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ROSIŃSKI MEASURES FOR TEMPERED STABLE AND RELATED ORNSTEIN–UHLENBECK PROCESSES*

BY

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Dedicated to the memory of Kazimierz Urbanik and his pioneering work on operator-decomposable distributions

Abstract. Several concrete parametric classes of tempered stable distributions are discussed in terms of explicit calculations of their *Rosiński measures.* The hope is that they will provide a family of concrete models useful in applied areas and for which the fitting can be done by parametric methods. Related Ornstein–Uhlenbeck processes are analyzed. The emphasis throughout the paper is on obtaining exact analytic formulas.

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1. INTRODUCTION

Ever since the pioneering applied work by Mandelbrot [13], and Montroll and Scher [16], Lévy stable processes enjoyed great popularity as a flexible modeling tool in the natural and economic sciences. However, their elegant scaling properties, which made them analytically pleasing, were also a problem when people tried to fit them to real data; the scaling could not be maintained at all scales, and the thick tail behavior was often impossible to verify rigorously. The remedy proposed first by Mantegna and Stanley [14], and others, was to keep the local behavior of the distributions Lévy stable-like but to truncate the tails. Different schemes for the truncation and their parametrization were proposed and discussed in the physical literature (see, e.g., [28], and [19], for a partial history of these efforts). All of those schemes suffered, however, from

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the lack of invariance under linear transformations for the distributions classes introduced, a property that Lévy stable distributions enjoyed. Such was the state of affairs until publication of Rosiński's papers [19]–[20] where a formal and elegant definition of the class of *tempered stable distributions and processes* was proposed. The latter did have invariance under the linear transformations as one of their most important structural properties.

The class of tempered stable distributions has an infinite-dimensional parametrization by a family of measures, which makes their fitting to real data a difficult task. For this reason, we embark in this paper on a fairly pedestrian project of developing a series of parametric models that fit into the general framework of tempered distributions but for which parametric statistical estimation procedures can be realistically developed. The emphasis is on obtaining explicit analytic formulas and on explicit calculations. Once those distributional models are developed, it is natural to study the corresponding Lévy processes and, more physically attractive, the corresponding Ornstein–Uhlenbek processes.

The paper begins, in Section 2, with brief preliminaries on infinitely divisible distributions in $\boldsymbol{R}^d$, and the subclass of self-decomposable distributions which were originally developed, among others, by Urbanik, see [29]–[31]. Here, the role of the cumulants and the cumulant functions, which will be paramount throughout the paper, is first explained.

Section 3 begins with the basic definitions of tempered stable distributions and introduces the concept of the Rosiński measure (R-measure). A subsection on fundamental properties of tempered stable distributions follows. The section ends with a discussion of the relation of this class, in one- and two-dimensional settings, to the previously studied smoothly truncated Lévy distributions. It is Section 4 where we introduce and discuss several low dimensional parametric tempering schemes using a variety of special functions: gamma, inverse Gaussian, fractional exponential, 1/3-stable and Bessel.

Finally, in Section 5 we turn to the multivariate tempered stable Ornstein–Uhlenbeck processes, their background driving Lévy processes (BDLP), and their stationary versions. An appendix on the multiple cumulant technique, not commonly seen in the literature, concludes the paper.

2. PRELIMINARIES

Let us begin with a random variable X having an infinitely divisible distribution P on $\boldsymbol{R}^d$, with a generating triple (Σ, M, b), and the characteristic function given by

$$\varphi(y) = \exp\Big(-\tfrac{1}{2}\langle y, \Sigma y\rangle + i\langle b, y\rangle + \int_{\boldsymbol{R}^d} \big(e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle \mathbb{1}_D(x)\big) M(dx)\Big),$$

where Σ is a $d \times d$ covariance matrix, $b \in \mathbf{R}^d$, $D = \{x : \|x\| \leqslant 1\}$, and M is a Lévy measure on $\mathbf{R}^d$, satisfying

$$M(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbf{R}^d} (\|x\|^2 \wedge 1) M(dx) < \infty.$$

Furthermore, throughout this paper, we are going to assume that

$$\int_{\|x\| > 1} \|x\|^m M(dx) < \infty$$

for some $m \geqslant 1$. This condition is equivalent (see Sato [21], Theorem 25.3) to the condition

$$\mathsf{E}\|X\|^m < \infty. \tag{2}$$

If, additionally,

$$\int_{\|x\| < 1} \|x\|^m M(dx) < \infty, \tag{3}$$

then we can rewrite the characteristic function $\varphi(y)$ in the centered form (cf., [21], p. 39)

$$\begin{aligned}\varphi(y) = \exp\Bigg(& iy^{\mathsf{T}} c_1 + i^2 \frac{1}{2!} y^{\mathsf{T}\otimes 2} c_2 + \ldots + i^k \frac{1}{k!} y^{\mathsf{T}\otimes m} c_m \\ & + \int_{\mathbf{R}^d} \Big(e^{i\langle y, x\rangle} - \sum_{j=0}^{m} \frac{i^j}{j!} y^{\mathsf{T}\otimes j} x^{\otimes j} \Big) M(dx) \Bigg) \\ = \exp\Bigg(& \sum_{j=1}^{m} \frac{i^j}{j!} y^{\mathsf{T}\otimes j} c_j + \int_{\mathbf{R}^d} \Big(e^{i\langle y, x\rangle} - \sum_{j=0}^{m} \frac{i^j}{j!} y^{\mathsf{T}\otimes j} x^{\otimes j} \Big) M(dx) \Bigg),\end{aligned}$$

where c_j is the j^{th} order cumulant of the random variable X, that is, the "moment" of order j of the Lévy measure M, and $\otimes$ and T stand for the Kronecker product and the transpose, respectively. Recall that if A is an $m \times n$ matrix, and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix

$$A \otimes B = \begin{bmatrix} a_{11} B & \ldots & a_{1n} B \\ \ldots & \ldots & \ldots \\ a_{m1} B & \ldots & a_{mn} B \end{bmatrix}.$$

If $m \geqslant 1$, then c_1 is the center of the measure P; if $m \geqslant 2$, then c_2 is the vector of the variances of X. Note that c_1 and c_2 are not uniquely determined by the Lévy measure M unless the distribution P has no Gaussian component ($\Sigma = 0$) and is centered ($c_1 = 0$).

The distribution P is said to be *self-decomposable* if, for every $\gamma > 1$, there exists a probability measure P_γ on $\mathbf{R}^d$ such that

$$\varphi(y) = \varphi(\gamma^{-1} y) \varphi_\gamma(y),$$

where φ_γ is the characteristic function of P_γ. A self-decomposable distribution P is necessarily infinitely divisible and, for any $\gamma > 1$, the measure P_γ is a uniquely determined infinitely divisible measure. An infinitely divisible distribution P is self-decomposable if and only if its Lévy measure M is of the form

$$M(B) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_B(ru)\, k_u(r) \frac{dr}{r}\, \sigma(du), \tag{4}$$

where S^{d-1} is the unit sphere in $\boldsymbol{R}^d$, $\sigma(du)$ is a finite measure on S^{d-1}, and $k_u(r)$ is decreasing in r, and measurable in u. In particular, for $d = 1$, the characteristic function of a self-decomposable measure is of the form

$$\varphi(y) = \exp\left(-\frac{1}{2}\Sigma y^2 + iby + \int_{-\infty}^{\infty} \left(e^{iyx} - 1 - iyx\mathbf{1}_{[-1,1]}(x)\right) k(x) \frac{dx}{|x|}\right),$$

where $\Sigma \geqslant 0$, $b \in \boldsymbol{R}$, and $k(x)$ is a nonnegative function, increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$, and such that

$$\int_{-\infty}^{\infty} (|x|^2 \wedge 1)\, k(x) \frac{dx}{|x|} < \infty,$$

see [29], [31], [30], [21], p. 109.

3. TEMPERED STABLE DISTRIBUTIONS

3.1. Basic definitions. A Gaussian-free ($\Sigma = 0$), self-decomposable probability distribution is said to be *tempered α-stable* if, in the polar representation (4) of its Lévy measure M, the function

$$k_u(r) = k(r\,|\,u)\, r^{-\alpha},$$

where $\alpha \in (0, 2)$, and $k(\cdot\,|\,u)$ is a completely monotone function with $k(0+\,|\,u) = 1$ and $k(\infty\,|\,u) = 0$. In other words, the Lévy measure of a tempered α-stable distribution is of the form

$$M(B) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_B(ru)\, k(r\,|\,u) \frac{dr}{r^{\alpha+1}}\, \sigma(du), \tag{5}$$

where $\sigma(du)$ is a finite measure on the unit sphere S^{d-1}. The parameter α is called the *index*, and the function k – the *tempering function.*

Obviously, the case of constant $k(r\,|\,u)$ corresponds to the classical α-stable distribution. The above concept of the tempered stable distribution is due to Rosiński [19]–[20], who made essential use of the fact that the tempering function k can be represented as the Laplace transform

$$k(r\,|\,u) = \int_0^\infty e^{-rs}\, Q(ds\,|\,u),$$

where $Q(ds\,|\,u)$ is a u-measurable family of probability measures on $\boldsymbol{R}_+$. The σ-weighted superposition of the measures $Q(ds\,|\,u)$ defines a finite measure

$$(6) \qquad Q(B) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_B(ru)\, Q(dr\,|\,u)\, \sigma(du)$$

on $\boldsymbol{R}^d$. The formula

$$(7) \qquad R(B) = \int_{\boldsymbol{R}_0^d} \mathbf{1}_B\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(dx), \qquad \boldsymbol{R}_0^d = \boldsymbol{R}^d \setminus \{0\},$$

defines a measure, equivalent to the measure Q which, in turn, can be expressed in terms of the measure R as follows:

$$Q(B) = \int_{\boldsymbol{R}_0^d} \mathbf{1}_B\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha R(dx).$$

Measures Q and R have been demonstrated (see Rosiński [19], [20]) to be of importance for different applications, with Q being particularly useful for the simulation of the tempered stable random variables. We shall refer to R as the *Rosiński measure*, or *R-measure*, for short. An R-measure R, together with a constant b, uniquely determines, via (4), a tempered α-stable measure. The class of all tempered α-stable measures (or, loosely speaking, related random variables) will be denoted by $TS_\alpha(R, b)$.

3.2. Properties of tempered stable distributions. We begin this subsection by listing the fundamental properties of $TS_\alpha(R, b)$.

