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PORTFOLIO DIVERSIFICATION WITH MARKOVIAN PRICES

BY

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Abstract. The problem of constructing impulsive rebalancing of portfolios, introduced by Pliska and Suzuki, is solved for models with general Markovian prices. Existence of the optimal strategy is established and its structure described. Quasi-variational inequalities determining the value function are deduced for multiplicative prices with general Lévy noise and the case of Poissonian noise is considered in some detail.

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1. INTRODUCTION

The present paper is an application of stochastic control theory to financial problems. We treat only mathematical aspects of the problem postponing numerical investigation to a future paper.

An important problem for portfolio managers is to respect the diversification requirement, that is to maintain proportions of the capital that should be invested in different asset groups, constant. It is impossible to rebalance a portfolio continuously, so it usually does not keep exactly to the required proportions. Therefore each manager has to come up with some algorithm to decide the moments of rebalancing.

Pliska and Suzuki [7], [8], elaborating the ideas of Leland [5], considered a model consisting of d assets, $d = 2$, whose prices satisfy

$$dS_t^i = S_t^i(\mu_i\, dt + \sigma_i\, dW_t), \quad i = 1, \ldots, d,$$

where W_t is an m-dimensional Brownian motion, σ_i are vectors, and μ_i are real numbers. A trading strategy was described by a d-dimensional adapted process $(N_t)_{t \geqslant 0}$ denoting the number of units of assets held at each moment. In [7], [8]

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both proportional and constant transaction costs were introduced. We will specify them in detail later.

In view of constant transaction costs, any trading strategy Π can be described by a sequence of transaction times (stopping times) $\tau_1, \tau_2, \ldots$ and resulting portfolio contents $N_{\tau_1}, N_{\tau_2}, \ldots$ The process N_t is *constant* between transaction times, i.e.

$$N_t = N_0 1_{t \in [0, \tau_1[} + \sum_{i=1}^{\infty} N_{\tau_i} 1_{t \in [\tau_i, \tau_{i+1}[}.$$

Let w be a *proportion process* linked up with the strategy Π by the formula

$$w_t = W\begin{pmatrix} N_t \\ S_t \end{pmatrix},$$

where, denoting by ns, $n = (n^1, \ldots, n^d)$, $s = (s^1, \ldots, s^d)$, the scalar product in $\boldsymbol{R}^d$, we have

$$W\begin{pmatrix} n \\ s \end{pmatrix} = \left(\frac{n^1 s^1}{ns}, \ldots, \frac{n^d s^d}{ns}\right).$$

The transaction costs are expressed in terms of proportions:

$$c(w, v) = K + k \sum_{i=1}^{d} |w^i - v^i|$$

for $K > 0$ and $k \geqslant 0$. This is a reasonable simplification that enables to incorporate transactions costs into a cost functional. Pliska and Suzuki introduced the cost functional

$$J(\Pi) = \boldsymbol{E}\left(\int_0^{\infty} e^{-\beta t} f(w_t)\, dt + \sum_{i=1}^{\infty} e^{-\beta \tau_i} c\left(W\begin{pmatrix} N_{\tau_i -} \\ S_{\tau_i} \end{pmatrix}, W\begin{pmatrix} N_{\tau_i} \\ S_{\tau_i} \end{pmatrix}\right) 1_{\tau_i < \infty}\right),$$

where $f\colon \boldsymbol{R}^d \to \boldsymbol{R}$ is a function measuring quality of the portfolio. They specified further that

$$f(w) = \lambda (w - w^*)' \sigma \sigma' (w - w^*) - (w - w^*)' \mu,$$

where σ is a matrix consisting of rows σ_i, $i = 1, \ldots, d$, $\mu = (\mu_1, \ldots, \mu_d) \in \boldsymbol{R}^d$, w^* is a target asset mix and $\lambda \in \boldsymbol{R}$.

In Pliska and Suzuki [7], [8], in the two-dimensional case, $d = m = 2$, the existence of optimal strategy and its characterization in terms of a continuation region was established. The results were based on the theory of quasi-variational inequalities (see [1]). Some financial consequences were discussed as well.

In the present paper we cover general Markovian price processes in all dimensions and examine several extensions of the results from [7], [8]. In particular, we deal with discontinuous price processes. More specifically we consider a market modeled by a general d-dimensional Markov process $(S_t)_{t \geqslant 0}$, with positive coordinates, representing price movements of different assets. Our cost functional, measuring quality of the portfolio at each moment, is determined by a continuous function f, defined on proportions. We prove in Section 2 (see Theorem 2.1) that under the minimal assumptions on the price process S and the function f there exists an optimal trading strategy Π minimizing the cost functional $J(\Pi)$. In Section 3 we consider the so-called multiplicative price processes, important in applications. Using Dynkin's result on Markov processes with transformed state spaces, we show that the optimal solution can be expressed in terms of the proportion processes (see Theorem 3.2). The result is then applied to price processes satisfying an Itô equation with Lévy noise. In addition, the precise form of the quasi-variational equality for the value function is established as well (see Proposition 3.3). More explicit cases are treated in Section 4. Here the noise process is assumed to be Poissonian with linear drift and the proportional cost for impulses is excluded. Sufficient conditions are given under which the continuation region is of a simple form. We use here the technique of quasi-variational inequalities and derive a transcendental equation (16). The solution to that equation indicates, under additional conditions, a point to which impulses should be performed. Finally we show (see Theorem 4.8) that one can construct a cost function f for which the optimal continuation region is given in advance. More details on quasi-variational inequalities for discontinuous process are given in the Appendix.

The paper is a rewritten version of the report [6].

2. EXISTENCE OF OPTIMAL STRATEGY

We approach the problem of finding optimal solution with the impulse control method. First we prove the existence of solution to the functional equation connected with our problem. Then we show that the obtained solution defines the optimal strategy.

We assume throughout this section that S_t is a Feller process and the function f is continuous.

To formulate the problem in a formal way we take

$$Y_t = \begin{pmatrix} N_t \\ S_t \end{pmatrix} \in \boldsymbol{R}_+^{2d}$$

as a controlled process. Certainly impulses change only the first coordinate; the second coordinate – representing asset prices – is present only for technical reasons. Our goal is to construct a process Y_t satisfying the following conditions:

- there exists an increasing sequence of stopping times $\tau_1, \tau_2, \ldots$ with $\tau_i \uparrow \infty$ such that the first coordinate of Y_t changes only at moments defined by $(\tau_i)_{i=1,2,\ldots}$;
- the second coordinate of Y_t is equal to an external price process S_t;
- the trading strategy encoded in Y_t is self-financing, i.e.

