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A PROOF OF GRABINER'S THEOREM ON NON-COLLIDING PARTICLES

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Abstract. A detailed proof of Grabiner's theorem [1] on the exact asymptotics of the time to collision for n independent Brownian motions is given.

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1. INTRODUCTION

Let $X_t = (X_t^1, \ldots, X_t^n)$ be a standard n-dimensional Brownian motion, and let $X_0 = \boldsymbol{x} = (x_1, \ldots, x_n)$. For simplicity, assume that $\boldsymbol{x}$ has ordered components, so $\boldsymbol{x} \in W$, where

$$W = \{\boldsymbol{x} \in \boldsymbol{R}^n \colon x_1 < x_2 < \ldots < x_n\}.$$

We define *collision time*

$$\tau = \inf\{t \geqslant 0 \colon X_t^i = X_t^j;\ i \neq j\}.$$

Denote the *Vandermonde determinant* $h(\boldsymbol{x})$ by

$$h(\boldsymbol{x}) = \det[\{x_i^{j-1}\}_{i,j=1}^n].$$

Grabiner in his work [1], Theorem 1, stated the following theorem, with a short descriptive proof using the reflection argument. In this note we give an elementary proof of this theorem.

THEOREM 1. *We have*

$$\lim_{t \to \infty} P_{\boldsymbol{x}}(\tau > t)\, t^{n(n-1)/4} = Ch(\boldsymbol{x}), \tag{1}$$

where

$$C = \frac{(2\pi)^{-n/2}}{\prod_{j=0}^{n-1} j!} \int_W \exp\left(-\frac{|y|^2}{2}\right) h(y)\, dy.$$

2. PROOF

We divide the proof into parts.

Following Karlin and McGregor [2] the density function for the Brownian motion starting at x to be at y at time t without having left W is

$$b_t(x, y) = \det[\{\phi_t(x_i - y_j)\}_{i,j=1}^n],$$

where

$$\phi_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

We can factor out $b_t(x, y)$ by

$$(2) \qquad b_t(x, y) = (2\pi t)^{-n/2} \exp\left(-\frac{|x|^2 + |y|^2}{2t}\right) \det\left[\left\{\exp\left(\frac{x_i y_j}{t}\right)\right\}\right].$$

For the asymptotics of (2) as $t \to \infty$ we need to study the asymptotic of the determinant

$$(3) \qquad \det\left[\left\{\exp\left(\frac{x_i y_j}{t}\right)\right\}_{i,j=1}^n\right].$$

By writing every element of the matrix into an exponential series we get

$$(4) \qquad \det\left[\left\{\exp\left(\frac{x_i y_j}{t}\right)\right\}_{i,j=1}^n\right] = \sum_{k=0}^{\infty} T_k t^{-k},$$

where

$$T_k = \sum_{\sigma \in S_n} \frac{\operatorname{sgn}(\sigma)}{k!} (x_1 y_{\sigma(1)} + \ldots + x_n y_{\sigma(n)})^k.$$

As usual, S_n denotes the set of permutations of $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ is the sign of permutation σ. The following lemma holds true:

LEMMA 2. *For* $i = 0, 1, \ldots, n(n-1)/2 - 1$ *we have* $T_i = 0$ *and*

$$T_{n(n-1)/2} = \frac{h(x) h(y)}{\prod_{j=0}^{n-1} j!}.$$

Proof. Note that

$$\begin{aligned} T_k &= \sum_{\sigma \in S_n} \frac{\operatorname{sgn}(\sigma)}{k!} (x_1 y_{\sigma(1)} + \ldots + x_n y_{\sigma(n)})^k \\ &= \sum_{\sigma \in S_n} \frac{\operatorname{sgn}(\sigma)}{k!} \sum_{\substack{k_1, \ldots, k_n \\ k_1 + \ldots + k_n = k}} \frac{k!}{k_1! \ldots k_n!} (x_1 y_{\sigma(1)})^{k_1} \ldots (x_n y_{\sigma(n)})^{k_n} \\ &= \sum_{\substack{k_1, \ldots, k_n \\ k_1 + \ldots + k_n = k}} \frac{1}{k_1! \ldots k_n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) (x_1 y_{\sigma(1)})^{k_1} \ldots (x_n y_{\sigma(n)})^{k_n}. \end{aligned}$$

Now we use the observation that if $k_i = k_j$ for $i \neq j$, then the pairs of permutations $(\sigma, (i, j)\circ\sigma)$ will cancel the inner sum above. This happens because signs of these permutations are opposite and factors are the same. Using this remark we infer that the coefficient T_k vanishes unless there exists a decomposition of the number k into a sum of n nonnegative integers $k_1, \ldots, k_n$ such that $k = k_1 + \ldots + k_n$, and $k_i \neq k_j$ for $i \neq j$. The smallest number for which such a decomposition exists is $n(n-1)/2 = 0+1+ \ldots +n-1$. This completes the proof of the first part.