(i) *The cumulant function representation.* The cumulant function κ_X (that is, the logarithm of the characteristic function) of a random variable X with a tempered stable distribution in $TS_\alpha(R, b)$ is uniquely given (see [19] for details) by the formula

$$(8) \qquad \kappa_X(y) = \int_{\boldsymbol{R}_0^d} \psi_\alpha(\langle y, x\rangle)\, R(dx) + i\langle y, b\rangle,$$

where

$$(9) \qquad \psi_\alpha(r) = \begin{cases} \Gamma(-\alpha)\,[(1-ir)^\alpha - 1], & 0 < \alpha < 1, \\ (1-ir)\log(1-ir) + ir, & \alpha = 1, \\ \Gamma(-\alpha)\,[(1-ir)^\alpha - 1 + i\alpha r], & 1 < \alpha < 2. \end{cases}$$

(ii) *Moments.* A random variable $X \in TS_\alpha(R, b)$ has always finite moments of order $m < \alpha$. If $m > \alpha$, then the moment assumption (1) can be expressed in terms of the R-measure as follows:

$$\mathsf{E}\,\|X\|^m < \infty$$

if and only if

$$\int_{\|x\|>1} \|x\|^m R(dx) < \infty. \tag{10}$$

If the support of R is a bounded set, then some exponential moments are also finite.

(iii) *Invariance of* $TS_\alpha(R, b)$ *under independent summation.* If random variables X and Y are independent, and such that $X \in TS_\alpha(R_1, b_1)$ and $Y \in TS_\alpha(R_2, b_2)$, then

$$X + Y \in TS_\alpha(R_1 + R_2, b_1 + b_2).$$

(iv) *Invariance of* $TS_\alpha(R, b)$ *under linear transformations.* If $X \in TS_\alpha(R, b)$, and V is an $m \times d$ matrix, then

$$VX \in TS_\alpha(R_V, b_V),$$

where $R_V(B) = R \circ V^{-1}(B) = R(x \mid Vx \in B)$, and $b_V = Vb$.

The above invariance property can be extended to linear functionals of the tempered stable Lévy processes as follows: Let Z_t, $t \geqslant 0$, be a Lévy process with $Z_1 \in TS_\alpha(R_Z, 0)$, and

$$X = \int_0^\tau g(s)\, dZ_s,$$

with a fixed $\tau > 0$, and a continuous (matrix-valued) function $g(s)$. Then the cumulant function of X,

$$\kappa_X(y) = \int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle) R_X(dx),$$

where

$$R_X(B) = \int_{\mathbf{R}_0^d} \int_0^\tau \mathbf{1}_B(g(s)x)\, ds R_Z(dx). \tag{11}$$

Thus (see, e.g., [12], [9]) the characteristic function of the linear functional X can be written in the form

$$\varphi_X(y) = \exp\Big[\int_0^\tau \kappa_{Z_1}(g(s)^{\mathrm{T}} y)\, ds\Big], \tag{12}$$

so that the cumulant function κ_X of X can be rewritten as follows:

$$\kappa_X(y) = \int_0^\tau \kappa_{Z_1}(g(s)^{\mathrm{T}} y)\, ds = \int_0^\tau \int_{\mathbf{R}_0^d} \psi_\alpha(\langle g(s)^{\mathrm{T}} y, x\rangle) R_Z(dx)\, ds$$

$$= \int_{\mathbf{R}_0^d} \int_0^\tau \psi_\alpha(\langle g(s)^{\mathrm{T}} y, x\rangle)\, ds R_Z(dx).$$

(v) *Gamma-like limit, for* $\alpha \to 0$. For each $r \neq 0$, and $\alpha \to 0$, the limit of the cumulant function $\psi_\alpha(r)$ in (9) is easily calculated:

$$\lim_{\alpha \to 0} \psi_\alpha(r) = -\log(1 - ir).$$

So, if a family R_α of R-measures converges weakly to an R-measure R_X, as $\alpha \to 0$, then the function

$$\kappa_X(y) = -\int_{\boldsymbol{R}_0^d} \log(1 - i\langle y, x\rangle) R_X(dx)$$

is the cumulant function of a random variable X. If $d = 1$ and we choose a fixed $\lambda > 0$, and the family of R-measures $R_\alpha = \lambda^\alpha a\delta_{1/\lambda}$, where δ is the Dirac measure, then the corresponding limiting distribution (of X) is the gamma distribution. Thus the definition of the class TS_α can be extended meaningfully to $\alpha = 0$.

(vi) *Simulations.* Rosiński [19] gives a series representation for random variables with distributions in $TS_\alpha(R, b)$ based on i.i.d. sequences of uniform, exponential and Q-distributed random variables. This series representation permits a convenient simulation of tempered stable Lévy processes and related Ornstein–Uhlenbeck processes.

(vii) *Cumulants.* The cumulants of order greater than one of a tempered stable distribution can be calculated purely in terms of its R-measure R; those of order one depend on the drift b as well.

LEMMA 1. *Suppose that the R-measure R satisfies the moment condition*

$$\int_{\boldsymbol{R}_0^d} \|x\|^m R(dx) < \infty \tag{13}$$

for an $m \geqslant 1$, in case of $0 < \alpha < 1$, and for an $m \geqslant 2$, when $1 \leqslant \alpha < 2$. Then the m^{th} order cumulant of the tempered stable random variable $X \sim TS_\alpha(R, b)$ is given by

$$c_m = \Gamma(m - \alpha)\mu_{R, \otimes m}, \tag{14}$$

where

$$\mu_{R, \otimes m} = \int_{\boldsymbol{R}_0^d} x^{\otimes m} R(dx).$$

Proof. First, note that the function ψ_α has the following series expansion:

$$\psi_\alpha(r) = \int_0^\infty (e^{irx} - 1 - irx) x^{-\alpha-1} e^{-x} dx$$

$$= \int_0^\infty \left(e^{irx} - \sum_{j=0}^m \frac{i^j}{j!} r^j x^j\right) x^{-\alpha-1} e^{-x} dx + \sum_{j=2}^m \frac{i^j}{j!} r^j \Gamma(j - \alpha),$$

which, for $\alpha = 1$, is understood as the limit for $\alpha \searrow 1$. The cumulant function κ_X has the form

$$\kappa_X(y) = \Gamma(-\alpha) \int_{\mathbf{R}_0^d} \left[(1 - i\langle y, x\rangle)^\alpha - \sum_{j=0}^{m} \frac{i^j}{j!} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)} \langle y, x\rangle^j \right] R(dx)$$

$$+ i\langle y, b\rangle + \sum_{j=2}^{m} \frac{i^j}{j!} \mu_{R,j}(y) \Gamma(j-\alpha),$$

where $\mu_{R,j}$ is the j^{th} moment of R, i.e.

$$\mu_{R,j}(y) = \int_{\mathbf{R}_0^d} \langle y, x\rangle^j R(dx) = y^{\mathsf{T}\otimes j} \int_{\mathbf{R}_0^d} x^{\otimes j} R(dx) = y^{\mathsf{T}\otimes j} \mu_{R,\otimes j}.$$

Hence, the j^{th} order cumulant c_j of the random variable X is given by the formula

$$c_j = \Gamma(j-\alpha) \mu_{R,\otimes j};$$

see the Appendix, and also [27], for the definition of the multiple moments $\mu_{R,\otimes j}$. ∎

The case of $m = 1$ is special. It is possible to find a random variable $X \sim TS_\alpha(R, b)$ with a finite first moment, but such that $\mu_{R,1} = \infty$ (see Sections 4.4 and 4.5). This fact explains why the condition (13) is stronger than the condition (10).

3.3. 1-D Smoothly Truncated Lévy Distributions. One of the early examples of TS_α processes were the 1-D Truncated Lévy Flights introduced by Mantegna and Stanley [14] as a model for random phenomena which exhibit at small scales properties similar to those of self-similar Lévy processes, but have distributions which at large scales have cutoffs and thus have finite moments of any order. Koponen [10], building on Mantegna and Stanley's ideas, defined the Smoothly Truncated Lévy Flights (STLF) which stressed the advantage of a nice analytic form. Independently, the same family of distributions was described earlier by Hougaard [8] in the context of a biological application. In this section, we discuss these special examples in the context of general tempered stable distributions.

The 1-D unit 'sphere' is the two-point set $S^0 = \{\pm 1\}$, and Koponen's *Smoothly Truncated Lévy Distribution* ($STLD_\alpha(a, p_1, \lambda)$) is defined as a tempered α-stable distribution with the tempering function (see Subsection 3.1)

$$k(r \mid \pm 1) = k(r) = \exp(-\lambda r), \qquad \lambda > 0, r > 0,$$

and the measure

$$\sigma(\{-1\}) = ap_1, \qquad \sigma(\{1\}) = ap_2,$$

where $a, p_1, p_2 > 0, p_1+p_2 = 1$. In this case, the polar representation (5) takes the form

$$k(r\,|\,u)\frac{dr}{r^{\alpha+1}}\sigma(du) = \begin{cases} ap_1\exp(\lambda r)\dfrac{dr}{|r|^{\alpha+1}}, & r<0,\\[2mm] ap_2\exp(-\lambda r)\dfrac{dr}{r^{\alpha+1}}, & r>0.\end{cases}$$

The measures Q and R (see (6) and (7), respectively) are given by the formulas

$$Q = ap_1\,\delta_{-\lambda}+ap_2\,\delta_\lambda \quad \text{and} \quad R = \lambda^\alpha a(p_1\,\delta_{-1/\lambda}+p_2\,\delta_{1/\lambda}),$$

respectively, where δ_λ denotes the Dirac measure at λ. In other words,

$$Q(B) = ap_1\,\mathbf{1}_B(-\lambda)+ap_2\,\mathbf{1}_B(\lambda)$$

and

$$R(B) = \lambda^\alpha a[p_1\,\mathbf{1}_B(-1/\lambda)+p_2\,\mathbf{1}_B(1/\lambda)].$$

The cumulant function κ_X of X (see (8)) is then given by

$$\kappa_X(u) = a\lambda^\alpha[p_1\,\psi_\alpha(-u/\lambda)+p_2\,\psi_\alpha(u/\lambda)]+iub, \tag{15}$$

with the function ψ_α defined in (9). Thus, for $m \geqslant 2$, the cumulants themselves are

$$\mathrm{Cum}_m(X) = a\lambda^\alpha\Gamma(m-\alpha)\big(p_1(-\lambda)^{-m}+p_2\lambda^{-m}\big) = a\lambda^{\alpha-m}\Gamma(m-\alpha)\big(p_1(-1)^m+p_2\big).$$

For a fixed $\lambda > 0$, as $\alpha \to 0$, the distribution $STLD_\alpha$ tends to the gamma distribution $\Gamma(a, \lambda)$. Indeed, for $0<a<1$, the Laplace transform ϕ_λ of $STLD_\alpha(a, 0, \lambda)$ is

$$\phi_\lambda(u) = \exp\big(a\lambda^\alpha\Gamma(-\alpha)\,[(1+u/\lambda)^\alpha-1]\big),$$

and

$$\lim_{\alpha\to 0}\exp\left(-a\Gamma(1-\alpha)\frac{(\lambda+u)^\alpha-\lambda^\alpha}{\alpha}\right) = \exp\big(-a\log(1+u/\lambda)\big) = (1+u/\lambda)^{-\alpha},$$

by the l'Hospital rule.

