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ON A PARTICULAR CLASS OF SELF-DECOMPOSABLE RANDOM VARIABLES: THE DURATIONS OF BESSEL EXCURSIONS STRADDLING INDEPENDENT EXPONENTIAL TIMES

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Abstract. The distributional properties of the duration of a recurrent Bessel process straddling an independent exponential time are studied in detail. Although our study may be considered as a particular case of Winkel's in [25], the infinite divisibility structure of these Bessel durations is particularly rich and we develop algebraic properties for a family of random variables arising from the Lévy measures of these durations.

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1. INTRODUCTION

1.1. The excursion durations Δ_α $(0 < \alpha < 1)$ of Bessel processes. Let $((R_t, t \geqslant 0), P^{(\alpha)})$ denote a Bessel process starting from 0, with dimension $d = 2(1-\alpha)$, $0 < d < 2$ (or $0 < \alpha < 1$). For any $t \geqslant 0$ let us define

$$(1.1) \qquad g_t^{(\alpha)} := \sup\{s \leqslant t;\, R_s = 0\} \quad \text{and} \quad d_t^{(\alpha)} := \inf\{s \geqslant t;\, R_s = 0\}$$

so that $\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)}$ is the length of the excursion above 0, straddling t, for the process $(R_u, u \geqslant 0)$.

We denote by $\mathfrak{e}$ a standard exponential variable, independent of $(R_u, u \geqslant 0)$. In a recent work, Fujita and Yor [12] studied the laws of

$$(1.2) \qquad \sup_{s \leqslant g_{\mathfrak{e}}^{(\alpha)}} R_s, \quad \sup_{s \leqslant \mathfrak{e}} R_s, \quad \sup_{s \leqslant d_{\mathfrak{e}}^{(\alpha)}} R_s.$$

Here, in a similar way, but focussing on durations rather than on heights, we shall study exhaustively the law of

$$\Delta_\alpha := \Delta_e^{(\alpha)} = d_e^{(\alpha)} - g_e^{(\alpha)}. \tag{1.3}$$

In a first step we compute the density f_{Δ_α} of Δ_α:

$$f_{\Delta_\alpha}(x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)}(1-e^{-x})\,1_{(x \geqslant 0)}, \tag{1.4}$$

and we prove that

$$E[\exp(-\lambda \Delta_\alpha)] = (1+\lambda)^\alpha - \lambda^\alpha \quad (\lambda \geqslant 0). \tag{1.5}$$

Note. We hope to devote another paper to the study of the remarkable properties of the subordinator $(\Delta_{1/2}(t),\, t \geqslant 0)$ whose value at time 1 is $\Delta_{1/2}$.

1.2. A general result by Winkel [25]. In fact, formulae (1.4) and (1.5) are a very particular case of a general result by Winkel [25], which we now describe.

Let $(\tau_l,\, l \geqslant 0)$ denote a subordinator with associated Bernstein function Φ, i.e.

$$E[\exp(-\lambda \tau_l)] = \exp(-l\Phi(\lambda)) \quad (\lambda,\, l \geqslant 0).$$

We define, for any $t \geqslant 0$,

$$L_t = \inf\{l : \tau_l > t\}, \tag{1.6}$$

$$O_t = \tau_{(L_t)} - t \text{ (the overshoot)}, \quad U_t = t - \tau_{(L_t)^-} \text{ (the undershoot)}, \tag{1.7}$$

and

$$\Delta_t = \tau_{(L_t)} - \tau_{(L_t)^-} = O_t + U_t. \tag{1.8}$$

For e, an independent standard exponential variable, Winkel computes (see Corollary 1 in [25]) the Laplace transform of the 7-tuple:

$$(\mathrm{e},\, L_\mathrm{e},\, U_\mathrm{e},\, O_\mathrm{e},\, \tau_{L_\mathrm{e}^-},\, \tau_{L_\mathrm{e}},\, \Delta_\mathrm{e}).$$

As a very partial result of this multidimensional formula, he obtains

$$E[\exp(-\lambda \Delta_\mathrm{e})] = \frac{\Phi(1+\lambda) - \Phi(\lambda)}{\Phi(1)} \quad (\lambda \geqslant 0). \tag{1.9}$$