$$(N_{\tau_i} - N_{\tau_i -}) S_{\tau_i} = 0;$$

- the portfolio wealth is always positive, i.e. $N_t S_t > 0$;
- the number of shares of each stock is non-negative (no borrowing of shares allowed), i.e. $N_t \geqslant 0$,

and minimizing the functional

$$(1) \qquad J(Y) = E^{Y_0}\left(\int_0^\infty e^{-\beta t} F\begin{pmatrix} N_t \\ S_t \end{pmatrix} dt + \sum_{i=1}^\infty e^{-\beta \tau_i} \tilde{C}(N_{\tau_i -}, N_{\tau_i}, S_{\tau_i}) 1_{\tau_i < \infty}\right),$$

where

$$F\begin{pmatrix} N_t \\ S_t \end{pmatrix} = f\left(W\begin{pmatrix} N_t \\ S_t \end{pmatrix}\right)$$

and

$$W\begin{pmatrix} N_t \\ S_t \end{pmatrix} = \left(\frac{N_t^1 S_t^1}{N_t S_t}, \ldots, \frac{N_t^d S_t^d}{N_t S_t}\right)$$

is the proportion function, $Y_0 = (N_0, S_0)$ is the initial point ($S_0 > 0$, $N_0 \geqslant 0$, $N_0 S_0 > 0$). The cost of impulses is defined as

$$\tilde{C}(N_1, N_2, S) = c\left(W\begin{pmatrix} N_1 \\ S \end{pmatrix}, W\begin{pmatrix} N_2 \\ S \end{pmatrix}\right).$$

Note that between impulses the dynamics of Y_t is governed by the semigroup

$$P_t^* v\begin{pmatrix} n \\ s \end{pmatrix} = P_t\left(v\begin{pmatrix} n \\ \cdot \end{pmatrix}\right)(s),$$

where $v \in C(\boldsymbol{R}^{2d}, \boldsymbol{R})$, and P_t is the semigroup for S_t.

To derive a functional equation connected with the problem (1) we recall the assumption that $\boldsymbol{P}(S_t > 0 \ \forall t) = 1$ and put

$$E = \left\{\begin{pmatrix} n \\ s \end{pmatrix} \in \boldsymbol{R}^{2d}\colon n \geqslant 0, \ n \neq 0, \ s > 0\right\}.$$

It is obvious that the process $\begin{pmatrix} N_t \\ S_t \end{pmatrix}$ starting from any point in E does not exit E.

For functions $v\colon E \to \boldsymbol{R}$ we write the equation

$$(2) \qquad v\begin{pmatrix} n \\ s \end{pmatrix} = \mathcal{K} v\begin{pmatrix} n \\ s \end{pmatrix} = \inf_\tau E^{(n,s)}\left[\int_0^\tau e^{-\beta t} F\begin{pmatrix} N_t \\ S_t \end{pmatrix} dt + e^{-\beta\tau} M v\begin{pmatrix} N_\tau \\ S_\tau \end{pmatrix}\right],$$

where the switching functional is given as

$$(3) \qquad M\phi\begin{pmatrix} n \\ s \end{pmatrix} = \inf\left\{\tilde{C}(n, n+\xi, s) + \phi\begin{pmatrix} n+\xi \\ s \end{pmatrix}\colon \begin{pmatrix} n+\xi \\ s \end{pmatrix} \in E, \ \xi s = 0\right\}.$$

The following theorem contains as a special case a result by Pliska and Suzuki [7] concerned with the case of S_t being a two-dimensional Black–Scholes process. Their method was based on the theory of quasi-variational inequalities (QVI) [1]. We deal directly with the equation (2).

THEOREM 2.1. *Assume that S_t is a Feller process and f is a bounded continuous function of E. Then there exists exactly one bounded continuous solution $v(n, s)$ to the equation* (2) *and the optimal strategy for the problem* (1) *is given by*

$$\tau_1 = \inf\{t \geqslant 0\colon\ Mv(N_t, S_t) = v(N_t, S_t)\},$$

$$\tau_i = \inf\{t > \tau_{i-1}\colon\ Mv(N_t, S_t) = v(N_t, S_t)\},$$

$$N_{\tau_i} \in \{n \in \boldsymbol{R}^d\colon\ (n, S_{\tau_i}) \in E,\ Mv(N_{\tau_{i-1}}, S_{\tau_i}) = v(n, S_{\tau_i}) + \tilde{C}(N_{\tau_{i-1}}, n, S_{\tau_i})\}.$$

Proof. In order to prove the existence of a unique solution to the equation (2) we recall a result from [9]; see also [4]. Define

$$h\begin{pmatrix} n \\ s \end{pmatrix} = E^{(n,s)}\left[\int_0^\infty e^{-\beta t} F\begin{pmatrix} N_t \\ S_t \end{pmatrix} dt\right]$$

and let $C^b(E)$ be the space of bounded continuous functions.

PROPOSITION 2.2. *Assume that* $\begin{pmatrix} N_t \\ S_t \end{pmatrix}$ *is a Feller process, $F \geqslant 0$, $h \in C^b(E)$, $\gamma h \leqslant M(0)$ for a positive constant γ, and M transforms $C^b(E)$ into $C^b(E)$. Then the equation* (2) *has exactly one solution $v \in C^b(E)$. Moreover, $\mathscr{K}^n h$ tends to v uniformly as $n \to \infty$.*

In our setting we have to weaken conditions of the above theorem. We define operators

$$\mathscr{K}^L v\begin{pmatrix} n \\ s \end{pmatrix} = \inf_\tau E^{(n,s)}\left[\int_0^\tau e^{-\beta t}\left(F\begin{pmatrix} N_t \\ S_t \end{pmatrix} + L\right) dt + e^{-\beta\tau} Mv\begin{pmatrix} N_\tau \\ S_\tau \end{pmatrix}\right]$$

for $L \in \boldsymbol{R}$. Thus $\mathscr{K}^0 = \mathscr{K}$.

LEMMA 2.3. *There exists a unique solution to the equation $v = \mathscr{K} v$ iff there exists a unique solution to the equation $v = \mathscr{K}^L v$. Moreover, if $\tilde{v}$ is the solution of $v = \mathscr{K}^L v$, then $\tilde{v} - L/\beta$ is the solution of $v = \mathscr{K} v$.*

Proof. Let $\tilde{v}$ be the solution of $v = \mathscr{K}^L v$. Then

$$\tilde{v}\begin{pmatrix} n \\ s \end{pmatrix} = \mathscr{K}^L \tilde{v}\begin{pmatrix} n \\ s \end{pmatrix}$$

$$= \inf_\tau E^{(n,s)}\left[\int_0^\tau e^{-\beta t}\left(F\begin{pmatrix} N_t \\ S_t \end{pmatrix} + L\right) dt + e^{-\beta\tau} Mv\begin{pmatrix} N_\tau \\ S_\tau \end{pmatrix}\right]$$

$$= \inf_{\tau} E^{(n,s)} \left[\int_0^{\tau} e^{-\beta t} F\begin{pmatrix} N_t \\ S_t \end{pmatrix} dt + \frac{L}{\beta} - \frac{L}{\beta} e^{-\beta\tau} + e^{-\beta\tau} Mv\begin{pmatrix} N_\tau \\ S_\tau \end{pmatrix} \right]$$

$$= \frac{L}{\beta} \inf_{\tau} E^{(n,s)} \left[\int_0^{\tau} e^{-\beta t} F\begin{pmatrix} N_t \\ S_t \end{pmatrix} dt + e^{-\beta\tau} M\left(v - \frac{L}{\beta}\right)\begin{pmatrix} N_\tau \\ S_\tau \end{pmatrix} \right].$$

Thus

$$\tilde{v} - L/\beta = \mathcal{K}(\tilde{v} - L/\beta).$$

A similar reasoning proves the second implication. ∎

As a corollary to the above results we obtain the following lemma.