It is an interesting observation that, for $0<\alpha<1$, the smooth truncation of the stable cumulant function results also in the smooth truncation of the stable probability density itself. More precisely, if $f(x)$ is the density function of a one-sided α-stable distribution, then $g_\lambda(x) = f(x)e^{-\lambda|x|}$ is the density function of some Smoothly Truncated Lévy Distribution and, vice versa, if $g_\lambda(x)$ is the density function of a Smoothly Truncated Lévy Distribution, then $f(x) = g_\lambda(x)e^{\lambda|x|}$ is an α-stable density function. Moreover, there is a one-to-one correspondence between the α-stable distributions and the λ-equivalence classes of Smoothly Truncated Lévy Distributions. The connection is given by a linear transformation of the variable of the cumulant functions.

For example, the density function for the one-sided 1/2-stable distribution is the *Inverse Gaussian* p.d.f. (see [25])

$$f(x) = c\frac{\exp(-c^2/(2x))}{x^{3/2}\sqrt{2\pi}}, \qquad x > 0,$$

with the cumulant function (see Sato [21], p. 13)

$$\kappa(y) = -c|y|^{1/2}(1 - i\,\mathrm{sign}(y)).$$

The corresponding density of the Smoothly Truncated Lévy Distribution ($\alpha = 1/2$) is

$$g_\lambda(x) = c\frac{\exp(-c^2/(2x) - \lambda x + c\sqrt{2\lambda})}{x^{3/2}\sqrt{2\pi}}, \qquad x > 0,$$

and it has the cumulant function

$$\kappa_\lambda(y) = -c\sqrt{2}\,[\sqrt{\lambda - iy} - \sqrt{\lambda}],$$

which, after substitution $c = 2a\sqrt{\pi}$, gives the representation (15) with $\alpha = 1/2$.

Fractional and multiscaling properties of $STLD_\alpha$ have been described in [28]. In particular, we have shown that, for a one-sided distribution in $STLD_\alpha(a, 0, \lambda)$, moments of any positive order ϱ (including fractional) have the asymptotics

$$\log \mathsf{E}(|X|^\varrho) \sim \begin{cases} \min(\varrho/\alpha, 1)\log a + c_1 & \text{as } a \to 0, \\ \varrho \log a + c_2 & \text{as } a \to \infty. \end{cases}$$

For the symmetric distribution in $STLD_\alpha(a, 1/2, \lambda)$,

$$\log \mathsf{E}(|X|^\varrho) \sim \begin{cases} \min(\varrho/\alpha, 1)\log a + C_1 & \text{as } a \to 0, \\ (\varrho/2) \log a + C_2 & \text{as } a \to \infty. \end{cases}$$

The above-quoted asymptotic results establish the multiscaling properties of the $STLD$s and the related Smoothly Truncated Lévy Flights.

3.4. 2-D Smoothly Truncated Lévy Distributions

Discrete tempering. Let $X = (X_1, X_2)$ be a tempered stable random variable in $\boldsymbol{R}^2$. Here, the unit 'sphere' $\boldsymbol{S}$ consists of the vectors $u_\vartheta = (u_1, u_2) = (\cos\vartheta, \sin\vartheta)$ and, for fixed elements $u_{\vartheta_1}, u_{\vartheta_2}, \ldots, u_{\vartheta_l} \in \boldsymbol{S}$ and constants $\lambda_1, \lambda_2, \ldots, \lambda_l > 0$, we define a *Smoothly Truncated Lévy Distribution* on $\boldsymbol{R}^2$, with index α, via the tempering function

$$k(r\,|\,u) = \exp(-\lambda_j r) \quad \text{for } u = u_{\vartheta_j},\ j = 1, 2, \ldots, l,$$

and zero otherwise, and a discrete measure

$$\sigma(\{e_{\vartheta_j}\}) = ap_j,$$

where $a, p_j > 0$, and $\sum p_j = 1$. The measure Q (see (6)) is then

$$Q = a \sum_{j=1}^{l} p_j \delta_{\lambda_j e_{\vartheta_j}},$$

and R-measure (see (7))

$$R = \sum_{j=1}^{l} \lambda_j^{\alpha} p_j \delta_{1/\lambda_j e_{\vartheta_j}}.$$

The cumulant function κ_X (see (8)) has now the representation

$$\kappa_X(y_1, y_2) = a \sum_{j=1}^{l} \lambda_j^{\alpha} p_j \psi_\alpha([y_1 \cos\vartheta_j + y_2 \sin\vartheta_j]/\lambda_j) + i\langle y, b\rangle$$

with the cumulants

$$\mathrm{Cum}_{m,n}(X_1, X_2) = a\Gamma(m+n-\alpha) \sum_{j=1}^{l} \lambda_j^{\alpha-m-n} p_j (\cos\vartheta_j)^m (\sin\vartheta_j)^n.$$

Remark 1. A 1-D projection of X, given by the formula $Y = v_1 X_1 + v_2 X_2$, can be viewed as a Generalized Smoothly Truncated Lévy Distribution on $\boldsymbol{R}$. Its cumulant function is of the form

$$\kappa_Y(y) = a \sum_{j=1}^{l} \lambda_j^{\alpha} p_j \psi_\alpha\big(yv \cos(\vartheta_j')/\lambda_j\big) + i\langle y, v\rangle,$$

where $\vartheta_j' = \vartheta_j - \arctan(v_2/v_1)$, and $v = \sqrt{v_1^2 + v_2^2}$. A particular case may be obtained by matching constants v_j to $\cos(\vartheta_j)$, and constants λ_j, to obtain a cumulant function of the form

$$\kappa_Y(y) = a \sum_{j=1}^{l} \lambda_j^{\alpha} p_j \psi_\alpha(yv_j) + i\langle y, v\rangle$$

with the cumulants

$$\mathrm{Cum}_m(X) = a\Gamma(m-\alpha) \sum_{j=1}^{l} \lambda_j^{\alpha-m} p_j v_j^m.$$

Continuous tempering. Let $X = (X_1, X_2)$ be a tempered stable random variable on $\boldsymbol{R}^2$, and $\sigma(d\vartheta) = g(\vartheta)\, d\vartheta$ be a finite measure on $[0, 2\pi)$. Define the tempering function k by the formula

$$k(r \mid \vartheta) = \exp\big(-\lambda(\vartheta) r\big),$$

where $\lambda(\vartheta) > 0$ and $r > 0$. Now, the polar representation (5) of .. has the form

$$k(r \mid \vartheta) \frac{dr}{r^{\alpha+1}} \sigma(d\vartheta) = \exp\big(-\lambda(\vartheta) r\big) \frac{dr}{r^{\alpha+1}} g(\vartheta)\, d\vartheta,$$

and hence

$$Q(ds\,|\,\vartheta) = \delta_{\lambda(\vartheta)}.$$

Putting $\theta_y = \arctan(y)$, we obtain the measures Q and R in the form

$$Q(B) = \int_{\mathbf{R}_0^2} \mathbf{1}_B\big(\lambda(\theta_{x_2/x_1})\,x/\|x\|\big)\,g(\theta_{x_2/x_1})\,dx,$$

$$R(B) = \int_{\mathbf{R}_0^2} \mathbf{1}_B\big(\lambda^{-1}(\theta_{x_2/x_1})\,x/\|x\|\big)\,\lambda(\theta_{x_2/x_1})^{\alpha}\,g(\theta_{x_2/x_1})\,dx.$$

The cumulant function now is

$$\kappa_X(y_1, y_2) = \int_0^{2\pi} \lambda(\vartheta)^{\alpha}\,\psi_{\alpha}\big([y_1\cos\vartheta + y_2\sin\vartheta]/\lambda(\vartheta)\big)\,g(\vartheta)\,d\vartheta$$

with cumulants

$$\mathrm{Cum}_{m,n}(X_1, X_2) = \Gamma(m+n-\alpha)\int_0^{2\pi} \lambda(\vartheta)^{\alpha-m-n}(\cos\vartheta)^m(\sin\vartheta)^n\,g(\vartheta)\,d\vartheta.$$

Remark 2. The 1-D projection $Y = v_1 X_1 + v_2 X_2$ discussed in Remark 1, with $v = \sqrt{v_1^2 + v_2^2}$, gives in the continuous case the cumulant function

$$\begin{aligned}\kappa_Y(y) &= \int_0^{2\pi} \lambda(\vartheta)^{\alpha}\,\psi_{\alpha}\left(\frac{v\cos(\vartheta-\theta_{v_2/v_1})}{\lambda(\vartheta)}\,y\right)g(\vartheta)\,d\vartheta\\ &= \int_0^{2\pi} \lambda(\vartheta)^{\alpha}\,\psi_{\alpha}\left(\frac{v\cos(\vartheta-\vartheta')}{\lambda(\vartheta)}\,y\right)g(\vartheta)\,d\vartheta,\end{aligned}$$

where $\vartheta' \in [0, 2\pi]$ and $v > 0$ are fixed, with the cumulants

$$\mathrm{Cum}_m(Y) = \Gamma(m-\alpha)\int_0^{2\pi} \lambda(\vartheta)^{\alpha-m}\big(v\cos(\vartheta-\vartheta')\big)^m\,g(\vartheta)\,d\vartheta.$$

4. TEMPERING VIA SPECIAL FUNCTIONS

The Smoothly Truncated Lévy Distributions were defined in the previous section through an exponential tempering functions which, in turn, determined the measures Q and R via the formula (7). In this section we will produce several other examples of tempering via special functions such as gamma, Bessel, α-stable density, etc. These models will approximate the stable distribution with different speeds and will have different probabilistic properties expressed by the behavior of their cumulants. All of the models will depend on parameter(s) λ; as the parameter goes to 0, the model converges to the Lévy stable distribution; some interesting new probability distributions arise in the process as well.