Hence, formula (1.5) is formula (1.9) applied to the subordinator $(\tau_l,\, l \geqslant 0)$ defined as

$$\tau_l = \inf\{t \geqslant 0 :\ L_t > l\},$$

where L_t denotes the local time at 0 for the Bessel process $(R_t,\, t \geqslant 0)$, i.e. $(\tau_l,\, l \geqslant 0)$ is a stable subordinator with index α. We note that from (1.9) we easily deduce the law of Δ_e:

$$(1.10)\qquad P(\Delta_e \in dx) = \frac{1-e^{-x}}{\Phi(1)}\nu(dx) + \frac{c}{\Phi(1)}\delta_0(dx),$$

where ν denotes the Lévy measure of the subordinator $(\tau_l,\ l \geqslant 0)$ which admits c as its translation coefficient. There again formula (1.4) is a particular case of (1.10) since the Lévy measure of the stable subordinator with index α is equal up to a multiplicative constant to $(dx/x^{\alpha+1})\,1_{(x>0)}$.

To summarize, the formulae (1.4) and (1.5) are doubly particular cases of the results of Winkel [25], since:

– here, the subordinator $(\tau_l,\ l \geqslant 0)$ is a particular one, namely the α-stable subordinator;

– our formula only discusses the law of the r.v. Δ_e, and not that of the 7-tuple

$$(e,\ L_e,\ U_e,\ O_e,\ \tau_{(L_e)^-},\ \tau_{L_e},\ \Delta_e).$$

1.3. The self-decomposability of the variable Δ_α $(0 < \alpha < 1)$. Recall that a random variable Δ is said to be *self-decomposable* if, for any $c \in]0, 1[$, there exists another variable $\Delta^{(c)}$ such that

$$(1.11)\qquad \Delta \overset{\text{(law)}}{=} c\Delta + \Delta^{(c)},$$

where Δ and $\Delta^{(c)}$ on the right-hand side are assumed independent. The class of self-decomposable laws (or variables) is a subclass of infinitely divisible laws; see, e.g., Sato [21].

In order to state our main result about the variable Δ_α, we need the following definition:

Let $a > 0$ and let K be a positive r.v. We shall say that $(Y_t,\ t \geqslant 0)$ is an (a, K) *compound Poisson process* (valued in $\boldsymbol{R}_+$) if

$$(1.12)\qquad Y_t := \sum_{i=1}^{N_t} K_i,$$

where $(K_1, K_2, \ldots)$ is a sequence of i.i.d. variables, distributed as K, and with $(N_t,\ t \geqslant 0)$ a Poisson process with parameter a independent of the sequence $(K_i,\ i = 1, 2, \ldots)$. In particular, N_t is a Poisson variable with parameter at.

THEOREM 1.1. *For any $\alpha \in]0, 1[$, we have:*

Point 1:

$$(1.13)\ \text{(i)}\qquad \Delta_\alpha \overset{\text{(law)}}{=} \frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}} \overset{\text{(law)}}{=} \frac{\gamma_{(1-\alpha)}}{U^{1/\alpha}},$$

where $\gamma_{(1-\alpha)}$ and $\beta_{(\alpha,1)}$ on the right-hand side are two independent r.v.'s with respective laws gamma $(1-\alpha)$ and beta $(\alpha, 1)$, and U denotes a uniform variable on $[0, 1]$, independent of $\gamma_{(1-\alpha)}$.

(ii) *The density of* Δ_α, *denoted here by* f_{Δ_α}, *is given by*

$$f_{\Delta_\alpha}(x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}(1-e^{-x})\, 1_{(x \geqslant 0)}. \tag{1.14}$$

(iii) *The Laplace transform of (the law of)* Δ_α *is*

$$E[\exp(-\lambda \Delta_\alpha)] = (1+\lambda)^\alpha - \lambda^\alpha \quad (\lambda \geqslant 0). \tag{1.15}$$

Point 2. (i) Δ_α *is self-decomposable, and the Lévy–Khintchine formula takes the form*

$$E[\exp(-\lambda \Delta_\alpha)] = \exp\left(-(1-\alpha)\int_0^\infty (1-e^{-\lambda x}) E[\exp(-xG_\alpha)] \frac{dx}{x}\right), \tag{1.16}$$

where G_α *denotes an r.v. with values in* $[0, 1]$, *and density*

$$f_{G_\alpha}(u) = \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha} - 2(1-u)^\alpha u^\alpha \cos(\pi\alpha) + u^{2\alpha}} 1_{[0,1]}(u). \tag{1.17}$$