LEMMA 2.4. *Assume that* $\begin{pmatrix} N_t \\ S_t \end{pmatrix}$ *is a Feller process,* $h \in C^b(E)$, *and* M *transforms* $C^b(E)$ *into* $C^b(E)$. *Let* F *be a function bounded from below by* $(-L)$. *If there exists a positive constant* γ *such that*

$$\gamma\left(h + L\int_0^{\infty} e^{-\beta t}\, dt\right) = \gamma\left(h + \frac{L}{\beta}\right) \leqslant M(0),$$

then there exists a unique solution $\tilde{v} \in C^b(E)$ *of the equation* $v = \mathcal{K}^L v$. *Moreover, the function* $v = \tilde{v} - L/\beta$ *is a unique solution of* (2).

To prove the existence of a solution to the equation (2) for our model notice that $\begin{pmatrix} N_t \\ S_t \end{pmatrix}$ is a Feller process because S_t is a Feller process and N_t is constant. Notice that proportions form a compact set D in $\boldsymbol{R}^d$:

$$D = \left\{(w^1, \ldots, w^d) \in \boldsymbol{R}^d\colon\ w^i \in [0, 1],\ \sum_{i=1}^{d} w^i = 1\right\}.$$

Therefore, a continuous function F, defined on proportions, must be bounded. Thus it is straightforward that $h \in C^b(E)$.

Let $L = \min(0, -\inf_{x \in E} F(x))$. Since $M(0) \geqslant K > 0$, one can easily find a positive constant γ such that $\gamma(h + L/\beta) \leqslant M(0)$. Continuity of the cost function $\tilde{C}$ and the multifunction mapping $\begin{pmatrix} n \\ s \end{pmatrix}$ into the set of possible impulse destinations imply that M transforms the set of continuous functions into itself. To show that M transforms $C^b(E)$ into $C^b(E)$ we take any function $g \in C^b(E)$ with $\alpha = \sup|g|$. Then

$$Mg\begin{pmatrix} n \\ s \end{pmatrix} \leqslant K + dk + \alpha \quad \text{and} \quad Mg\begin{pmatrix} n \\ s \end{pmatrix} \geqslant K - \alpha,$$

so $Mg \in C^b(E)$. Therefore, by Lemma 2.4, there exists a unique continuous and bounded function v, that is, the solution to $v = \mathcal{K}v$. Thus, we have proved the first assertion of Theorem 2.1.

Now, we characterize an optimal impulse control. Since we know that there exists a unique solution v to the functional equation (2), we have to prove that the infimum in (2) is attained by some stopping time (this would be the moment of the impulse) and that we can find a transaction (change of N_t) that should be made at this moment. It is well known (see Bensoussan and Lions [1], Zabczyk [9]) that the optimal stopping time is given by $\tau = \inf\{t \geqslant 0: Y_t \in Z\}$, where $Z = \{y \in \boldsymbol{R}^{2d}: v(y) = Mv(y)\}$. We only have to prove that for each $(n, s) \in E$ there exists $\xi \in \boldsymbol{R}^d$ such that $(n+\xi, s) \in E$, $\xi s = 0$, and

$$Mv\binom{n}{s} = \tilde{C}(n, n+\xi, s) + v\binom{n+\xi}{s}.$$

Fix $(n, s) \in E$. Both functions $\tilde{C}$ and v are continuous (v is also bounded). We first prove that the infimum is taken over a closed set. In fact, this set can be written as

$$A = \{\xi: (n+\xi, s) \in E, \xi s = 0\} = \{\xi: n+\xi \geqslant 0, \xi s = 0\} \setminus \{-n\}.$$

The self-financing condition $\xi s = 0$ assures that $(-n) \notin A$, so A is closed. Now take a sequence $\xi_k \in A$ such that

$$\tilde{C}(n, n+\xi_k, s) + v\binom{n+\xi_k}{s} \to Mv\binom{n}{s}.$$

If $\|\xi_k\| \not\to \infty$, then ξ_k admits a subsequence converging to some $\xi \in E$. Otherwise, $\|\xi_k\| \to \infty$. From the self-financing condition and the equivalence of all norms on $\boldsymbol{R}^d$ we obtain $C(n, n+\xi_k, s) \geqslant K + \beta (ns)^{-1} \|\xi_k\|$ for some $\beta \in \boldsymbol{R}_+$. Hence the boundedness of v implies that $C(n, n+\xi_k, s) + v(\xi_k) \to \infty$, which leads to a contradiction. For completeness of the proof we shall show that $\tau_i \to \infty$ a.s. Notice that each impulse adds a cost of size at least K. Since the value function v is bounded, an infinite number of transactions in finite time is impossible — its discounted transaction costs would sum up to infinity. This completes the proof of Theorem 2.1. ∎

3. MULTIPLICATIVE PRICE PROCESSES

In this section we assume that the price process is multiplicative, i.e.

$$S^i(\gamma s, t) = \gamma S^i(s, t), \quad \gamma \in \boldsymbol{R},\ s \in \boldsymbol{R}^d,\ s \geqslant 0,\ t \geqslant 0,\ i = 1, \ldots, d, \tag{4}$$

where $(S^i(s, t))_{t \geqslant 0}$ denotes an i-th coordinate of a price process starting from the point s,

$$S(s, 0) = s.$$

An important example of a multiplicative positive price process is a solution to the Itô equation

$$dS^i(s,t) = S^i(s,t)\,dZ^i(t), \quad i = 1, \ldots, d,$$
$$S(s,0) = s, \quad s \in \mathbf{R}^d,\ s \geqslant 0, \tag{5}$$

for a Lévy process $(Z^1, \ldots, Z^d)$ with jumps greater than -1.

3.1. General case. We will show that the proportion process linked up with S_t is Markovian and argue that the control problem (1) formulated in terms of proportions has an optimal solution. Let D be a simplex of proportions defined as in Section 2:

$$D = \{(w^1, \ldots, w^d) \in [0,1]^d \colon \sum_{i=1}^{d} w^i = 1\}.$$

The process $N(t)$ is constant, so, intuitively, we can incorporate it into $S(s,t)$ using (4). We define $T\colon \mathbf{R}^d_+ \to D$ by

$$T(S) = \left(\frac{S^1}{S^1 + \ldots + S^d}, \ldots, \frac{S^d}{S^1 + \ldots + S^d}\right).$$

Then $w(t) = T(S(\tilde{s}, t))$, where $\tilde{s} = (N^1(0)S^1(0), \ldots, N^d(0)S^d(0))$, and obviously $w(t)$ is indifferent to scaling of the initial condition $\tilde{s}$,

$$T(S(\tilde{s}, t)) = T(S(\gamma\tilde{s}, t)) \quad \text{for scalar } \gamma \neq 0. \tag{6}$$

We introduce an operator T^* acting on functions $f\colon D \to \mathbf{R}$ in the following way:

$$(T^* f)(s) = f(T(s)), \quad s \in \mathbf{R}^d_+.$$

THEOREM 3.1. *Let $\mathscr{A}$ be a generator for the positive price process S, i.e. almost all trajectories of $(S(s,t))_{t \geqslant 0}$ are positive for a positive initial condition s. Then the proportion process is Markov with the generator $\tilde{\mathscr{A}}$ given by*

$$(\tilde{\mathscr{A}} f)(w) = (\mathscr{A}(T^* f))(w), \quad w \in D.$$

Proof. The proof uses Theorem 10.13 in Dynkin [3]. We have to show a few properties of the map T with respect to the transition function of $S(s,t)$. We denote by $\mathscr{B}$ the Borel σ-field in $\mathbf{R}^d_+$ and by $\tilde{\mathscr{B}}$ the Borel σ-field in D. Let $P(t, s, \Gamma)$ be a transition function for the process $S(t)$, $\Gamma \in \mathscr{B}$. We have to verify the following conditions:

(i) $T(\mathbf{R}^d_+) = D$,

(ii) $T(\mathscr{B}) \subseteq \tilde{\mathscr{B}}$,

(iii) for all $s, s' \in \mathbf{R}^d$ such that $Ts = Ts'$ and $\Gamma \in \tilde{\mathscr{B}}$ we have

$$P(t, s, T^{-1}\Gamma) = P(t, s', T^{-1}\Gamma).$$

The properties (i) and (ii) are straightforward; only the third one requires some consideration. If $Ts = Ts'$, then there exists a scalar $\gamma \neq 0$ such that $s = \gamma s'$. Therefore, $T(S(s, t)) = T(S(s', t))$ by (6). Hence, Theorem 10.13 in Dynkin [3] implies that

$$T^* \tilde{\mathscr{A}} = \mathscr{A} T^*.$$

Take $f\colon D \to \boldsymbol{R}$, $s \in \boldsymbol{R}^d$, $s \geqslant 0$, and notice that

$$\big(T^*(\tilde{\mathscr{A}} f)\big)(s) = \big(\mathscr{A}(T^* f)\big)(s), \quad (\tilde{\mathscr{A}} f)\big(T(s)\big) = \big(\mathscr{A}(T^* f)\big)(s),$$
$$(\tilde{\mathscr{A}} f)(w) = \big(\mathscr{A}(T^* f)\big)(T^{-1} w),$$

where $w = T(s)$ and $T^{-1} w$ is any element of the counterimage of w, for example s. We can simplify the formula further by noting that $w \in T^{-1} w$. Hence $(\tilde{\mathscr{A}} f)(w) = \big(\mathscr{A}(T^* f)\big)(w)$. ■

We can reformulate our problem solely in the language of the proportion process. Our trading strategy Π consists of a sequence of stopping times $\tau_1, \tau_2, \ldots$ and changes of the proportion process at these times $\tilde{w}_1, \tilde{w}_2, \ldots$ Since the proportion process must be defined on $\boldsymbol{R}_+$ almost everywhere, we take on the following interpretation of the trading strategy which would allow us to write clearly the cost functional: $w(t) = w(\tilde{w}_i, t - \tau_i)$ for $t \in]\tau_i, \tau_{i+1}]$.

We do not have to limit possible impulses (as in the previous case) to satisfy the self-financing condition. It is possible to reach any proportion starting from an arbitrary one and satisfying the self-financing condition. Hence, the functional takes the form

$$J(\Pi) = \boldsymbol{E}\Big(\int_0^\infty e^{-\beta t} f\big(w(t)\big)\,dt + \sum_{i=1}^\infty e^{-\beta\tau_i} c\big(w(\tau_i), \tilde{w}_i\big) 1_{\tau_i < \infty}\Big). \tag{7}$$

We can use a similar approach to that in Section 2 to prove a counterpart of Theorem 2.1.

THEOREM 3.2. *Assume that w is a Feller process and f is a continuous function on D. Then there exists exactly one bounded continuous solution $v(w)$ to the equation*

$$\begin{aligned} v(x) &= \inf_\tau \boldsymbol{E}\Big(\int_0^\tau e^{-\beta t} f\big(w(x, t)\big)\,dt + e^{-\beta\tau} \tilde{M} v\big(w(x, \tau)\big)\Big), \\ \tilde{M} v(x) &= \inf_{y \in D} \big(v(y) + c(x, y)\big), \end{aligned} \tag{8}$$

and the optimal strategy for the problem (1) *is given by*

$$\tau_1 = \inf\{t \geqslant 0\colon\ \tilde{M} v(w_t) = v(w_t)\},$$

$$\tau_i = \inf\{t > \tau_{i-1}\colon\ \tilde{M} v(w_t) = v(w_t)\}.$$

The size of the impulse at the moment τ_i *is any number from the set*

$$\{w \in D\colon\ \tilde{M}v(w_{\tau_i}) = v(w) + c(w_{\tau_i}, w)\}.$$

Notice that if S is a Feller process, then so is w. Take $g \in C^b(D)$, $t \geqslant 0$, and consider $\tilde{g}(x) = \boldsymbol{E} g\big(w(x, t)\big)$, $x \in D$. The function $\tilde{g}$ can be written in terms of the price process

$$\tilde{g}(x) = \boldsymbol{E} g\big(T\big(S(x, t)\big)\big).$$

Moreover, $g \circ T \in C^b(\tilde{E})$, so $\tilde{g} \in C^b(\tilde{E})$, where $\tilde{E} = [0, \infty[^d \setminus \{0\}$ and $D \subseteq \tilde{E}$.

To find explicit solutions to the equation (8) it is convenient to rewrite it in a differential form as a suitable quasi-variational inequality (QVI). We change the state space in order to have a non-empty interior. We remove the last coordinate and take

$$D = \{w^1, \ldots, w^{d-1} \in [0, 1]^{d-1}\colon\ \sum_{i=1}^{d-1} w^i \leqslant 1\},$$

$$c(u, w) = K + k \sum_{i=1}^{d-1} |u^i - w^i| + k \Big|\sum_{i=1}^{d-1} (u^i - w^i)\Big| \qquad \text{for } K > 0,\ k \geqslant 0.$$

We denote by $\tilde{\mathscr{A}}$ the generator for the proportion process in the new state space and make obvious modifications to the function f. We introduce a switching functional

$$\tilde{M}v(w) = \inf_{u \in D} \big(v(u) + c(w, u)\big) \tag{9}$$

for any function $v\colon D \to \boldsymbol{R}$. The QVI related to the cost functional (7) takes the form

$$\min\big(\tilde{\mathscr{A}}v(w) - \beta v(w) + f(w),\ \tilde{M}v(w) - v(w)\big) = 0, \qquad w \in D. \tag{10}$$

3.2. Lévy noise models. Let the price process be two-dimensional with the second coordinate being always 1 and the first satisfying the Itô equation

$$\begin{aligned} dS(s, t) &= H\big(S(s, t-)\big)\, d\zeta(t), \\ S(s, 0) &= s, \qquad s \in \boldsymbol{R},\ s > 0. \end{aligned} \tag{11}$$

Here $\zeta(t)$ is a Lévy process with the Fourier transform

$$\boldsymbol{E} \exp\big(-is\zeta(t)\big) = \exp\big(-t\psi(s)\big),$$

$$\psi(s) = \tfrac{1}{2}\sigma^2 s^2 - i\mu s - \int_{\boldsymbol{R}} (e^{isy} - 1 - 1_{|y| \leqslant 1}\, isy)\, \nu(dy),$$

where $\sigma \in \boldsymbol{R}_+$, $\mu \in \boldsymbol{R}$ and ν is a σ-finite measure satisfying

$$\int_{\boldsymbol{R}} (1 \wedge y^2)\, \nu(dy) < \infty.$$

We assume that H is chosen in such a way that (11) has a unique weak solution for any initial condition $s \geqslant 0$. It can be easily verified that $S(s, t)$ is a multiplicative process (cf. (4)) only if H is a linear function. Therefore, without any loss of generality we assume that $H(x) = x$.