In the following we consider only a one-sided ($p_1 = 0$) measure σ on the unit ball $S^0 = \{-1, 1\}$. The two-sided measure can be handled as in the case of

Smoothly Truncated Lévy Distribution described in Subsection 3.3: the two-sided measure is given by the formula

$$\sigma(\{-1\}) = ap_1, \quad \sigma(\{1\}) = ap_2,$$

where a, $p_1, p_2 > 0$, $p_1+p_2 = 1$, while the tempering does not depend on the direction, i.e. $k(r\,|\pm 1) = k(r)$. Subsection 3.4 gives also a general method for extending the results of this section to a family of more flexible models.

4.1. Mixed tempering. In this subsection we consider a truncated tempering scheme which permits control of the closeness to the α-stable distribution through two parameters λ_1 and λ_2. So, let the tempering function $k(r)$ be

$$k(r) = (1+\lambda_2 r)^{-\beta}\exp(-\lambda_1 r), \quad \lambda_1, \lambda_2 > 0, \; r > 0.$$

Then $k(r)$ is the Laplace transform of the function

$$Q(dx) = \frac{(x-\lambda_1)_+^{\beta-1}}{\lambda_2^\beta \Gamma(\beta)}\exp\left(-\frac{[x-\lambda_1]_+}{\lambda_2}\right)dx,$$

so that, in view of (6), the Q-measure takes the form

$$Q(B) = \int_0^\infty \mathbf{1}_B(x)\frac{(x-\lambda_1)_+^{\beta-1}}{\lambda_2^\beta \Gamma(\beta)}\exp\left(-\frac{[x-\lambda_1]_+}{\lambda_2}\right)dr,$$

and the R-measure (see (7)) is given by the formula

$$\begin{aligned} R(B) &= \int_{\mathbf{R}_+} \mathbf{1}_B\left(\frac{\operatorname{sign}(x)}{|x|}\right)|x|^\alpha Q(dx) \\ &= \int_{\lambda_1}^\infty \mathbf{1}_B\left(\frac{1}{x}\right)|x|^\alpha \frac{(x-\lambda_1)^{\beta-1}}{\lambda_2^\beta \Gamma(\beta)}\exp\left(-\frac{x-\lambda_1}{\lambda_2}\right)dx \\ &= \int_0^{1/\lambda_1} \mathbf{1}_B(y)\, y^{-\beta-\alpha-1}\frac{(1-\lambda_1 y)^{\beta-1}}{\lambda_2^\beta \Gamma(\beta)}\exp\left(-\frac{1/y-\lambda_1}{\lambda_2}\right)dy. \end{aligned}$$

Since the density ρ of the R-measure

$$\rho(y) = \mathbf{1}_{0<y<1/\lambda_1}\, y^{-\beta-\alpha-1}\frac{(1-\lambda_1 y)^{\beta-1}}{\lambda_2^\beta \Gamma(\beta)}\exp\left(-\frac{1/y-\lambda_1}{\lambda_2}\right)$$

is zero outside of a finite interval, all the moments, and the moment generating function, exist (see [19]–[20]). The cumulants c_m are then calculated from the Lévy measure M, and, for $m > \alpha$, we have

$$\begin{aligned} c_m &= \int_0^\infty r^{m-\alpha-1}(1+\lambda_2 r)^{-\beta}\exp(-\lambda_1 r)\,dr \\ &= \lambda_2^{-\beta}\int_0^\infty r^{m-\alpha-1}(1/\lambda_2+r)^{-\beta}\exp(-\lambda_1 r)\,dr \\ &= \Gamma(m-\alpha)\,\lambda_2^{\alpha-m}\,U(m-\alpha,\, m-\alpha+1-\beta,\, \lambda_1/\lambda_2), \end{aligned}$$

where U is the Kummer function of the second kind, also known as the confluent hypergeometric function of the second kind ${}_1F_1$ (see [18], 2.3.6.9). Naturally, if λ_2 tends to zero, then c_m is close to the cumulant of Lévy Flights since

$$U(m-\alpha,\, m-\alpha+1-\beta,\, \lambda_1/\lambda_2) = (\lambda_1/\lambda_2)^{\alpha-m}\big(1+O(\lambda_2/\lambda_1)\big).$$

4.2. Gamma tempering. In this subsection the exponential, smoothly truncated tempering is replaced by a slower-decaying gamma tempering. So, let the tempering function $k(r)$ be

$$k(r) = (1+\lambda r)^{-\beta}, \qquad r > 0.$$

Then $Q(ds\,|\,y)$ is the gamma distribution with the Laplace transform

$$q(x) = \frac{x^{\beta-1}}{\lambda^\beta \Gamma(\beta)} \exp(-x/\lambda).$$

It is straightforward to check that the measures Q and R (see (6) and (7), respectively) are given by the formulas

$$Q(dx) = \mathbf{1}_{x>0} \frac{x^{\beta-1}}{\lambda^\beta \Gamma(\beta)} \exp(-x/\lambda)\, dx,$$

and

$$\begin{aligned} R(B) &= \int_{\mathbf{R}_0} \mathbf{1}_B\left(\frac{\operatorname{sign}(x)}{|x|}\right)|x|^\alpha Q(dx) \\ &= \int_0^\infty \mathbf{1}_B\left(\frac{1}{x}\right)\frac{x^{\beta+\alpha-1}}{\lambda^\beta \Gamma(\beta)} \exp(-x/\lambda)\, dx \\ &= \int_0^\infty \mathbf{1}_B(y)\frac{y^{-\beta-\alpha-1}}{\lambda^\beta \Gamma(\beta)} \exp[-1/(\lambda y)]\, dy. \end{aligned}$$

The density of the R-measure, with $0 < \beta+\alpha$, is thus

$$\rho(y) = \mathbf{1}_{y>0} \frac{y^{-\beta-\alpha-1}}{\lambda^\beta \Gamma(\beta)} \exp[-1/(\lambda y)],$$

and is the reciprocal (or inverse) of the Γ-density. The moments of order $m < \beta+\alpha$ of the R-measure exist, and

$$\mu_{R,m} = \frac{\lambda^{\alpha-m}\Gamma(\alpha+\beta-m)}{\Gamma(\beta)}$$

with the cumulants of the form

$$c_m = \frac{\lambda^{\alpha-m}\Gamma(m-\alpha)\Gamma(\alpha+\beta-m)}{\Gamma(\beta)}.$$

4.3. General Inverse Gaussian tempering. The Generalized Inverse Gaussian (GIG) distribution has the density

$$q(x) = \frac{(a/b)^{h/2}}{2K_h(\sqrt{ab})} x^{h-1} \exp(-\tfrac{1}{2}[ax+bx^{-1}]),$$

where K_h is the modified Bessel function of the second kind with index h. Notice that after replacing the parameters a and b by the parameters a/λ and λb, respectively, the density takes the form

$$q(x) = \frac{(a/b)^{h/2}\lambda^{-h}}{2K_h(\sqrt{ab})} x^{h-1} \exp(-\tfrac{1}{2}[ax/\lambda+\lambda bx^{-1}]),$$

where K_h is the modified Bessel function of the second kind, and the parameter λ can now be used for tuning the TS_α distribution close to the α-stable distribution. Let $h < 0$; then the Laplace transform $k(r)$ of $q(x)$ is well known (see Halgreen [7]):

$$k(r) = \exp\left(-\int_{a/2}^{\infty} bg_{-h}(2xb-ab)\ln\left(1+\frac{\lambda r}{x}\right)dx\right),$$

where

$$g_{-h}(x) = 2\left(\pi^2 x\,[J_{|h|}^2(\sqrt{x})+N_{|h|}^2(\sqrt{x})]\right)^{-1},$$

J_h and N_h are Bessel functions (see [6], Chapter 8). Observe that, for fixed a, b, and h, the limit of $k(r)$, as $\lambda \to 0$, is equal to 1. The measure Q is concentrated on the half-line $(0, \infty)$, and

$$Q(dx) = q(x)\,dx,$$

while the corresponding R-measure has the density ρ of the form

$$\rho(x) = \frac{(a/b)^{h/2}\lambda^{-h}}{2K_h(\sqrt{ab})} x^{-\alpha-h-1} \exp\left(-\tfrac{1}{2}[(a/\lambda)x^{-1}+b\lambda x]\right).$$

The moments of the R-measure are

$$\begin{aligned}\mu_{R,m} &= \frac{(a/b)^{h/2}\lambda^{-h}}{2K_h(\sqrt{ab})}\int_0^\infty x^{m-\alpha-h-1}\exp\left(-\tfrac{1}{2}[(a/\lambda)x^{-1}+b\lambda x]\right)dx \\ &= \left(\frac{a}{b\lambda^2}\right)^{(m-\alpha)/2}\frac{K_{m-\alpha-h}(\sqrt{ab})}{K_h(\sqrt{ab})}\end{aligned}$$

(see [6], 3.471.9). In particular, if $h = -1$ and $\alpha = 1/2$, then

$$\mu_{R,m} = \left(\frac{a}{b\lambda^2}\right)^{(m-\alpha)/2}\frac{K_{m+1/2}(\sqrt{ab})}{K_h(\sqrt{ab})},$$

where

$$K_{m+1/2}(y) = \sqrt{\frac{\pi}{2y}}\, e^{-y} \sum_{k=0}^{m} \frac{(m+k)!}{k!\,(m-k)!\,(2y)^k}$$

(see [6], 3.468). Similar results can be obtained for any integer h and $\alpha = 1/2$ or $3/2$.

