_1(D_2+D_3)$ is given (see the integral representation of the previous proposition) by the following Kampé de Fériet series:

$$F^{1:1;2}_{1:0;1}\left[\begin{matrix} a: & b; & c, d; \\ b+c: & -; & a; \end{matrix}\ s, s\right], \tag{4.6}$$

which can be reduced to ${}_1F_0(a-c; -; s)\,{}_2F_1(b+c+d-a, c; b+c; s)$. We thank E. D. Krupnikov for this remark (personal communication). ■

PROPOSITION 4.3. *Let us consider the following five independent Dufresne random variables with respective laws:*

- $D_1 \sim D(2c;\, 1+2c)$, *i.e.* $\beta_{2c,1}$;
- $D_2 \sim D(\boldsymbol{a};\, \boldsymbol{b})$;
- $D_3, D'_1, D'_2 \sim D(c, \boldsymbol{a};\, 1+c, \boldsymbol{b})$, *where* $\boldsymbol{a} = (a_1, \ldots, a_{n+1})$ *and* $\boldsymbol{b} = (b_1, \ldots, b_n)$.

Then

$$D_1(D_2+D_3) \sim D'_1+D'_2. \tag{4.7}$$

Proof. The Laplace transform of $D_1(D_2+D_3)$ is given (see the integral representation of Proposition 4.1) by the following Kampé de Fériet series:

$$F^{1:2+r;1+r}_{1:1+r;r}\left[\begin{matrix} 2c: & c, \boldsymbol{a}; & \boldsymbol{a}; \\ 1+2c: & 1+c, \boldsymbol{b}; & \boldsymbol{b}; \end{matrix}\ s, s\right], \tag{4.8}$$

which can be reduced to $\left({}_{r+2}F_{r+1}(c, \boldsymbol{a}; c+1, \boldsymbol{b}; s)\right)^2$. We thank E. D. Krupnikov for this remark (personal communication). ■

COROLLARY 4.1. *Let us consider the following five independent mono- and bilateral Dufresne random variables with respective laws:*

- $D_1 \sim D(2c;\, 1+2c)$, *i.e.* $\beta_{2c,1}$;
- $\Lambda_2 \sim \lambda_{c+1}(\boldsymbol{a};\, \boldsymbol{b})$;
- $\Lambda_3, \Lambda'_1, \Lambda'_2 \sim \lambda_c(\boldsymbol{a};\, \boldsymbol{b})$, *where* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are of length* r.

Then

$$D_1(\Lambda_2+\Lambda_3) \sim \Lambda'_1+\Lambda'_2. \tag{4.9}$$

Proof. The Laplace transform of $D_1(\Lambda_2+\Lambda_3)$ is given by the following Kampé de Fériet series:

$$F^{1:2r+1;2r+1}_{1:2r;2r}\left[\begin{matrix} 2c: & c, \boldsymbol{a}/2, (\boldsymbol{a}+1)/2; & 1+c, \boldsymbol{a}/2, (\boldsymbol{a}+1)/2; \\ 1+2c: & \boldsymbol{b}/2, (\boldsymbol{b}+1)/2; & \boldsymbol{b}/2, (\boldsymbol{b}+1)/2; \end{matrix}\ s^2, s^2\right], \tag{4.10}$$

which, as before, can be reduced to $\left({}_{2r+1}F_{2r}(c, \boldsymbol{a}/2, (\boldsymbol{a}+1)/2; \boldsymbol{b}/2, (\boldsymbol{b}+1)/2; s^2)\right)^2$. ■

5. RANDOM DIFFERENCE EQUATIONS

We present now a series of examples concerning the random induction:

$$X_{n+1} \sim A_n(X_n + Y_n), \quad n \in N,$$

where X_n, Y_n are independent Dufresne random variables and A_n are products of independent beta variables.