PROPOSITION 3.3. *The generator* $\tilde{\mathscr{A}}$ *for the proportion process for the price process* (11) *has the form*:

$$\tilde{\mathscr{A}}u(w) = \tfrac{1}{2}\sigma^2 w\big(u''(w)(1-w)^3 - 2u'(w)(1-w)^2\big) + \mu w u'(w)(1-w)$$
$$+\int_{\boldsymbol{R}} \left(u\left(\frac{w+wy}{1+wy}\right) - u(w) - 1_{|y|\leqslant 1}\, wyu'(w)(1-w)\right)\nu(dy), \qquad u \in]0, 1[,$$
$$\tilde{\mathscr{A}}u(0) = \tilde{\mathscr{A}}u(1) = 0,$$

for $u \in C^2(\boldsymbol{R})$.

Proof. Following Bichteler [2], we write the generator $\mathscr{A}$ for $S(s, t)$. Let $u \in C^2(0, 1)$. Then

$$\mathscr{A}u(s) = \tfrac{1}{2}\sigma^2 su''(s) + \mu su'(s) + \int_{\boldsymbol{R}} \big(u(s+sy) - u(s) - 1_{|y|\leqslant 1}\, syu'(s)\big)\nu(dy).$$

Let

$$T(s) = \frac{s}{s+1}.$$

As the price of the second instrument is equal to 1, the proportion process is identical with $T(S(s, t))$. We apply Theorem 3.1 and observe that

$$\frac{d}{ds}u\left(\frac{s}{s+1}\right) = u'\left(\frac{s}{s+1}\right)\frac{1}{(s+1)^2},$$
$$\frac{d^2}{ds^2}u\left(\frac{s}{s+1}\right) = u''\left(\frac{s}{s+1}\right)\frac{1}{(s+1)^4} - 2u'\left(\frac{s}{s+1}\right)\frac{1}{(s+1)^3}.$$

Moreover, $T^{-1}(w) = w/(1-w)$, $s+1 = 1/(1-w)$, which implies our result for $w \in (0, 1)$.

We extend the generator to the points 0, 1 in an obvious way. These points are stable for the process, i.e. the process cannot move away from them, so $\tilde{\mathscr{A}}u(0) = \tilde{\mathscr{A}}u(1) = 0$. ∎

On this stage we can write a QVI for the problem of optimal asset allocation:

$$\text{(12)} \qquad \begin{aligned} \min\big(\tilde{\mathscr{A}}v(w) - \beta v(w) + f(w),\ Mv(w) - v(w)\big) &= 0, \qquad w \in]0, 1[, \\ \min\big(f(w) - \beta v(w),\ Mv(w) - v(w)\big) &= 0, \qquad w = 0, 1, \end{aligned}$$

where

$$Mv(w) = K + \inf_{u\in[0,1]} \big(k|u-w|+v(u)\big), \qquad K > 0,\ k \geqslant 0.$$

Hence, we conclude that the optimal strategy is described by an impulse region $\{Mv-v = 0\}$.

4. MULTIPLICATIVE POISSONIAN PRICES

For further considerations we restrict ourselves to the case where prices are driven by a Poisson process. We specify

$$\zeta(t) = N(t)-\gamma t,$$

where $N(t)$ is a Poisson process with intensity λ and $\gamma\in \boldsymbol{R}$. The characteristics (μ, σ, ν) of this Lévy process is the following: $\mu = \lambda-\gamma$, $\sigma = 0$, $\nu(\{1\}) = \nu(\boldsymbol{R}) = \lambda$. By Proposition 3.3 the generator for the proportion process is given by

$$\tilde{\mathscr{A}}u(w) = \lambda\left(u\left(\frac{2w}{w+1}\right)-u(w)\right)-\gamma u'(w)\,w(1-w).$$

We write a QVI for the problem of optimal asset allocation:

$$\begin{aligned}(13)\qquad &\min\left(\lambda\left(v\left(\frac{2w}{w+1}\right)-v(w)\right)\right.\\ &\qquad\left.-\gamma v'(w)\,w(1-w)-\beta v(w)+f(w),\ Mv(w)-v(w)\right) = 0, \qquad w\in\,]0,1[,\\ &\min\big(f(w)-\beta v(w),\ Mv(w)-v(w)\big) = 0, \qquad w = 0, 1,\end{aligned}$$

where

$$Mv(w) = K + \inf_{u\in[0,1]} \big(k|u-w|+v(u)\big), \qquad K > 0,\ k \geqslant 0.$$

Moreover, the optimal strategy is described by an impulse region $\{Mv-v = 0\}$. However, we have to prove that the QVI (13) has an appropriately smooth bounded solution and that this solution satisfies the functional equation

$$(14)\qquad v(x) = \inf_{\tau} E\Big(\int_0^{\tau} e^{-\beta s} f\big(w(x,s)\big)\,ds + e^{-\beta\tau} Mv\big(w(x,\tau)\big)\Big).$$

The function $v(x)$ defines an optimal strategy as stated in Theorem 3.2.

The following theorem could be deduced from general results of Bensoussan and Lions [1], chapter 3, although a non-degenerate diffusion term is required there. We present here a direct proof.

THEOREM 4.1. *Assume that* $\gamma \neq 0$. *Let* v *be a bounded continuous function defined on* $[0, 1]$, *piecewise* C^1 *and with finite left- and right-hand derivatives. If* v *satisfies* (13) *at* 0 *and* 1 *and almost everywhere in* $]0, 1[$, *then* v *solves* (14).

Proof. We shall use Theorem 5.2 in the Appendix. First observe that $\mathscr{D}(\tilde{\mathscr{A}}) \supset C^1$. Moreover, there exists a sequence v_n of C^1-functions converging to v in sup-norm such that $v_n = v$, $v_n' = v'$ everywhere but intervals of Lebesgue measure converging to 0. We introduce

$$h(w) = \lambda\left(v\left(\frac{2w}{w+1}\right) - v(w)\right) - \gamma v'(w)\, w(1-w)$$

for $w \in [0, 1]$ such that v' is defined and continuous in w. The process $w(x, t)$ has a non-zero drift, so $P(t, x, \{v_n \neq v,\ v_n' \neq v'\}) \to 0$ as $n \to \infty$ for any $t > 0$, which proves condition (iii) of Theorem 5.2. Moreover, a similar argument shows that every set of Lebesgue measure zero in $]0, 1[$ is of $\tilde{\mathscr{A}}$-measure zero, so (iv) is satisfied. Therefore

$$Y(x, t) = \int_0^t e^{-\beta s} f(w(x, s))\, ds + e^{-\beta\tau} v(w(x, t))$$

is a submartingale, so $EY(x, \sigma) \geqslant EY(x, 0) = v(x)$ for any stopping time σ (f is bounded). The process

$$Z(w, t) = \int_0^t e^{-\beta s} f(w(x, s))\, ds + e^{-\beta\tau} Mv(w(x, t))$$

satisfies $Z(x, t) \geqslant Y(x, t)$ since $Mv(x) - v(x) \geqslant 0$. This leads to the conclusion that $EZ(x, \sigma) \geqslant v(x)$ for any stopping time σ. To complete the proof it suffices to show that there exists an optimal stopping time $\tau^*(x)$ such that $EZ(x, \tau^*(x)) = v(x)$. It is true for $\tau^*(x) = \inf\{t \geqslant 0\colon Mv(w(x, t)) = v(w(x, t))\}$. ∎

Previous considerations imply the following

COROLLARY 4.2. $v(w)$ *is a unique bounded continuous solution to* (14) *and it is the value function for the problem of minimizing* (7). *Theorem* 3.2 *shows how to construct the optimal strategy.*

From now on we assume, without any loss of generality (see Lemma 2.3), that $f(w)$ is a positive function. We derive two results.