4.4. Fractional exponential. The function

$$k(r) = \frac{1}{(\lambda r)^\beta + 1}$$

is the Laplace transform of the fractional exponential function $q(x)$. The moments up to the order $\alpha+\beta$ exist. In particular, for $\beta = 1/2$, we have

$$Q(dx) = q(x)\,dx \quad \text{with } q(x) = \sqrt{\frac{\pi}{\lambda x}} - \frac{\pi}{\lambda} e^{x/\lambda} \operatorname{erfc}(\sqrt{x/\lambda}),$$

where erfc is the complementary error function (see [17]). In this case, the R-measure is of the form

$$\begin{aligned} R(B) &= \int_{\boldsymbol{R}_+} \mathbf{1}_B\left(\frac{1}{x}\right) x^\alpha q(x)\,dx \\ &= \int_{\boldsymbol{R}_+} \mathbf{1}_B\left(\frac{1}{x}\right)\left[\sqrt{\frac{\pi}{\lambda x}} - \frac{\pi}{\lambda} e^{x/\lambda} \operatorname{erfc}(\sqrt{x/\lambda})\right] x^\alpha\,dx \\ &= \int_{\boldsymbol{R}_+} \mathbf{1}_B(x)\left[\sqrt{\frac{\pi x}{\lambda}} - \frac{\pi}{\lambda} e^{1/(x\lambda)} \operatorname{erfc}(1/\sqrt{\lambda x})\right] \frac{dx}{x^{2+\alpha}} \end{aligned}$$

with the density

$$p(x) = \frac{1}{x^{2+\alpha}}\left[\sqrt{\frac{\pi x}{\lambda}} - \frac{\pi}{\lambda} e^{1/(x\lambda)} \operatorname{erfc}(1/\sqrt{\lambda x})\right].$$

The verification of the existence of cumulants (moments) of order $m < \alpha + 1/2$ is here straightforward since, for large x, we have the asymptotics

$$\sqrt{\frac{\pi}{x}} - \pi e^x \operatorname{erfc}(\sqrt{x}) \sim \sqrt{\frac{\pi}{4x^3}}$$

(see [6], 8.254). For small x,

$$\sqrt{\frac{\pi}{x}} - \pi e^x \operatorname{erfc}(\sqrt{x}) \sim \sqrt{\frac{\pi}{x}},$$

so that, if $\alpha > 1/2$, then the measure R is not finite. The condition (13) is fulfilled only in some cases so that some assumptions have to be made before formula (14) can be used to calculate the cumulants.

Thus, for $1/2 < \alpha < 1$, the first order cumulant (the expectation)

$$c_1 = \frac{2\pi \lambda^{\alpha-1}}{\sin(2\pi[1-\alpha])},$$

and, for $3/2 < \alpha < 2$, the second order cumulant (the variance)

$$c_2 = \frac{2\pi \lambda^{\alpha-2}}{\sin(2\pi[2-\alpha])}.$$

Of course, one can try to calculate the cumulants directly from the Lévy measure M, but the basic formula

$$c_m = \int_{\mathbf{R}_+} x^m M(dx) = \int_0^\infty \frac{r^{m-\alpha-1}}{\sqrt{\lambda r}+1} dr$$

works only for $\alpha < m < \alpha + 1/2$ (since assumption (3) is not satisfied), although we know that c_m exists for all $m < \alpha + 1/2$.

4.5. 1/3-stable tempering. Another example of the tempering function for which explicit calculations are possible is the case when $q(x)$ is the density of the one-sided 1/3-stable distribution, that is

$$q(x) = \frac{\sqrt{\lambda}}{3\pi} x^{-3/2} K_{1/3}\left(\frac{2\sqrt{\lambda}}{3^{3/2}\sqrt{x}}\right), \quad x > 0,$$

where $K_{1/3}(x)$ is the modified Bessel function of the second kind with order 1/3 (see Subsection 4.3). Its Laplace transform is

$$k(r) = \exp\left(-(\lambda r)^{1/3}\right).$$

In this context the R-measure and its density are easily determined as

$$\begin{aligned} R(B) &= \int_{\mathbf{R}_+} \mathbf{1}_B\left(\frac{1}{x}\right) x^\alpha q(x)\, dx = \frac{\sqrt{\lambda}}{3\pi} \int_{\mathbf{R}_+} \mathbf{1}_B\left(\frac{1}{x}\right) x^{\alpha-3/2} K_{1/3}\left(\frac{2\sqrt{\lambda}}{3^{3/2}\sqrt{x}}\right) dx \\ &= \frac{\sqrt{\lambda}}{3\pi} \int_{\mathbf{R}_+} \mathbf{1}_B(x)\, x^{-\alpha-1/2} K_{1/3}\left(\frac{2\sqrt{\lambda}}{3^{3/2}}\sqrt{x}\right) dx. \end{aligned}$$

The R-measure is finite if $\alpha < 1/3$, since its density $\rho(x) \sim x^{(1/3-\alpha)-1}$ around zero. Under the assumption $2m - 2\alpha \pm 1/3 > -1$ (see [1], 11.4.22), the cumulants are then calculated as moments of the R-measure:

$$\begin{aligned} \mu_{R,m} &= \frac{\sqrt{\lambda}}{3\pi} \int_{\mathbf{R}_+} x^{m-\alpha-1/2} K_{1/3}\left(\frac{2\sqrt{\lambda}}{3^{3/2}}\sqrt{x}\right) dx \\ &= \frac{2\sqrt{\lambda}}{3\pi} \int_{\mathbf{R}_+} x^{2m-2\alpha} K_{1/3}\left(\frac{2\sqrt{\lambda}}{3^{3/2}} x\right) dx \end{aligned}$$

$$= \frac{2\sqrt{\lambda}}{3\pi}\left(\frac{2\sqrt{\lambda}}{3^{3/2}}\right)^{2\alpha-2m-1} \int_{\mathbf{R}_+} x^{2m-2\alpha} K_{1/3}(x)\,dx$$

$$= \frac{2\lambda^{\alpha-m}}{3\pi}\left(\frac{2}{3^{3/2}}\right)^{2\alpha-2m-1} 2^{2m-2\alpha-1}\,\Gamma\left(\frac{2m-2\alpha+4/3}{2}\right)\Gamma\left(\frac{2m-2\alpha+2/3}{2}\right)$$

$$= \frac{\lambda^{\alpha-m}\,3^{1/2-3\alpha+3m}}{2\pi}\Gamma\left(m-\alpha+\frac{2}{3}\right)\Gamma\left(m-\alpha+\frac{1}{3}\right).$$

The assumption implies that $\alpha-m < 1/3$, so α must be less than 1/3 for evaluation of the $\mu_{R,m}$, $0 \leqslant m < \max(0, \alpha-1/3)$.

For the cumulants of an arbitrary integer order m we have the formula

$$c_m = \Gamma(m-\alpha)\,\mu_{R,m} = 3\frac{\lambda^{\alpha-m}\,3^{3(m-\alpha)-1/2}}{2\pi}\Gamma(m-\alpha)\,\Gamma\left(m-\alpha+\frac{2}{3}\right)\Gamma\left(m-\alpha+\frac{1}{3}\right)$$

$$= 3\lambda^{\alpha-m}\,\Gamma(3\,[m-\alpha]),$$

unless $\alpha < 4/3$. If $\alpha > 4/3$, the first cumulant ($m = 1$) can be calculated directly from the Lévy measure:

$$c_1 = \int_{\mathbf{R}_+} xM(dx) = \int_{\mathbf{R}_+} r\exp\left(-(\lambda r)^{1/3}\right)\frac{dr}{r^{\alpha+1}}$$

$$= \int_{\mathbf{R}_+} r^{-\alpha}\exp\left(-(\lambda r)^{1/3}\right)dr = 3\lambda^{\alpha-1}\,\Gamma(3\,[1-\alpha]),$$

so that, for all integers m,

$$c_m = 3\lambda^{\alpha-m}\,\Gamma(3\,[m-\alpha]).$$

This is the case when (3) is satisfied but (10) is not.

4.6. Bessel tempering. The function

$$k(r) = \frac{1}{(\lambda r+1)^{\eta+1}}\exp\left(-\beta\frac{r}{r+1/\lambda}\right)$$

is the Laplace transform $k(r)$ of the Bessel density

$$q(x) = \lambda^{-1}\exp(-\beta-x/\lambda)\sqrt{\left(\frac{x}{\lambda\beta}\right)^{\eta}}\,I_\eta(2\sqrt{\beta x/\lambda}),$$

where λ, β, η are positive and I_η is the modified Bessel function of the first kind (see [5], Chapter XIII, Section 3[1]). The corresponding R-measure takes the

[1] See also http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html

form

$$R(B) = \int_{\mathbf{R}_+} \mathbf{1}_B\left(\frac{1}{x}\right) x^\alpha q(x)\,dx$$

$$= \lambda^{-1-\eta/2}\beta^{-\eta/2}e^{-\beta}\int_{\mathbf{R}_+} \mathbf{1}_B\left(\frac{1}{x}\right) x^{\alpha+\eta/2}\exp(-x/\lambda)\, I_\eta(2\sqrt{\beta x/\lambda})\,dx$$

$$= \lambda^{-1-\eta/2}\beta^{-\eta/2}e^{-\beta}\int_{\mathbf{R}_+} \mathbf{1}_B(x)\, x^{-\alpha-\eta/2-2}\exp(-1/[\lambda x])\, I_\eta(2\sqrt{\beta/[\lambda x]})\,dx.$$

The limiting behavior of I_η is known ([1], 9.6.7 and 9.7.1) as

$$I_\eta(x) \sim \begin{cases} (\frac{1}{2}x)^\eta/\Gamma(\eta+1) & \text{if } x \text{ is small,} \\ e^x/\sqrt{2\pi x} & \text{if } x \text{ is large.} \end{cases}$$

Hence the moments of order $m < \alpha+3\eta/2+1$ exist and the corresponding cumulants can be calculated by formula (14) (see [6], 6.643.2, for the expression in terms of the confluent hypergeometric functions).

5. TEMPERED STABLE ORNSTEIN–UHLENBECK PROCESSES

The well-known connection between self-decomposable distribution and Ornstein–Uhlenbeck processes is described in, e.g., [21] and [32]. Here, we consider the stationary Ornstein–Uhlenbeck process X_t given by the usual moving average

$$X_t = \int_{-\infty}^{t} e^{-\gamma(t-s)}\,dZ_s,$$

where $\gamma > 0$, and Z_t is usually called the *Background Driving Lévy Process* (BDLP) for X_t; see Kwapień and Woyczyński [11] for an exposition of the theory of stochastic integrals with respect to the general Lévy processes and semimartingales.

In this case the cumulant function κ_{X_t} of X_t is expressed in terms of the cumulant function κ_{Z_1} of Z_t as follows:

$$\kappa_{X_t}(y) = \int_{-\infty}^{t} \kappa_{Z_1}(e^{-\gamma(t-s)}y)\,ds. \tag{16}$$

Throughout this section we shall assume that the second-order moments of Z_s (and X_s) exist and that the processes are centered.