PROPOSITION 5.1. *Let $0 < a < 1$, where $a \neq 1/2$. Let X_n, Y_n, A_n, B_n be independent random variables with respective laws:*

$$X_n \sim D(a, a+1/2; 2a), \qquad Y_n \sim D(1-a, 3/2-a; 2(1-a))$$

and

$$A_n \sim D(a, a-1/2; 1, 2a) \text{ for } 1/2 < a < 1,$$

$$B_n \sim D(a, 1/2-a; 1, 2(1-a)) \text{ for } 0 < a < 1/2.$$

Then for $1/2 < a < 1$

$$X_{n+1} \sim A_n(X_n + Y_n) \tag{5.1}$$

and for $0 < a < 1/2$

$$Y_{n+1} \sim B_n(X_n + Y_n). \tag{5.2}$$

Proof. The Laplace transform of $A_n(X_n + Y_n)$ is given (see the integral representation of the previous proposition) by the following Kampé de Fériet series:

$$F^{2:2;2}_{2:1;1}\left[\begin{matrix} a, a+1/2: & a, a+1/2; & 1-a, 3/2-a; & \\ 1, 2a: & 2a; & 2(1-a); & \end{matrix} s, s\right], \tag{5.3}$$

which can be reduced to ${}_2F_1(a, a+1/2; 2a; s)$ according to Srivastava and Karlsson (1985), p. 32, equation (51). The Laplace transform of $B_n(X_n + Y_n)$ is reduced to ${}_2F_1(1-a, 3/2-a; 2(1-a); s)$. ∎

PROPOSITION 5.2. *Let $a > 0$, $b > 0$, and $\max(a, b) < c < a+b$. Let X_n, Y_n, A_n be independent random variables with respective laws:*

$$D(a, b; c), \ D(c-a-b; -), \ D(a, b; c-a, c-b).$$

Then

$$X_{n+1} \sim A_n(X_n + Y_n). \tag{5.4}$$

Proof. The Laplace transform of $A_n(X_n + Y_n)$ is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

$$F^{2:2;1}_{2:1;0}\left[\begin{matrix} a, b: & a, b; & c-a-b; \\ c-b, c-a: & c; & -; \end{matrix} \; s, s\right], \tag{5.5}$$

which can be reduced to ${}_2F_1(a, b; c; s)$ according to Srivastava and Karlsson (1985), p. 28, equation (34). ■

PROPOSITION 5.3. *Let $a > 0$ and $b > 0$ be such that $1/2 < a+b < 1$. Let X_n, Y_n, A_n be independent random variables with respective laws:*

$$D(a, b; a+b+1/2),\ D(a, b; a+b-1/2),\ D(a, b, 2a+2b-1; 2a, 2b, a+b).$$

Then

$$X_{n+1} \sim A_n(X_n+Y_n). \tag{5.6}$$

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given (see the integral representation in Proposition 4.1) by

$$F^{3:2;2}_{3:1;1}\left[\begin{matrix} a, b, 2(a+b)-1: & a, b; & a, b; \\ 2a, 2b, a+b: & a+b+1/2; & a+b-1/2; \end{matrix} \; s, s\right], \tag{5.7}$$

which can be reduced to ${}_2F_1(a, b; a+b+1/2; s)$ according to Srivastava and Karlsson (1985), p. 29, equation (37). ■

PROPOSITION 5.4. *Let $0 < a < 1/2$ and $1 < b < 1+2a$. Let X_n, Y_n, A_n be independent Dufresne random variables of respective laws*

$$D(a, b-1; a+b-1/2),\ D(a, b; a+b-1/2),$$
$$D\big(a, b-1, 2(a+b-1); a+b-1, 2a, 2b\big).$$

Then

$$X_{n+1} \sim A_n(X_n+Y_n) \tag{5.8}$$

for $1/2 > a > 0$ and $1 < b < 1+2a$.