LEMMA 4.3. *If* f *is non-decreasing on* $[0, 1]$, *then* v *is non-decreasing. If* f *is non-increasing on* $[0, 1]$, *then* v *is non-increasing.*

Proof. We sketch the proof of the first fact. The proof of the second one is analogous. Observe that if $x \geqslant x'$, the proportion process satisfies

$w(x,t) \geqslant w(x',t)$. Denote by v_0 the potential defined by

$$v_0(x) = \boldsymbol{E}\int_0^\infty e^{-\beta s} f(w(x,s))\,ds, \qquad x \in [0,1].$$

Hence v_0 is a non-decreasing function. Let

$$v_n(x) = \inf_\tau \boldsymbol{E}\Big(\int_0^\tau e^{-\beta s} f(w(x,s))\,ds + e^{-\beta\tau} M v_{n-1}(w(x,\tau))\Big), \qquad x \in [0,1].$$

We can show by induction that v_n is a non-decreasing function. By Proposition 2.2, v_n converges uniformly to the value function v, so v is non-decreasing. ∎

LEMMA 4.4. *If the difference between minimum and maximum of f is smaller than βK, the continuation region spans the whole interval* $[0,1]$.

Proof. Let $\underline{f} = \min_{u\in[0,1]} f(u)$, $\bar{f} = \max_{u\in[0,1]} f(u)$, The value function v has trivial bounds $\underline{f} \leqslant \beta v(x) \leqslant \bar{f}$, $x \in [0,1]$. Hence $Mv(x) > v(x)$ for all $x \in [0,1]$, which implies that the optimal strategy prevents any impulses. ∎

Assume now that there are no proportional costs, i.e. $k = 0$. In this case all impulses aim at the same target point $u^* \in [0,1]$ at which the function v attains its minimum. Hence, if f is non-decreasing, impulses can only occur in some interval $[b_0, 1]$ and they aim at $w = 0$ (minimum of v). We know, by a direct calculation, that $v(0) = f(0)/\beta$. The potential of f at 1 equals $f(1)/\beta$. Hence, the impulse interval is non-empty if and only if $v(1) > v(0) + \beta K$. The same reasoning applies to the case of non-increasing f.

4.1. Recursive formulae. We derive a solution to (13) for a specific case of a non-empty impulse region around 1 and the absence of proportional transaction costs $k = 0$. We do not require monotonicity of f. However, we assume that $f \geqslant 0$, which is no restriction (see Lemma 2.3).

We construct an iterative procedure to find the solution to the QVI (13). We set $v_0(w) = H$, $H \in \boldsymbol{R}$, for $w \in [b_0, 1]$. The function v_0 is undefined outside of the interval $[b_0, 1]$. A pair $H, b_0 \in \boldsymbol{R} \times [0,1]$ is used as an index for the set of solutions.

To formulate the lemma we need to define a sequence

$$b_{n+1} = \frac{b_n}{2-b_n}, \qquad n = 0, 1, \ldots,$$

and introduce the following equation being a differential part of (13):

$$\lambda\left(v\left(\frac{2w}{w+1}\right) - v(w)\right) - \gamma v'(w)\,w(1-w) - \beta v(w) + f(w) = 0. \tag{15}$$

We note that the sequence b_n is strictly decreasing with the limit equal to 0.

LEMMA 4.5. *Assume that* v_n *is defined on* $[b_n, 1]$ *and satisfies* (15) *for* $w \in [b_n, b_0]$. *We define* v_{n+1} *on* $[b_{n+1}, 1]$ *by the formula*

$$v_{n+1}(w) = \nu(w)\left(\frac{v_n(b_n)}{\nu(b_n)} - \int_w^{b_n} \frac{\lambda v_n(2u/(u+1)) + f(u)}{\gamma\nu(u)u(1-u)}\,du\right), \qquad w \in [b_{n+1}, b_n[,$$

$$v_{n+1}(w) = v_n(w), \qquad w \in [b_n, 1],$$

where

$$\xi = \frac{\lambda+\beta}{\gamma} \quad \text{and} \quad \nu(w) = \left(\frac{1-w}{w}\right)^{\xi}.$$

Then v_{n+1} *satisfies* (15) *for* $w \in [b_{n+1}, b_0]$.

Proof. We can easily see that

$$v_{n+1}(w) = \left(\frac{1-w}{w}\right)^{\xi}\left(\frac{v_n(b_0)}{((1-b_0)/b_0)^{\xi}} - \int_w^{b_0} \frac{\lambda v_n(2u/(u+1)) + f(u)}{\gamma((1-u)/u)^{\xi}u(1-u)}\,du\right), \quad w \in [b_{n+1}, b_0[.$$

We solve first the following equation:

$$\lambda v\left(\frac{2w}{w+1}\right) + f(w) - \lambda v(w) - \gamma v'(w)w(1-w) - \beta v(w) = 0, \qquad w \in [b_{n+1}, b_n],$$

$$v(w) = v_n(w), \qquad w \in [b_n, 1].$$

Its homogeneous version is of the form

$$-(\lambda+\beta)v(w) - \gamma v'(w)w(1-w) = 0,$$

which we simplify to

$$-\xi v(w) = v'(w)w(1-w) \quad \text{for } \xi = (\lambda+\beta)/\gamma.$$

We obtain the solution

$$v(w) = c\left(\frac{1-w}{w}\right)^{-\xi}$$

By setting $c = c(w)$ and plugging into the generic equation we obtain

$$c'(w) = \frac{\lambda v(2w/(w+1)) + f(w)}{\gamma((1-w)/w)^{\xi}w(1-w)}. \quad \blacksquare$$

Remark. The function $\nu(w)$ is unbounded on $[0, 1]$. It converges to ∞ as $w \to 0$ and to 0 as $w \to 1$.

We formulate conditions under which the optimal control is determined by the numbers $0 \leqslant a < c < b \leqslant 1$ and consists of making impulses to c when the proportion process exits from the interval $[a, b]$. Such strategies will be denoted by $\Pi_{a,b,c}$.

Let $v_{b_0,H}(w)$ be the limit of v_n with the initial condition $v_0(w) = H$, $w \in [b_0, 1]$, in the sense that $v_{b_0,H}(w) = v_n(w)$, $w \in [b_n, 1]$. Define

$$b^*_{b_0,H} = \sup\{w < b_0\colon\ v_{b_0,H}(w) = H\}.$$

The following theorem might be useful for numerical treatment of the problem:

THEOREM 4.6. *Assume that the following conditions hold*:

(i) $\inf_{w \in [b^*_{b_0,H}, 1]} v_{b_0,H}(w) = H - K$;

(ii) $\sup_{w \in [b^*_{b_0,H}, 1]} v_{b_0,H}(w) = H$;

(iii) $f(w) \geqslant \beta H$, $w \in [b_0, 1]$;

(iv) $f(w) + \lambda v_{b_0,H}(2w/(1+w)) \geqslant (\lambda+\beta) H$, $w \in [b^*_{b_0,H}/(2-b^*_{b_0,H}),\ b^*_{b_0,H}]$;

(v) $f(w) \geqslant \beta H$, $w \in [0,\ b^*_{b_0,H}/(2-b^*_{b_0,H})]$.