To study the finite-dimensional distributions and higher order spectra of X_t assume

$$\mathsf{E}|Z_1|^m < \infty,$$

and write

$$c_{Z,m} = \mathrm{Cum}_m(Z_1).$$

Then X_t is stationary of order m, i.e., for each $t, h_1, \ldots, h_{m-1}$,

$$\operatorname{Cum}(X_t, X_{t+h_1}, \ldots, X_{t+h_{m-1}}) = \operatorname{Cum}(X_0, X_{h_1}, \ldots, X_{h_{m-1}}).$$

The Fourier transform of the cumulants gives the m^{th} order spectrum S_m of X_t. In our case, S_m exists and is given by

$$S_m(\omega_1, \omega_2, \ldots, \omega_{m-1}) = \frac{c_{Z,m}}{\prod_{j=1}^{m}(\gamma - i\omega_j)},$$

where $\omega_m = \sum_{j=1}^{m-1} \omega_j$ (see [4] and [26]). In particular, if $m = 2$, then the spectrum is

$$S_2(\omega) = \frac{c_{Z,2}}{|\gamma - i\omega|^2}.$$

Notice that

$$S_2(\omega) = \frac{c_{Z,2}}{2\gamma} \int_{-\infty}^{\infty} e^{-i\omega h - \gamma|h|}\, dh,$$

and conclude that the covariance function of X_t is

$$C_X(h) = \operatorname{Cov}(X_t, X_{t+h}) = e^{-\gamma|h|} \frac{\operatorname{Var}(Z_1)}{2\gamma}.$$

In general, the symmetry of the cumulant implies that the support of the $\operatorname{Cum}(X_0, X_{h_1}, \ldots, X_{h_{m-1}})$ is the set $\{0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_{m-1}\}$, which directly leads to the result

$$\operatorname{Cum}(X_t, X_{t+h_1}, \ldots, X_{t+h_{m-1}}) = \exp\Big(-\gamma \sum_{j=1}^{m-1} h_j\Big) \frac{c_{Z,m}}{m\gamma},$$

where $0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_{m-1}$.

The finite-dimensional distributions of the stationary Ornstein–Uhlenbeck process X_t are also determined by the BDLP process Z_t and by γ. Indeed, the joint cumulant function of

$$(X_t, X_{t+h_1}, \ldots, X_{t+h_{m-1}})$$

is given by

$$\begin{aligned}\kappa_{t,t+h_1,\ldots,t+h_{m-1}}&(y_1, y_2, \ldots, y_m)\\ &= \kappa_{X_0}\Big(y_1 + \sum_{j=1}^{m-1} \exp(-\gamma h_j)\, y_{j+1}\Big) + \sum_{j=1}^{m-1} \kappa_{I(\Delta h_j)}\Big(\sum_{n=j}^{m-1} y_{n+1}\Big),\end{aligned}$$

where $0 \leqslant h_1 \leqslant h_2 \leqslant \ldots \leqslant h_{m-1}$, $\Delta h_j = h_{j+1} - h_j$, and

$$I(h) = \int_0^h e^{-\gamma s}\, dZ_s.$$

The cumulant functions of X_0 and $I(\Delta h_j)$ are obtained from the general formula (12). Hence

$$\kappa_{t,t+h_1,\ldots,t+h_{m-1}}(y_1, y_2, \ldots, y_m)$$
$$= \int_0^\infty \kappa_{Z_1}\Big(e^{-\gamma s}\Big[y_1+\sum_{j=1}^{m-1}\exp(-\gamma h_j)y_{j+1}\Big]\Big)ds+\sum_{j=1}^{m-1}\int_0^{\Delta h_j}\kappa_{Z_1}\Big(e^{-\gamma s}\sum_{n=j}^{m-1}y_{n+1}\Big)ds.$$

The easy consequence of this formula is that the distribution of X_0 and γ determine the finite-dimensional distributions of X_t. The basic example here is the Gaussian Ornstein–Uhlenbeck process: If either X_t or Z_t is Gaussian, then both Lévy measures M_X and M_Z are zero, $b = 0$, and

$$C_X(h) = e^{-\gamma h}\operatorname{Var}(Z_1)/2\gamma.$$

All higher order cumulants are zero.

5.1. BDLP for tempered stable $TS_\alpha(R, b)$. From now onwards we concentrate our attention on the Ornstein–Uhlenbeck processes for which either X_t, or its BDLP Z_t, is a Gaussian-free, and centered tempered stable process.

First, consider the case when X_t is a $TS_\alpha(R, 0)$ process with the cumulant function

$$\kappa_X(y) = \int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle)R_X(dx),$$

where ψ_α is given by (9). The general formula (17) makes it possible to express the cumulant function κ_Z of the BDLP in terms of measure R_X. A similar result in terms of Lévy densities has been obtained in [3].

LEMMA 2. *Let X_t be a $TS_\alpha(R, 0)$ process with the cumulant function*

$$\kappa_X(y) = \int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle)R_X(dx),$$

where ψ_α is given by (9). *Then, for any $0 < \alpha < 2$, the cumulant function κ_Z of the associated BDLP is given by the formula*

$$\kappa_Z(y) = \gamma \int_{\mathbf{R}_0^d} \xi_\alpha(\langle y, x\rangle)R_X(dx), \tag{17}$$

where

$$\xi_\alpha(r) = r\frac{d\psi_\alpha(r)}{dr}.$$

Moreover, if the R-measure of X has a differentiable density ρ_X such that the gradient

$$\nabla_x[x\rho_X(x)] = \left[\frac{\partial}{\partial x_1}x\rho_X(x), \frac{\partial}{\partial x_2}x\rho_X(x), \ldots, \frac{\partial}{\partial x_d}x\rho_X(x)\right]$$

is continuous at zero, then the BDLP is a tempered stable process with

$$\kappa_Z(y) = \int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle)\rho_Z(x)\,dx,$$

where the R-density ρ_Z is given by

$$\rho_Z(x) = -\gamma \operatorname{Tr} \nabla_x [\rho_X(x) x] = -\gamma [d\rho_X(x) + [\nabla_x \rho_X(x)] x]. \tag{18}$$

In the one-dimensional case, $d = 1$, we have

$$\rho_Z(x) = -\gamma\left[\rho_X(x) + x\frac{d\rho_X(x)}{dx}\right] \quad \text{and} \quad \kappa_Z(y) = \int_{\mathbf{R}_0} \psi_\alpha(yx)\rho_Z(x)\,dx.$$

Proof. In the polar representation the Lévy measure of X is

$$M(B) = \int_{S^{d-1}} \sigma(du) \int_0^\infty \mathbf{1}_B(ru)\,k(r\,|\,u)\frac{dr}{r^{\alpha+1}},$$

and the Lévy measure of the BDLP takes the form

$$\begin{aligned} N(B) &= -\gamma \int_{S^{d-1}} \sigma(du) \int_0^\infty \mathbf{1}_B(ru)\,d\left[\frac{k(r\,|\,u)}{r^\alpha}\right] \\ &= -\gamma \int_{S^{d-1}} \sigma(du) \int_0^\infty \mathbf{1}_B(ru)\left(r\frac{\partial k(r\,|\,u)}{\partial r} - \alpha k(r\,|\,u)\right)\frac{dr}{r^{\alpha+1}}. \end{aligned} \tag{19}$$

Now, the second term in the last integral

$$\gamma \int_{S^{d-1}} \sigma(du) \int_0^\infty \mathbf{1}_B(ru)\,\alpha k(r\,|\,u)\frac{dr}{r^{\alpha+1}}$$

provides the representation (8). The representation of the first term containing $\partial k(r\,|\,u)/\partial r$ needs some extra work. We have

$$\frac{\partial k(r\,|\,u)}{\partial r} = -\int_0^\infty s e^{-rs} Q(ds\,|\,u),$$

so that

$$\begin{aligned} &\int_{S^{d-1}} \sigma(du) \int_0^\infty \mathbf{1}_B(ru)\,r\frac{\partial k(r\,|\,u)}{\partial r}\frac{dr}{r^{\alpha+1}} \\ &= -\int_{S^{d-1}} \sigma(du) \int_0^\infty \mathbf{1}_B(ru)\,r \int_0^\infty s e^{-rs} Q(ds\,|\,u)\frac{dr}{r^{\alpha+1}} \\ &= -\int_{S^{d-1}} \sigma(du) \int_0^\infty \left(\int_0^\infty \mathbf{1}_B\left(\frac{t}{s}u\right) e^{-t} t^{-\alpha}\,dt\right) s^\alpha Q(ds\,|\,u) \\ &= -\int_0^\infty \int_{\mathbf{R}_0} \mathbf{1}_B\left(t\frac{x}{\|x\|^2}\right)\|x\|^\alpha Q(dx)\,e^{-t} t^{-\alpha}\,dt = -\int_{\mathbf{R}_0} \int_0^\infty \mathbf{1}_B(tx)\,e^{-t} t^{-\alpha}\,dt R(dx). \end{aligned}$$

Proceeding in the footsteps of Rosiński's result [19], we notice that, for $0 < \alpha < 1$,

$$\alpha\psi_\alpha(r) + \psi_{\alpha-1}(r) = ir\Gamma(1-\alpha)(1-ir)^{\alpha-1},$$

and, for $1 < \alpha < 2$,

$$\alpha\psi_\alpha(r)+\psi_{\alpha-1}(r) = ir\Gamma(1-\alpha)[(1-ir)^{\alpha-1}-1].$$

Hence, in both cases we obtain

$$\alpha\psi_\alpha(r)+\psi_{\alpha-1}(r) = r\frac{d\psi_\alpha(r)}{dr}.$$

Note that the case $\alpha-1 < 0$ does not create any difficulties here. Let us put

$$\xi_\alpha(s) = s\frac{d\psi_\alpha(s)}{ds},$$

and observe that $\xi_\alpha(s)$ has the limit zero at zero. Utilizing (8) we obtain the cumulant function

$$\kappa_Z(y) = \gamma \int_{\boldsymbol{R}_0^d} \alpha\psi_\alpha(\langle y, x\rangle)+\psi_{\alpha-1}(\langle y, x\rangle) R(dx) = \gamma \int_{\boldsymbol{R}_0^d} \xi_\alpha(\langle y, x\rangle) R(dx).$$