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

$$F^{3:2;2}_{3:1;1}\left[\begin{matrix} a, b-1, 2(a+b-1): & a, b-1; & a, b; \\ a+b-1, 2a, 2b-1: & a+b-1/2; & a+b-1/2; \end{matrix} \; s, s\right], \tag{5.9}$$

which can be reduced to ${}_2F_1(a, b-1; a+b-1/2; s)$ according to Srivastava and Karlsson (1985), p. 29, equation (39). ■

PROPOSITION 5.5. *Let $a, b > 1/2$. Let X_n, Y_n, A_n be independent Dufresne random variables with respective laws:*

$$D(a+1/2, b+1/2; a+b+1/2),\ D(a-1/2, b-1/2; a+b-1/2),$$
$$D(a, b, a+b; 2a-1/2, 2b-1/2, 2a+2b-1).$$

Then

$$X_{n+1} \sim A_n(X_n + Y_n). \tag{5.10}$$

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

$$F^{3:2;2}_{3:1;1}\left[\begin{matrix} a,\, b,\, a+b: & a+1/2,\, b+1/2; & a-1/2,\, b-1/2; \\ 2a-1/2,\, 2b-1/2,\, 2(a+b)-1: & a+b+1/2; & a+b-1/2; \end{matrix}\; s,\, s\right], \tag{5.11}$$

which can be reduced to ${}_2F_1(a+1/2, b+1/2; a+b+1/2; s)$ according to Srivastava and Karlsson (1985), p. 30, equation (40). ∎

PROPOSITION 5.6. *Let* $0 < 2b < a < b+1$. *Let* X_n, Y_n, A_n *be independent Dufresne random variables with respective laws*:

$$D(a,\, 1+a/2,\, b;\, 1+a-b,\, a/2),\quad D(1;\, -),$$

i.e. exponential random variables, and

$$D(a,\, b,\, 1+a/2;\, a+b+1,\, b+a/2,\, 2+3a/2),$$

i.e. products of three independent beta variables. Then

$$X_{n+1} \sim A_n(X_n + Y_n). \tag{5.12}$$

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

$$F^{3:3;1}_{3:2;0}\left[\begin{matrix} a,\, b,\, 1+a/2: & a,\, 1+a/2,\, b; & 1; \\ 1+a,\, 1+b,\, a/2: & a/2,\, 1-b+a; & -; \end{matrix}\; s,\, s\right], \tag{5.13}$$

which can be reduced to ${}_3F_2(a, 1+a/2, b; 1+a-b, a/2; s)$ according to Lavoie and Grondin (1994), p. 395, equation (7). ∎

PROPOSITION 5.7. *Let* $0 < 2b < a < b+1$, $b+c < a$, $c > 0$. *Let* X_n, Y_n, A_n *be independent Dufresne random variables with respective laws*:

$$D(a,\, 1+a/2,\, b,\, c,\, a-b-c;\, 1+a-b,\, 1+a-c,\, 1+b+c,\, a/2),\quad D(1;\, -),$$

i.e. exponential random variables, and

$$D\big(a,\, b,\, 1+a/2,\, c,\, a-b-c;\, a+b+1,\, b+a/2,\, 2+3a/2,\, 2c+1,\, 1+2(a-b-c)\big),$$

i.e. products of five independent beta variables. Then

$$X_{n+1} \sim A_n(X_n + Y_n). \tag{5.14}$$

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given (see the integral representation in Proposition 4.1) by the following Kampé de Fériet series:

$$F^{5:5;1}_{5:4;0}\left[\begin{matrix} a,\, b,\, 1+a/2,\, c,\, a-b-c: & a,\, 1+a/2,\, b,\, c,\, a-b-c; & 1; \\ 1+a,\, 1+b,\, a/2,\, 1+c,\, 1+a-b-c: & 1+a-b,\, 1+a-c,\, 1+b+c,\, a/2; & -; \end{matrix}\; s,\, s\right], \tag{5.15}$$

which can be reduced to

$$ {}_5F_4(a,\ 1+a/2,\ b,\ c,\ a-b-c;\ 1+a-b,\ 1+a-c,\ 1+b+c,\ a/2;\ s) $$

according to Lavoie and Grondin (1994), p. 398, equation (16). ∎

PROPOSITION 5.8. *Let* $0 < b < c < a < b+1$. *Let* X_n, Y_n, A_n *be independent Dufresne random variables with respective laws:*

$$ X_n \sim D\big(a,\ b,\ 1+ab/(a+b-c);\ 1+c,\ ab/(a+b-c)\big), \qquad Y_n \sim D(1;\ -) $$