Then

$$v(w) = 1_{\{w \geqslant b^*_{b_0,H}\}} v_{b_0,H}(w) + 1_{\{w < b^*_{b_0,H}\}} H$$

is a solution to (13). *Moreover*, $\Pi_{b^*_{b_0,H}, b_0, c}$, *where*

$$c = \arg\inf_{w \in [b^*_{b_0,H}, 1]} v_{b_0,H}(w),$$

is an optimal strategy.

Proof. The conditions guarantee that v is piecewise C^1 with finite left- and right-hand derivatives and satisfies (13) at all points but $(b_n)_{n \in N}$. By Theorem 4.1, v is a value function for the control problem of minimizing (7). ■

Remark. If $f(b_0) > \beta H$, then $v'(b_0-) < 0$ and condition (ii) is satisfied.

4.2. Transcendental equation. If we know *a priori* that the function v attains its minimum in the first interval $[b_1, b_0]$, then the target point u^* can be characterized by the following transcendental equation:

$$\frac{1}{v(u^*)}\frac{\lambda H + f(u^*)}{\lambda+\beta} = \frac{H}{v(b_0)} - \int_{u^*}^{b_0} \frac{\lambda H + f(u)}{\gamma v(u) u(1-u)}\, du. \tag{16}$$

To obtain the above equation we observe that $v'(u^*) = 0$. From (15) we have

$$\lambda H - (\beta+\lambda) v(u^*) + f(u^*) = 0.$$

Hence

$$v(u^*) = \frac{\lambda H + f(u^*)}{\lambda+\beta}$$

and we take the formula for $v(u^*)$ from Lemma 4.5.

If we assume that

$$\frac{1}{v(u)}\frac{\lambda H + f(u)}{\lambda+\beta}$$

is decreasing in u, then the equation (16) has at most one solution. We recall that $f \geqslant 0$, which makes the expression under integrand non-negative.

Similar but more complicated equations can be obtained if $[b_1, b_0]$ is replaced by $[b_{k+1}, b_k]$.

4.3. Models with given impulse regions. In this section we will show that for any region J in $[0, 1]$ there exists a cost functional (1) such that the optimal control impulse region is exactly J.

We introduce a family of functions:

- $(g_b^{(1)})_{b\in[0,1)} \subset C^1([0, 1])$, $g_b^{(1)}|_{(b,1]} \in [a, 1)$, $g_b^{(1)}(b) = 1$, $\frac{d}{dw} g_b^{(1)}(b) = 0$,
- $(g_{l,r,\alpha}^{(2)})_{0 \leqslant l < r \leqslant 1, \alpha\in\boldsymbol{R}_+} \subset C^1([0, 1])$, $g_{l,r,\alpha}^{(2)}|_{(l,r)} \in [a, 1)$,
$$g_{l,r,\alpha}^{(2)}(l) = g_{l,r,\alpha}^{(2)}(r) = 1, \quad \frac{d}{dw} g_{l,r,\alpha}^{(2)}(l) = 0, \quad \frac{d}{dw} g_{l,r,\alpha}^{(2)}(r) = \alpha,$$
- $(g_{r,\alpha}^{(3)})_{0 \leqslant r \leqslant 1, \alpha\in\boldsymbol{R}_+} \subset C^1([0, 1])$, $g_{r,\alpha}^{(3)}|_{[0,r)} \in [a, 1)$, $g_{r,\alpha}^{(3)}(r) = 1$,
$$\frac{d}{dw} g_{r,\alpha}^{(3)}(r) = \alpha$$

for some $a \in (0, 1)$.

Now, we proceed with the construction of the function f starting from the right end. We assume that in the impulse region the value function v is equal to 1. We will guarantee that the value function is bounded by 1. Hence, by setting the impulse cost $K = 1 - \min v$, we obtain the solution to the QVI. In the following lemma we show how to extend the function f so as to keep to the required impulse region. First we introduce the notation: $[a, b] \prec c$ if $a < c$ and $b < c$. Analogously, $[a, b] \prec [c, d]$ if $[a, b] \prec c$ and $[a, b] \prec d$.

Lemma 4.7. *Assume that f and the value function v are defined on $[b, 1]$ and $v|_{[b,\tilde{b}]} = 1$ for some $\tilde{b} > b$. For any interval $0 \preccurlyeq [l, r] \prec b$ there exists an extension of f to $[l, 1]$ such that v is a solution to the QVI on $[l, 1]$ with $[l, r]$ being a part of the impulse region and $]r, b[$ being a part of a continuation region*:

$$v|_{(r,b)} < 1, \quad v|_{[l,r]} = 1, \quad f(w) - (\beta + \lambda) + \lambda v\left(\frac{2w}{1+w}\right) \geqslant 0, \quad w \in [l, r].$$

Proof. We set

$$\alpha = \frac{\lambda + \beta - \lambda v(2b/(1+b)) - f(b)}{\gamma b(1-b)}.$$

We extend f on $[r, b)$ in such a way that $v|_{[r,b)} = g_{r,b,\alpha}^{(2)}|_{[r,b)}$, i.e.

$$f(w) = \gamma v'(w)\, w(1-w) + \beta v(w) + \lambda\left(v(w) - v\left(\frac{2w}{1+w}\right)\right), \quad w \in [r, b).$$

The function f is continuous on $[r, b)$. Moreover, the condition for the derivative $v'(b) = \alpha$ implies the continuity in b.

To define f on $[l, r)$ we have to assure that

$$f(w)-(\beta+\lambda)+\lambda v\left(\frac{2w}{1+w}\right) \geqslant 0, \quad w \in [l, r]. \tag{17}$$

We check that

$$f(r)-(\beta+\lambda)+\lambda v\left(\frac{2r}{1+r}\right) = 0,$$

since $v'(r) = 0$. We extend f to $[l, r)$ in any way that guarantees that the inequality (17) is satisfied and the continuity of f holds. ∎

THEOREM 4.8. *Let $(I_n)_{n=1,\ldots,N}$ be a family of closed intervals in $[0, 1]$ with non-void interior satisfying $I_{n+1} \prec I_n$. Then there exists a function f and the impulse cost $K > 0$ such that $\bigcup_{n=1,\ldots,N} I_n$ is the impulse region of the optimal strategy.*

Proof. If $I_1 = [b, 1]$, we set $v|_{[b,1]=1}$, $f|_{[b,1]} = \beta$. Otherwise, $I_1 = [l, r] \prec 1$. We put

$$v|_{[r,1]} = g_r^{(1)}, \quad v|_{[l,r]} = 1,$$

taking appropriate f as in Lemma 4.7. For next intervals, excluding the last, we apply the lemma. Let $I_N = [l, r]$ be the last interval. If $l = 0$, then we apply the lemma. Otherwise, we proceed as follows. We take

$$v|_{[0,l)} = g_{l,\alpha}^{(3)}, \quad \alpha = \frac{\lambda+\beta-\lambda v(2l/(1+l))-f(l)}{\gamma l(1-l)}.$$

We define f appropriately, as in Lemma 4.7. For v to be a solution to the QVI, we have to specify the impulse cost K. We put $K = 1-\min v$. Now, we observe that f is a continuous function on $[0, 1]$, $v \in [a, 1]$, and v is a solution to the QVI. ∎

Theorem 4.8 can be generalized to the case of an infinite number of disjoint intervals with non-empty interior converging to 0.