In the one-dimensional case, $d = 1$,

$$\kappa_Z(y) = \gamma \int_{\boldsymbol{R}_0} \xi_\alpha(yx)\rho_X(x)\,dx = \gamma \int_{\boldsymbol{R}_0} \frac{d\psi_\alpha(r)}{dr}\bigg|_{r=ys} ys\rho_X(s)\,ds,$$

and splitting the domain of integration into $(-\infty, 0)$ and $(0, \infty)$, and integrating by parts,

$$\int_0^\infty \frac{d\psi_\alpha(ys)}{ds} s\rho_Z(s)\,ds = -\int_0^\infty \psi_\alpha(ys)\frac{d}{ds}[s\rho_X(s)]\,ds,$$

we obtain

$$\frac{d}{dx}x\rho_X(x) = \rho_X(x)+x\frac{d\rho_X(x)}{dx}.$$

Therefore, putting

$$\rho_Z(x) = -\gamma\left[\rho_X(x)+x\frac{d\rho_X(x)}{dx}\right],$$

we see that the density ρ_Z of the R-measure of Z satisfies the equation

$$\kappa_Z(y) = \int_{\boldsymbol{R}_0} \psi_\alpha(yx)\rho_Z(x)\,dx.$$

Now, we prove the case $d = 2$; in the general case $d > 2$ the argument is almost the same, the only difference being that the factor r^{d-1} has to be replaced by the Jacobian. So, we have

$$\kappa_Z(y) = -\gamma \int_{\boldsymbol{R}_0^d} \xi_\alpha(\langle y, x\rangle) R(dx) = -\gamma \int_{\boldsymbol{R}_0^d} \xi_\alpha(\langle y, x\rangle)\rho_X(x)\,dx.$$

Fix a $y \neq 0$, and perform a rotation so that the first axis points in the direction of y, and change the integral to the polar coordinate system:

$$\int_{\mathbf{R}_0^2} \xi_\alpha(\langle y, x\rangle)\rho_X(x)\,dx$$

$$= \int_0^\infty \int_0^{2\pi} \xi_\alpha(\|y\| r\cos\omega)\rho_X(r\cos\omega, r\sin\omega)\,r\,dr\,d\omega$$

$$= \int_0^\infty \int_0^{2\pi} \|y\| r\cos\omega \left.\frac{d\psi_\alpha(s)}{ds}\right|_{s=\|y\|r\cos\omega} \rho_X(r\cos\omega, r\sin\omega)\,r\,dr\,d\omega$$

$$= \int_0^\infty \int_0^{2\pi} \frac{d\psi_\alpha(\|y\| r\cos\omega)}{dr}\rho_X(r\cos\omega, r\sin\omega)\,r^2\,dr\,d\omega.$$

Integrating by parts we obtain

$$-\int_0^\infty \int_0^{2\pi} \frac{d\psi_\alpha(\|y\| r\cos\omega)}{dr}\rho_X(r\cos\omega, r\sin\omega)\,r^2\,dr\,d\omega$$

$$= \int_0^\infty \int_0^{2\pi} \psi_\alpha(\|y\| r\cos\omega)\Big[2\rho_X(r\cos\omega, r\sin\omega)$$

$$+ r\cos\omega\frac{\partial}{\partial x_1}\rho_X(r\cos\omega, r\sin\omega) + r\sin\omega\frac{\partial}{\partial x_2}\rho_X(r\cos\omega, r\sin\omega)\Big]\,r\,dr\,d\omega$$

$$= \int_{\mathbf{R}_0^2} \psi_\alpha(\langle y, x\rangle)\{2\rho_X(x) + [\nabla_x\rho_X(x)]\,x\}\,dx,$$

and (18) follows. ■

If the R-density of X is differentiable, then the calculation of κ_Z is straightforward; otherwise we have to use the formula (17).

Example 1. Let X be $STLF_{1/2}(a, 0, \lambda)$ with the cumulant function

$$\kappa_X(u) = -2a\sqrt{2\pi\lambda}\,[\sqrt{1-iu/\lambda}-1];$$

see Section 3.3. Then the cumulant function of the BDLP Z is

$$\kappa_Z(u) = ia\gamma u(1-iu/\lambda)^{-1/2}.$$

5.2. Stationary multivariate Ornstein–Uhlenbeck process with tempered stable BDLP. Consider now a stationary Ornstein–Uhlenbeck process X_t with parameter γ and BDLP Z_t, where $Z_1 \in TS_\alpha(R_Z, 0)$. The cumulant function κ_{Z_1} of the BDLP is given in terms of the R-measure:

$$\kappa_{Z_1}(y) = \int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle)\,R_Z(dx);$$

see (9) for the definition of the function ψ_α. For each t, the cumulant function of X_t can be written in the form

$$\kappa_{X_t}(y) = \int_{-\infty}^{t} \kappa_{Z_1}(e^{-\gamma(t-s)} y)\,ds = \int_{-\infty}^{t} \int_{\mathbf{R}_0^d} \psi_\alpha(\langle e^{-\gamma(t-s)} y, x\rangle) R_Z(dx)\,ds$$

$$= \int_{\mathbf{R}_0^d} \int_0^\infty \psi_\alpha(\langle e^{-\gamma s} y, x\rangle)\,ds R_Z(dx).$$

Now, formula (11), and property (iii) in Subsection 3.2 imply

$$\kappa_{X_t}(y) = \int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle) R_X(dx)$$

with the R-measure

$$R_X(B) = \int_0^\infty R_Z(e^{\gamma s} B)\,ds = \frac{1}{\gamma}\int_1^\infty R_Z(sB)\frac{ds}{s}.$$

Hence X_t is tempered stable. If $d = 1$, it follows that the R-density $\rho_X(x)$ exists and

$$\rho_X(x) = \begin{cases} (\gamma x)^{-1} R_Z([x, \infty)), & x > 0, \\ (\gamma |x|)^{-1} R_Z((-\infty, x]), & x < 0. \end{cases}$$

Hence $\rho_X(x)$ fulfils the equation

$$\rho_Z(x) = -\gamma \frac{d}{dx}[x\rho_X(x)], \tag{20}$$

as long as ρ_X is differentiable and $\frac{d}{dx}[x\rho_X(x)]$ is continuous at zero.

EXAMPLE 2. Let Z_1 be an STLF with the R-measure

$$R_Z = \lambda^\alpha a(p_1 \delta_{-1/\lambda} + p_2 \delta_{1/\lambda}).$$

Here (20) does not apply, and we use (11) instead. The R-measure of the stationary Ornstein–Uhlenbeck process X_t with parameter γ is

$$R_X(B) = \frac{1}{\gamma}\int_{\mathbf{R}_0}\int_0^1 \mathbf{1}_B(sx)\frac{ds}{s} R_Z(dx) = \frac{\lambda^\alpha a}{\gamma}\int_0^1 p_1 \mathbf{1}_B\left(-\frac{s}{\lambda}\right) + p_2 \mathbf{1}_B\left(\frac{s}{\lambda}\right)\frac{ds}{s}$$

$$= \frac{\lambda^{\alpha+1} a}{\gamma}\int_0^\lambda p_1 \mathbf{1}_B(-s) + p_2 \mathbf{1}_B(s)\frac{ds}{s}.$$

Hence the R-density of X_t is

$$\rho_X(x) = \frac{\lambda^{\alpha+1} a}{\gamma x}(p_2 \mathbf{1}_{0<x\leqslant\lambda} - p_1 \mathbf{1}_{-\lambda\leqslant x<0}).$$

Below, we generalize the above result for a stationary multivariate Ornstein–Uhlenbeck process X_t. Let X_t be given by the stochastic differential equation system

$$dX_t = -QX_t\,dt + dZ_t, \tag{21}$$

where Q is a real $d \times d$ matrix such that the real parts of all eigenvalues are positive ($Q \in \mathcal{M}_+(\mathbf{R}^d)$). Then it is well known (see [22], [24], [23]) that there exists a stationary solution X_t of the equation (21), which is given by

$$X_t = \int_{-\infty}^{t} e^{-Q(t-s)}\,dZ_s,$$

with the cumulant function

$$\kappa_X(u) = \int_0^\infty \kappa_{Z_1}(\exp(-sQ^\top)u)\,ds, \tag{22}$$

i.e. the cumulant function of X_t is determined by that of Z_1. This result should be compared with those of [2] as well.

THEOREM 1. *Suppose $Z_1 \in TS_\alpha(R_Z, 0)$; then the multivariate stationary Ornstein–Uhlenbeck process X_t, with matrix $Q \in \mathcal{M}_+(\mathbf{R}^d)$, and BDLP Z_t, is tempered stable. The R-measure of X_t is*

$$R_X(B) = \int_0^\infty R_Z(e^{Qs}B)\,ds.$$

The cumulant function takes the form

$$\kappa_X(y) = \int_{\mathbf{R}_0^d} \eta_\alpha(y, x, Q)\,R_Z(dx),$$

where $\eta_\alpha(y, x, Q)$ is given by the equation

$$\eta_\alpha(y, x, Q) = \int_0^\infty \psi_\alpha(\langle \exp(-sQ^\top)y, x\rangle)\,ds.$$

For each $x, y \in \mathbf{R}_0^d$, $\eta_\alpha(y, x, Q)$ fulfils the equation

$$\psi_\alpha(\langle y, x\rangle) = [\nabla_y \eta_\alpha(y, x, Q)]\,Q^\top y.$$

Moreover, if the R-measure of Z has density ρ_Z, and the differential equation

$$\rho_Z(y) = -\operatorname{Tr}\nabla_y[\rho_X(y)Q^\top y] = -\rho_X(y)\operatorname{Tr}Q - [\nabla_y \rho_X(y)]\,Q^\top y$$

has a solution ρ_X such that $\nabla_x[\rho_X(x)x]$ is continuous at zero, then X_t has R-density ρ_X. The cumulants of X_t are given in terms of the cumulants of Z_1 by the formula

$$\underline{\mathrm{Cum}}_m(\underline{X}_t) = \sum_{k=1}^{m} \left(I^{\otimes(k-1)} \otimes (Q^\top)^{-1} \otimes I^{\otimes(m-k)}\right)\underline{\mathrm{Cum}}_m(\underline{Z}_1),$$

as long as $\underline{\mathrm{Cum}}_m(\underline{Z}_1)$ exists.