(they are exponential variates) and

$$ A_n \sim D\big(a,\ b,\ 1+ab/(a+b-c);\ a+b+1,\ b\big(1+a/(a+b-c)\big),\ 2+a\big(1+b/(a+b-c)\big)\big) $$

(a product of three independent beta variables). Then

$$ X_{n+1} \sim A_n(X_n+Y_n). \tag{5.16} $$

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given (see integral representation in Proposition 4.1) by the following Kampé de Fériet series:

$$ F^{3:3;1}_{3:2;0}\left[\begin{matrix} a,\ b,\ 1+ab/(a+b-c): & a,\ b,\ 1+ab/(a+b-c); & 1; \\ 1+a,\ 1+b,\ ab/(a+b-c): & 1+c,\ ab/(a+b-c); & -; \end{matrix}\ s,\ s\right], \tag{5.17} $$

which can be reduced to ${}_3F_2\big(a,\ b,\ 1+ab/(a+b-c);\ 1+c,\ ab/(a+b-c);\ s\big)$ according to Lavoie and Grondin (1994), p. 397, equation (13). ∎

In the next corollary we offer an example among other possible choices of interrelations between the parameters a, b, c when they fulfill the reduction formula (5.19) below.

COROLLARY 5.1. *Let* $b > 0$, $a+b > c$, $a = b+1/2$, $c = b+\frac{1}{2}\sqrt{2b+1}$ *and let* X_n, Y_n, A_n *be independent mono- and bilateral Dufresne random variables such that*

$$ X_n \sim \lambda_{1+ab/(a+b-c)}(2b;\ 1+2c), \qquad Y_n \sim \lambda_1(-;\ -) $$

(they are double exponential random variables) and

$$ A_n \sim D\big(a,\ b,\ 1+ab/(a+b-c);\ a+b+1,\ b\big(1+a/(a+b-c)\big),\ 2+a\big(1+b/(a+b-c)\big)\big) $$

(a product of three independent beta variables). Then

$$ X_{n+1} \sim A_n(X_n+Y_n). \tag{5.18} $$

Proof. The Laplace transform of $A_n(X_n+Y_n)$ is given by the following Kampé de Fériet series:

$$ F^{3:3;1}_{3:2;0}\left[\begin{matrix} a,\ b,\ 1+ab/(a+b-c): & a,\ b,\ 1+ab/(a+b-c); & 1; \\ 1+a,\ 1+b,\ ab/(a+b-c): & 1+c,\ ab/(a+b-c); & -; \end{matrix}\ s^2,\ s^2\right], \tag{5.19} $$

which can be reduced to ${}_3F_2(a, b, 1+ab/(a+b-c); 1+c, ab/(a+b-c); s^2)$ according to the previous reference. ■

6. RANDOM ARITHMETICAL ENCOUNTERS OF THIRD KIND

In this section we consider four independent random variables B_1, B_2, G_1, G_2 with beta distribution of third kind and gamma distribution, and we give the distributions of the random variables $Z = (G_1 G_2)(B_1 + B_2)$ for some values of the parameters.

PROPOSITION 6.1. *Let a, b, g_1, g_2 be positive numbers, B_1, B_2, G_1, G_2 be positive independent random variables such that $G_j \sim \gamma_{g_j}, j = 1, 2$, and put $Z = (G_1 G_2)(B_1 + B_2)$ and $F_Z(s) = E(e^{isZ})$. Then:*

If $B_1 \sim \beta^{(3)}_{a+1/2, a+1/2}$ and $B_2 \sim \beta^{(3)}_{b+1/2, b+1/2}$, then

(6.1) $F_Z(s) = {}_4F_3(g_1, g_2, \tfrac{1}{2}(a+b), \tfrac{1}{2}(a+b)+1; a+1/2, b+1/2, a+b; -s^2)$.