To compute the coefficient $T_{n(n-1)/2}$, we must notice that decompositions of the number $n(n-1)/2$ into a sum of nonnegative integers are only permutations of the set $\{0, 1, \ldots, n-1\}$. Thus

$$T_{n(n-1)/2} = \sum_{\substack{k_1,\ldots,k_n \\ \Sigma k_i = n(n-1)/2}} \frac{1}{k_1! \ldots k_n!} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)(x_1 y_{\sigma(1)})^{k_1} \ldots (x_n y_{\sigma(n)})^{k_n}$$

$$= \frac{1}{\prod_{j=0}^{n-1} j!} \sum_{\theta\in S_n} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)(x_1 y_{\sigma(1)})^{\theta(1)-1} \ldots (x_1 y_{\sigma(n)})^{\theta(n)-1}$$

$$= \frac{1}{\prod_{j=0}^{n-1} j!} \sum_{\theta\in S_n} x_1^{\theta(1)-1} \ldots x_n^{\theta(n)-1} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, y_{\sigma(1)}^{\theta(1)-1} \ldots y_{\sigma(n)}^{\theta(n)-1}.$$

In the last expression we recognize a determinant, so we obtain

$$T_{n(n-1)/2} = \frac{1}{\prod_{j=0}^{n-1} j!} \sum_{\theta\in S_n} x_1^{\theta(1)-1} \ldots x_n^{\theta(n)-1} \det[\{y_i^{\theta(j)-1}\}_{i,j=1}^n]$$

$$= \frac{1}{\prod_{j=0}^{n-1} j!} \sum_{\theta\in S_n} x_1^{\theta(1)-1} \ldots x_n^{\theta(n)-1} \operatorname{sgn}(\theta) \det[\{y_i^{j-1}\}_{i,j=1}^n].$$

Hence we get

$$T_{n(n-1)/2} = \frac{h(x)h(y)}{\prod_{j=0}^{n-1} j!}. \quad \blacksquare$$

Proof of Theorem 1. Using Lemma 2 we write

$$P_x(\tau > t)\, t^{n(n-1)/4} = \int_W b_t(x, y)\, dy$$

$$= t^{n(n-1)/4} \int_W (2\pi t)^{-n/2} \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) \det\left[\left\{\exp\left(\frac{x_i y_j}{t}\right)\right\}_{i,j=1}^n\right] dy$$

$$= \frac{t^{n(n-3)/4}}{(2\pi)^{n/2}} \int_W \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) \sum_{k=0}^{\infty} T_k t^{-k}\, dy.$$

Since the first coefficients vanish, we get

$$P_x(\tau > t)\, t^{n(n-1)/4} = \frac{t^{n(n-3)/4}}{(2\pi)^{n/2}} \int_W \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) \sum_{k=n(n-1)/2}^{\infty} T_k t^{-k}\, dy$$

$$(5) \qquad = \frac{t^{n(n-3)/4}}{(2\pi)^{n/2}} \int_W \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) T_{n(n-1)/2}\, t^{-n(n-1)/2}\, dy$$

$$(6) \qquad + \frac{t^{n(n-3)/4}}{(2\pi)^{n/2}} \int_W \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) \sum_{k=n(n-1)/2+1}^{\infty} T_k t^{-k}\, dy.$$

Consider the first element (5) of the sum above:

$$(7) \qquad \frac{t^{n(n-3)/4}}{(2\pi)^{n/2}} \int_W \exp\left(-\frac{|x|^2+|y|^2}{2t}\right) \frac{t^{-n(n-1)/2}}{\prod_{j=0}^{n-1} j!} h(x)\, h(y)\, dy$$

$$= \frac{t^{-n(n+1)/4}}{(2\pi)^{n/2}} \frac{h(x)}{\prod_{j=0}^{n-1} j!} \exp\left(-\frac{|x|^2}{2t}\right) \int_W \exp\left(-\frac{|y|^2}{2t}\right) h(y)\, dy.$$

Since $h(\alpha x) = \alpha^{n(n-1)/2} h(x)$, we see that (7) is of the form

$$\frac{1}{(2\pi)^{n/2}} \frac{h(x)}{\prod_{j=0}^{n-1} j!} \exp\left(-\frac{|x|^2}{2t}\right) \int_W \exp\left(-\frac{|y|^2}{2t}\right) h(y)\, dy.$$

We now show that (6) tends to 0 as $t \to \infty$. After some simple calculations we infer that (6) is equal to

$$\frac{1}{(2\pi)^{n/2}} \sum_{k=n(n-1)/2+1}^{\infty} t^{-n(n-1)/4-k/2} \exp\left(-\frac{|x|^2}{2t}\right) \int_W \exp\left(-\frac{|y|^2}{2}\right) T_k\, dy,$$

and it tends to 0 as $t \to \infty$. Thus we have (1). ■

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