5. APPENDIX A

We state and prove here an auxiliary result needed in the proof of Theorem 4.1.

Let $X(t, x)$ be a Markov process on the space $(E, \mathscr{E})$ with respect to the filtration $(\mathscr{F}_t)$ and a semigroup (P_t). By $(\mathscr{A}, \mathscr{D}(\mathscr{A}))$ we denote its generator. We formulate and prove a general result giving the probabilistic interpretation of the solution (in some sense, specified later) to the equation

$$\mathscr{A}v(w)-\beta v(w)+f(w) = 0 \ (\geqslant 0, \ \leqslant 0).$$

DEFINITION 5.1. A set $B \in \mathscr{E}$ is of *null $\mathscr{A}$-measure* if

$$\forall x \in E \ \forall t > 0 \ P_t 1_B(x) = 0.$$

THEOREM 5.2. *Let $v\colon E \to \boldsymbol{R}$ be a continuous function such that there exists a sequence of functions $v_n \in \mathscr{D}(\mathscr{A})$ and a function h satisfying*

(i) $v_n \to v$ *in* sup-norm;

(ii) h *is defined* $\mathscr{A}$-*a.s.*,

(iii) $$\boldsymbol{E}\int_0^t e^{-\beta s}\mathscr{A}v_n\big(X(s, x)\big)\,ds \to \boldsymbol{E}\int_0^t e^{-\beta s} h\big(X(s, x)\big)\,ds;$$

(iv) $h(x) - \beta v(x) + f(x) \geqslant 0$ $\mathscr{A}$-*a.s.*, *for a continuous function* $f\colon E \to \boldsymbol{R}$.
Then

$$Y(t, x) := e^{-\beta t} v\big(X(t, x)\big) - v(x) + \int_0^t e^{-\beta s} f\big(X(x, s)\big)\,ds$$

is a submartingale (if it is well-defined and integrable).

For the proof of the theorem we will need the well-known lemma:

LEMMA 5.3. *Let $Z(t)$, $t \geqslant 0$, be an adapted and measurable process in $\boldsymbol{R}^d$. For any Borel function $f\colon \boldsymbol{R}_+ \times \boldsymbol{R}^d \to \boldsymbol{R}$,*

$$\boldsymbol{E}\Big(\int_a^b f\big(u, Z(u)\big)\,du \,|\, \mathscr{F}_a\Big) = \int_a^b \boldsymbol{E}\Big(f\big(u, Z(u)\big) \,\big|\, \mathscr{F}_a\Big)\,du$$

if the left- or right-hand side exists.

Proof of Theorem 5.2. The proof consists of two parts. First we show that $\boldsymbol{E}Y(t, x) > 0$. Then we use the Markov property of $X(t, x)$ to show that it is a submartingale. We have

$$\frac{d}{dt} P_t v_n = P_t \mathscr{A} v_n,$$

since v_n is in the domain of $\mathscr{A}$. Hence

$$\frac{d}{dt} e^{-\beta t} P_t v_n = e^{-\beta t} P_t \mathscr{A} v_n - \beta e^{-\beta t} P_t v_n.$$

We integrate the above equation and we obtain

$$e^{-\beta t} P_t v_n - v_n = \int_0^t e^{-\beta s} P_s \mathscr{A} v_n\,ds - \beta \int_0^t e^{-\beta s} P_s v_n\,ds.$$

Changing the order of integration we get

$$\boldsymbol{E}\Big(e^{-\beta t} v_n\big(X(t, x)\big) - v_n(x) - \int_0^t e^{-\beta s}\Big(\mathscr{A}v_n\big(X(s, x)\big) - \beta v_n\big(X(s, x)\big)\Big)\,ds\Big) = 0.$$

We let $n \to \infty$ and by (i) and (iii) we get

$$E\Big(e^{-\beta t} v\big(X(t, x)\big) - v(x) - \int_0^t e^{-\beta s}\big(h\big(X(s, x)\big) - \beta v\big(X(s, x)\big)\big)\, ds\Big) = 0.$$

Using the condition (iv) we have $-(h-\beta v) \leqslant f$, so

$$EY(t, x) = E\Big(e^{-\beta t} v\big(X(t, x)\big) - v(x) + \int_0^t e^{-\beta s} f\big(X(x, s)\big)\, ds\Big) \geqslant 0,$$

which can be written equivalently:

$$e^{-\beta t} P_t v(x) - v(x) + \int_0^t e^{-\beta u} P_u f(x)\, du \geqslant 0. \tag{18}$$

We shall show that $Y(t, x)$ is a submartingale. We take $0 \leqslant s < t$ and write

$$\begin{aligned} &E\big(Y(x, t) - Y(x, s) \,|\, \mathscr{F}_s\big) \\ &= E\Big(e^{-\beta t} v\big(X(t, x)\big) - e^{-\beta s} v\big(X(s, x)\big) + \int_s^t e^{-\beta u} f\big(X(u, x)\big)\, du \,\Big|\, \mathscr{F}_s\Big) \\ &= E\big(e^{-\beta t} v\big(X(t, x)\big) \,\big|\, \mathscr{F}_s\big) - E\big(e^{-\beta s} v\big(X(s, x)\big) \,\big|\, \mathscr{F}_s\big) + E\Big(\int_s^t e^{-\beta u} f\big(X(u, x)\big)\, du \,\Big|\, \mathscr{F}_s\Big). \end{aligned}$$

From the Markov property of $X(t, x)$ we get

$$E\big(e^{-\beta t} v\big(X(t, x)\big) \,\big|\, \mathscr{F}_s\big) = e^{-\beta t} P_{t-s} v\big(X(s, x)\big).$$

Lemma 5.3 implies

$$E\Big(\int_s^t e^{-\beta u} f\big(X(u, x)\big)\, du \,\Big|\, \mathscr{F}_s\Big) = \int_s^t e^{-\beta u} P_{u-s} f\big(X(s, x)\big)\, du.$$

Combining the above results and using (18) we obtain

$$\begin{aligned} &E\big(Y(x, t) - Y(x, s) \,|\, \mathscr{F}_s\big) \\ &= e^{-\beta s}\Big(e^{-\beta(t-s)} P_{t-s} v\big(X(s, x)\big) - v\big(X(s, x)\big) + \int_s^t e^{-\beta u} P_{u-s} f\big(X(s, x)\big)\, du\Big) \geqslant 0, \end{aligned}$$

which completes the proof. ■

COROLLARY 5.4. *Under the assumptions of Theorem 5.2, if $h(x) - \beta v(x) + f(x) \leqslant 0$ $(= 0)$ $\mathscr{A}$-a.s., then $Y(t, x)$ is a supermartingale (martingale).*

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