In particular:

(1) *for $g_1 = 1/2$, $b = 0$ we have $Z \sim \lambda_{g_2}(a; 2a)$;*

(2) *for $g_1 = 3/2$, $b = 1$ we have $Z \sim \lambda_{g_2}(a+1; 2a+1)$.*

Proof. The Fourier transform of $B_1 + B_2$ is

$$ {}_0F_1(-; a+1; -s^2/4)\, {}_0F_1(-; b+1; -s^2/4) $$
$$ = {}_2F_3(\tfrac{1}{2}(a+b), \tfrac{1}{2}(a+b)+1; a+1/2, b+1/2, a+b; -s^2); $$

this equality comes from Erdelyi et al. (1954), p. 185, formula 2. Therefore

$$ F_Z(s) = \frac{1}{\Gamma(g_1)\Gamma(g_2)} \times $$
$$ \int_0^\infty \int_0^\infty e^{-x-y} x^{g_1-1} y^{g_2-1}\, {}_2F_3(\tfrac{1}{2}(a+b), \tfrac{1}{2}(a+b)+1; a+\tfrac{1}{2}, b+\tfrac{1}{2}, a+b; -xys^2)\, dx dy. $$

From Gradsteyn and Ryzhyk (1980), formula 7.52.9, we get (6.44). Replacing the parameters by the indicated values and using the possible reduction of ${}_4F_3$ to the canonical form ${}_3F_2$ of the lambda law, we complete the proof of parts (1) and (2). ■

PROPOSITION 6.2. *Let a, b be positive numbers and let $G_1, G_2; G'_1, G'_2$ be four gamma independent random variables with respective parameters $a, b; a, b$. If $G_0 \sim \gamma_{a+b}$ is independent of the lambda variable $\Lambda_0 \sim \lambda_1(b; b+1/2)$, then*

$$ \tfrac{1}{2}(G_1 G_2 - G'_1 G'_2) \sim G_0^2 \Lambda_0. $$

Proof. The Laplace transforms of $G_1 G_2$ and $G'_1 G'_2$ are ${}_2F_0(a, b; -; s)$; the Laplace transform of the half of their difference is

(6.2) $$ {}_2F_0(a, b; -; s/2)\, {}_2F_0(a, b; -; -s/2) = {}_4F_1\left(a, b, \frac{a+b}{2}, \frac{a+b+1}{2}; a+b; s^2\right); $$

this equality comes from Erdelyi et al. (1954), p. 186, formula 4. Therefore for the Laplace transform

$$L_{G_0^2 \Lambda_0}(s) = \frac{1}{\Gamma(a+b)} \int_0^\infty e^{-x} x^{a+b-1} {}_2F_1\left(a, b; a+b; (xs)^2\right) dx dy.$$

From Gradsteyn and Ryzhyk (1980), formula 7.525.1, we get the right-hand side of the previous relation, which completes the proof. ■

PROPOSITION 6.3. *Let d be a positive parameter and assume that*

- $B_1 \sim D(1; 2+d/2)$, *i.e.* $\beta_{1,1+d/2}$;
- $G_1 \sim D(2; -)$, *i.e.* γ_2;
- $G_2 \sim D(d; -)$, *i.e.* γ_d;
- $G_0 \sim D(1; -)$, *i.e. an exponential variable*;
- $D_0 \sim D(1+d, d/2; 2+d/2)$.

Then

$$B_1(G_2 - G_1) \sim D_0 - G_0.$$

Proof. The Laplace transform of the product of a beta variable and the difference of two gamma variables $B_1(G_2-G_1)$ is

$$F^{1:1;1}_{1:0;0}\left[\begin{matrix} a: & c; & d; \\ a+b: & -; & -; \end{matrix} \; -s, s\right] = \frac{1}{B(a, b)} \int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{(1-sx)^d (1+sx)^c} dx$$

(see Exton (1978), p. 39, formula 2.1.5.7). Therefore using the transformation of the Kampé de Fériet functions:

$$F^{1:1;1}_{1:0;0}\left[\begin{matrix} a: & c; & d; \\ a+b: & -; & -; \end{matrix} \; -s, s\right] \tag{6.3}$$

$$= \frac{1}{(1-s)^{c-b}} F^{1:2;1}_{1:1;0}\left[\begin{matrix} a+b-c: & a, d; & b; \\ a+b: & a+b-c; & -; \end{matrix} \; -s, s\right].$$

(see Exton (1978), p. 39, formula 2.1.5.7) we reduce it for $a = 1+d/2$, $c = 2$, $b = 1$ to the form