Proof. Start with the equation (22)

$$\kappa_X(y) = \int_0^\infty \kappa_{Z_1}(\exp(-sQ^\mathsf{T})y)\,ds = \int_0^\infty \int_{\mathbf{R}_0^d} \psi_\alpha(\langle \exp(-sQ^\mathsf{T})y, x\rangle) R_Z(dx)\,ds$$

$$= \int_{\mathbf{R}_0^d} \int_0^\infty \psi_\alpha(x^\mathsf{T} \exp(-sQ^\mathsf{T})y)\,ds R_Z(dx).$$

Let us set

$$\eta_\alpha(y, x) = \int_0^\infty \psi_\alpha(x^\mathsf{T} \exp(-sQ^\mathsf{T})y)\,ds,$$

and apply the gradient operator ∇_y

$$\nabla_y \eta_\alpha(y, x) = \int_0^\infty \frac{d\psi_\alpha(r)}{dr}\bigg|_{r = x^\mathsf{T}\exp(-sQ^\mathsf{T})y} x^\mathsf{T} \exp(-sQ^\mathsf{T})\,ds.$$

Now, let us differentiate $\psi_\alpha(x^\mathsf{T} e^{-Qs} y)$ with respect to s,

$$\frac{d\psi_\alpha(x^\mathsf{T} \exp(-sQ^\mathsf{T})y)}{ds} = -\frac{d\psi_\alpha(r)}{dr}\bigg|_{r = x^\mathsf{T}\exp(-sQ^\mathsf{T})y} x^\mathsf{T} \exp(-sQ^\mathsf{T})\, Q^\mathsf{T} y,$$

and integrate,

$$\psi_\alpha(\langle y, x\rangle) = \int_0^\infty \frac{d\psi_\alpha(r)}{dr}\bigg|_{r = x^\mathsf{T}\exp(-sQ^\mathsf{T})y} x^\mathsf{T} \exp(-sQ^\mathsf{T})\,ds\, Q^\mathsf{T} y = [\nabla_y \eta_\alpha(y, x)]\, Q^\mathsf{T} y,$$

assuming $Q^\mathsf{T} u \neq 0$. Let us consider

$$\int_{\mathbf{R}_0^d} \eta_\alpha(y, x) \operatorname{Tr} \nabla_y [\rho_X(y) Q^\mathsf{T} y]\,dy = -\operatorname{Tr} \int_{\mathbf{R}_0^d} [\nabla_y \eta_\alpha(y, x)]\, Q^\mathsf{T} y \rho_X(y)\,dy$$

$$= -\int_{\mathbf{R}_0^d} \psi_\alpha(\langle y, x\rangle) \rho_X(y)\,dy.$$

Now we conclude that

$$\rho_Z(y) = -\operatorname{Tr} \nabla_y [\rho_X(y) Q^\mathsf{T} y] = -\rho_X(y) \operatorname{Tr} Q - [\nabla_y \rho_X(y)]\, Q^\mathsf{T} y.$$

Note that the multivariate stationary Ornstein–Uhlenbeck process X_t is Q-decomposable; see Masuda [15] for details. The cumulants for X_t are (see the Appendix for the relevant definitions)

$$D_u^{\otimes k} \kappa_X(y) = \int_{\mathbf{R}_0^d} D_u^{\otimes k} \eta_\alpha(y, x) R(dx),$$

$$D_u^{\otimes} \kappa_X(y) = \int_{\mathbf{R}_0^d} \int_0^\infty D_u^{\otimes} \psi_\alpha(x^\mathsf{T} \exp(-sQ^\mathsf{T})y)\,ds R(dx)$$

$$= \int_{\mathbf{R}_0^d} \int_0^\infty \frac{d\psi_\alpha(r)}{dr}\bigg|_{r = x^\mathsf{T}\exp(-Qs)y} \exp(-sQ^\mathsf{T})\,ds\, x R(dx).$$

Putting $y=0$ we obtain

$$\mathsf{E}X_t = \int_0^\infty \exp(-sQ^{\mathsf{T}})\,ds\,\mathsf{E}Z_1 = (Q^{\mathsf{T}})^{-1}\,\mathsf{E}Z_1.$$

Now

$$\begin{aligned}
D_{\boldsymbol{u}}^{\otimes 2}\kappa_X(y) &= \int_{\boldsymbol{R}_0^d}\int_0^\infty D_{\boldsymbol{u}}^{\otimes 2}\psi_\alpha\big(x^{\mathsf{T}}\exp(-sQ^{\mathsf{T}})\,y\big)\,ds R(dx)\\
&= \int_{\boldsymbol{R}_0^d}\int_0^\infty D_{\boldsymbol{u}}^{\otimes}\left[\exp(-sQ^{\mathsf{T}})\,x\left.\frac{d\psi_\alpha(r)}{dr}\right|_{r=x^{\mathsf{T}}\exp(-Qs)y}\right] ds R(dx)\\
&= \int_{\boldsymbol{R}_0^d}\int_0^\infty [\exp(-sQ^{\mathsf{T}})\,x\otimes I]\left.\frac{d^2\psi_\alpha(r)}{dr^2}\right|_{r=x^{\mathsf{T}}\exp(-Qs)y}\exp(-sQ^{\mathsf{T}})\,x\,ds R(dx)\\
&= \int_{\boldsymbol{R}_0^d}\int_0^\infty [\exp(-sQ^{\mathsf{T}})\otimes\exp(-sQ^{\mathsf{T}})]\left.\frac{d^2\psi_\alpha(r)}{dr^2}\right|_{r=x^{\mathsf{T}}\exp(-Qs)y} x^{\otimes 2}\,ds R(dx).
\end{aligned}$$

Again, putting $u=0$, we obtain

$$\underline{\mathrm{Cum}}_2(\underline{X}_t) = [(Q^{\mathsf{T}})^{-1}\otimes I + I\otimes(Q^{\mathsf{T}})^{-1}]\,\underline{\mathrm{Cum}}_2(\underline{Z}_1),$$

since

$$\int_0^\infty [\exp(-sQ^{\mathsf{T}})\otimes\exp(-sQ^{\mathsf{T}})]\,ds = (Q^{\mathsf{T}})^{-1}\otimes I + I\otimes(Q^{\mathsf{T}})^{-1}.$$

In the general case $\underline{\mathrm{Cum}}_m(\underline{X}_t)$ is then obtained by a standard induction. ∎

6. APPENDIX. MULTIPLE CUMULANTS

Introduce the notation $\partial/\partial\boldsymbol{u}^{\mathsf{T}} = \nabla_{\boldsymbol{u}}$. The operator $D_{\boldsymbol{u}}^{\otimes}$ is defined as

$$D_{\boldsymbol{u}}^{\otimes}\boldsymbol{\phi} = \mathrm{Vec}\left(\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}^{\mathsf{T}}}\right)^{\mathsf{T}} = \mathrm{Vec}\begin{bmatrix}\frac{\partial\phi_1}{\partial u_1} & \frac{\partial\phi_1}{\partial u_2} & \cdots & \frac{\partial\phi_1}{\partial u_d}\\ \frac{\partial\phi_2}{\partial u_1} & \ddots & & \vdots\\ \vdots & & \ddots & \vdots\\ \frac{\partial\phi_m}{\partial u_1} & \cdots\cdots & & \frac{\partial\phi_m}{\partial u_d}\end{bmatrix}^{\mathsf{T}},$$

which is a column vector of order md. We can also write $D_{\boldsymbol{u}}^{\otimes}$ in terms of the Kronecker product:

$$\begin{aligned}
D_{\boldsymbol{u}}^{\otimes}\boldsymbol{\phi} &= \mathrm{Vec}\left(\boldsymbol{\phi}\frac{\partial}{\partial\boldsymbol{u}^{\mathsf{T}}}\right)^{\mathsf{T}} = \mathrm{Vec}\left(\frac{\partial}{\partial\boldsymbol{u}}\boldsymbol{\phi}^{\mathsf{T}}\right)\\
&= [\phi_1(\boldsymbol{u}),\ \phi_2(\boldsymbol{u}),\ \ldots,\ \phi_m(\boldsymbol{u})]^{\mathsf{T}}\otimes\left[\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2},\ldots,\frac{\partial}{\partial u_d}\right]^{\mathsf{T}}.
\end{aligned}$$

If we repeat the differentiation $D_{\boldsymbol{u}}^{\otimes}$ twice, we obtain

$$D_{\boldsymbol{u}}^{\otimes 2}\boldsymbol{\phi} = D_{\boldsymbol{u}}^{\otimes}(D_{\boldsymbol{u}}^{\otimes}\boldsymbol{\phi}) = \operatorname{Vec}\left[\left(\boldsymbol{\phi}\otimes\frac{\partial}{\partial\boldsymbol{u}}\right)\frac{\partial}{\partial\boldsymbol{u}^{\mathsf{T}}}\right]^{\mathsf{T}}$$

$$= \boldsymbol{\phi}\otimes\left(\frac{\partial}{\partial\boldsymbol{u}}\right)^{\otimes 2} = \boldsymbol{\phi}\otimes\frac{\partial}{\partial\boldsymbol{u}^{\otimes 2}},$$

and, in general (assuming the existence of derivatives of order k), the k^{th} derivative is given by

$$D_{\boldsymbol{u}}^{\otimes k}\boldsymbol{\phi} = D_{\boldsymbol{u}}^{\otimes}(D_{\boldsymbol{u}}^{\otimes k-1}\boldsymbol{\phi}) = [\phi_1(\boldsymbol{u}), \phi_2(\boldsymbol{u}), \ldots, \phi_m(\boldsymbol{u})]^{\mathsf{T}}\otimes\left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_d}\right]^{\mathsf{T}\otimes k}.$$

Let $\varphi(y)$ be the characteristic function of the random variable $\underline{X}$. Then the multiple moment of order k is defined by

$$\mu_{\otimes j} = (-i)^k D_y^{\otimes k}\varphi(y)|_{y=0},$$

and the multiple k^{th}-order cumulant is

$$c_k = \underline{\mathrm{Cum}}_k(\underline{X}) = (-i)^k D_y^{\otimes k}\log\varphi(y)|_{y=0},$$

see [27] for details. Note that

$$\underline{\mathrm{Cum}}_2(\underline{X}_1, \underline{X}_2) = \mathsf{E}[(\underline{X}_1 - \mathsf{E}\underline{X}_1)\otimes(\underline{X}_2 - \mathsf{E}\underline{X}_2)] = \operatorname{Vec}\operatorname{Cov}(\underline{X}_2, \underline{X}_1). \tag{23}$$

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