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix} a+b-c: & 1+d/2, d; & 1; \\ a+b: & d/2; & -; \end{matrix} \; -s, s\right] = {}_2F_1(1+d, a+b-c; a+b; s)$$

according to Lavoie and Grondin (1994). Thus the proof is completed. Note that $d = 2$ is the only symmetrical case. ■

7. SUMMARY TABLES

parameters conditions	law D_0	law D'_1	law D'_2
$a > 1/2$ $0 < b < a-1/2$	$2a,\ 2b,\ a-1/2$ $a+b+1/2,\ 2a-1$	b —	$b,\ a-b-1/2$ $a+b+1/2$
$a > 0$ $1/2 < b < a+1/2$	$2a+1,\ b+1/2$ $2a-b+3/2$	1 —	$2a,\ a+1,\ b-1/2$ $a,\ a-b+3/2$
	law Λ_0	law Λ'_1	law Λ'_2
	$4(3;\ 6)$ $8(7;\ 13)$	$1(-;\ -)$ $2(-;\ -)$	$3(1;\ 6)$ $6(3;\ 12)$

Then for all these cases:

$$D_0 \sim D'_1 + D'_2. \tag{7.1}$$

parameters conditions	law D_1	law D_2	law D'_1	law D'_2
$0 < a,\ b < 1$ $\max(a,\ b) < c < \min(1+a,\ 1+b)$	$a,\ b$ c	$1-a,\ 1-b$ $2-c$	$c-a,\ c-b$ c	$a+1-c,\ b+1-c$ $2-c$
parameters conditions	law Λ_1	law Λ_2	law Λ'_1	law Λ'_2
$0 < a < 1/2$ $a+1/2 < c < a+1$	$c+1/2$ $2a;\ 2c$	$3/2-c$ $1-2a;\ 3-2c$	$c+1/2$ $2(c-a)-1;\ 2-c$	$3/2-c$ $2(1+a-c);\ 3-2c$

Then for all these cases:

$$D_1 + D_2 \sim D'_1 + D'_2. \tag{7.2}$$

parameters conditions	law D_1	law D_2	law D_3	law D'_1	law D'_2
	$2;\ 5/2$	$1;\ -$	$1,\ 1,\ 1;\ 2,\ 2$	$1,\ 1;\ 3/2$	$1,\ 1,\ 1;\ 3/2,\ 2$
	$2;\ 3$	$1,\ 1;\ 3/2$	$1,\ 1,\ 1;\ 3/2,\ 2$	$1,\ 1,\ 1;\ 3/2,\ 2$	$1,\ 1,\ 1;\ 3/2,\ 2$
$0 < d,\ c < a < b+c$	$a,\ b+c$	$b;\ -$	$c,\ d;\ a$	$a-c;\ -$	$b+c+d-a,\ c;\ b+c$
$c,\ \mathbf{a},\ \mathbf{b} > 0$	$2c;\ 1+2c$	$\mathbf{a};\ \mathbf{b}$	$c,\ \mathbf{a};\ 1+c,\ \mathbf{b}$	$c,\ \mathbf{a};\ 1+c,\ \mathbf{b}$	$c,\ \mathbf{a};\ 1+c,\ \mathbf{b}$
parameters conditions	law D_1	law Λ_2	law Λ_3	law Λ'_1	law Λ'_2
$c,\ \mathbf{a},\ \mathbf{b} > 0$	$2c;\ 1+2c$	$\lambda_{c+1}(\mathbf{a};\ \mathbf{b})$	$\lambda_c(\mathbf{a};\ \mathbf{b})$	$\lambda_c(\mathbf{a};\ \mathbf{b})$	$\lambda_c(\mathbf{a};\ \mathbf{b})$

Note. $\mathbf{a}$ and $\mathbf{b}$ are of length $r+1$ and r, respectively.

Then for all these cases:

(7.3)
$$D_1(D_2+D_3) \sim D_1'+D_2'.$$

law of Λ_0	law of G_1	law of G_2	law of Λ_1	law of Λ_2
$\lambda_g(a;\,2a)$	$g;\ -$	$1/2;\ -$	$\beta^{(3)}_{1/2,1/2}$	$\beta^{(3)}_{a+1/2,a+1/2}$
$\lambda_g(a+1;\,2a+1)$	$g;\ -$	$3/2;\ -$	$\beta^{(3)}_{3/2,3/2}$	$\beta^{(3)}_{a+1/2,a+1/2}$

Then for all these cases:

(7.4)
$$\Lambda_0 \sim G_1 G_2(\Lambda_1+\Lambda_2).$$

law of G_1	law of G_1'	law of G_2	law of G_2'	law of G_0	law of Λ_0
a	a	b	b	$a+b$	$\lambda_1(b;\,b+1/2)$

Then for all these cases:

(7.5)
$$\tfrac{1}{2}(G_1 G_2 - G_1' G_2') \sim G_0^2 \Lambda_0.$$

law of B_1	law of G_1	law of G_2	law of G_0	law of D_0
$1,\ 1+d/2$	2	d	1	$1+d,\ d/2;\ 2+d/2$

Then for all these cases:

(7.6)
$$B_1(G_2-G_1) \sim D_0-G_0.$$

parameters conditions	law X_n	law Y_n	law A_n
$1/2<a<1$	$a,\ a+1/2$ $2a$	$1-a,\ 3/2-a$ $2(1-a)$	$a,\ a-1/2$ $1,\ 2a$
$0\le a<1/2$	$1-a,\ 3/2-a$ $2(1-a)$	$a,\ a+1/2$ $2a$	$a,\ 1/2-a$ $1,\ 2(1-a)$
$a>0$ and $b>0$ $\max(a,b)<c<a+b$	$a,\ b$ c	$c-a-b$ $-$	$a,\ b$ $c-a,\ c-b$
$a>0$ and $b>0$ $1/2<a+b<1$	$a,\ b$ $a+b+1/2$	$a,\ b$ $a+b-1/2$	$a,\ b,\ 2a+2b-1$ $2a,\ 2b,\ a+b$
$0<a<1/2$ $1<b<1+2a$	$a,\ b-1$ $a+b-1/2$	$a,\ b$ $a+b-1/2$	$a,\ b,\ 2(a+2b-1)$ $2a,\ 2b,\ a+b-1$
$a,\ b>1/2$	$a+1/2,\ b+1/2$ $a+b+1/2$	$a-1/2,\ b-1/2$ $a+b-1/2$	$a,\ b,\ a+b$ $2a-1/2,\ 2b-1/2,\ 2a+2b-1$

law of (Y_n is $D(1; -)$) X_n	law of A_n
parameters, conditions	$0 < 2b < a < b+1$
$a,\ 1+a/2,\ b$ $1+a-b,\ a/2$	$a,\ b,\ 1+a/2$ $a+b+1,\ b+a/2,\ 2+3a/2$
parameters, conditions	$0 < 2b < a < b+1$ and $b+c < a,\ c > 0$
$a,\ b,\ 1+ab/(a+b-c)$ $1+a-b,\ 1+a-c,\ 1+b+c,\ a/2$	$a,\ b,\ 1+a/2,\ c,\ a-b-c$ $a+b+1,\ b+a/2,\ 2+3a/2,\ 2c+1,\ 1+2(a-b-c)$
parameters, conditions	$0 < b < c < a < b+1$
$a,\ b,\ 1+ab/(a+b-c)$ $1+c,\ ab/(a+b-c)$	$a,\ b,\ 1+ab/(a+b-c)$ $a+b+1,\ b(1+a/(a+b-c)),\ 2+a(1+b/(a+b-c))$
law of (Y_n is $\lambda_1(-; -)$) X_n	law of A_n
parameters, conditions	$b > 0,\ a+b > c,\ a = b+1/2,\ c = b+\frac{1}{2}\sqrt{2b+1}$
$\lambda_{1+ab/(a+b-c)}$ $(2b; 1+2c)$	$a,\ b,\ 1+ab/(a+b-c)$ $a+b+1,\ b(1+a/(a+b-c)),\ 2+a(1+b/(a+b-c))$

Then for all these cases:

$$X_{n+1} \sim A_n(X_n + Y_n). \tag{7.7}$$

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