(ii) *The r.v.* G_α *is characterized by its Stieltjes transform*

$$\begin{aligned} S(f_{G_\alpha})(\lambda) &:= \int_0^1 \frac{f_{G_\alpha}(u)}{\lambda+u} du = E\left(\frac{1}{\lambda+G_\alpha}\right) \\ &= \frac{\alpha}{1-\alpha} \frac{\lambda^{\alpha-1} - (1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - \lambda^\alpha} \quad (\lambda \geqslant 0) \end{aligned} \tag{1.18}$$

or, equivalently, by

$$E[\exp(-\lambda \mathrm{e} G_\alpha)] = E\left(\frac{1}{1+\lambda G_\alpha}\right) = \frac{\alpha}{1-\alpha} \frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \quad (\lambda \geqslant 0). \tag{1.19}$$

Point 3. *Define (the law of) the r.v.*

$$K_\alpha \overset{(law)}{=} \mathrm{e}/G_\alpha, \tag{1.20}$$

where e *and* G_α *on the right-hand side are assumed independent. In particular,*

$$P(K_\alpha \geqslant x) = P\left(\frac{\mathrm{e}}{G_\alpha} \geqslant x\right) = P(\mathrm{e} \geqslant xG_\alpha) = E[\exp(-xG_\alpha)].$$

(i) *There exists a* $(1-\alpha, K_\alpha)$ *positive compound Poisson process* $(Y_t, t \geqslant 0)$ *such that*

$$\Delta_\alpha \overset{(law)}{=} \int_0^\infty e^{-t} dY_t. \tag{1.21}$$

(ii) Δ_α *satisfies the affine equation*

$$\Delta_\alpha \overset{(law)}{=} U^{1/(1-\alpha)}(\Delta_\alpha + K_\alpha), \tag{1.22}$$

where U, Δ_α and K_α on the right-hand side are assumed independent, and U is uniformly distributed on $[0, 1]$.

We note that decompositions such as (1.22), and below (1.69), were also studied in Jurek [14].

1.4. Some properties of the r.v.'s G_α $(0 \leqslant \alpha \leqslant 1)$. Recall that, for any $\alpha \in]0, 1[$, the r.v. G_α is defined either via its density (1.17) or via its Stieltjes transform (1.18) (or (1.19)).

THEOREM 1.2. Point 1. *The law of $G_{1/2}$ is beta $(\frac{1}{2}, \frac{1}{2})$, i.e. $G_{1/2}$ is arc-sine distributed*:

$$f_{G_{1/2}}(u) = \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}} 1_{[0,1]}(u). \tag{1.23}$$

Point 2. *Let $p \geqslant 2$ denote an integer, and let $B_1, \ldots, B_{p-1}$ be a sequence of $p-1$ independent variables such that, for any $i = 1, 2, \ldots, p-1$, B_i is distributed as beta $(i/p, 1-i/p)$. Let ε_p denote a variable which is uniformly distributed on $\{1, 2, \ldots, p-1\}$ and is independent of the sequence $((B_i), i = 1, \ldots, p-1)$. Then, for $\alpha = 1/p$, we have*

$$G_\alpha = G_{1/p} \overset{(law)}{=} B_{\varepsilon_p}, \tag{1.24}$$

i.e.

$$f_{G_{1/p}}(u) = \frac{1}{\pi(p-1)} \sum_{i=1}^{p-1} \sin\left(\frac{\pi i}{p}\right) u^{i/p-1} (1-u)^{-i/p} 1_{[0,1]}(u) \tag{1.25}$$

$$= \frac{1}{\pi(p-1)} \sum_{i=1}^{p-1} \sin\left(\frac{\pi i}{p}\right) u^{-i/p} (1-u)^{i/p-1} 1_{[0,1]}(u). \tag{1.26}$$

Point 3:

$$G_\alpha \overset{(law)}{=} 1 - G_\alpha. \tag{1.27}$$

Point 4. *As $\alpha \to 1$, G_α converges in law to an r.v., denoted by G_1, which is uniformly distributed on* $[0, 1]$.

Point 5. *As $\alpha \to 0$, G_α converges in law to an r.v., denoted by G_0, which satisfies*:

(1.28) (i)
$$f_{G_0}(u) = \frac{1}{\pi}\Big(\int_0^1 (\sin(\pi\beta))\, u^{\beta-1} (1-u)^{-\beta} d\beta\Big) 1_{[0,1]}(u)$$
$$= \frac{1}{u(1-u)} \frac{1}{\pi^2 + \big(\log((1-u)/u)\big)^2} 1_{[0,1]}(u);$$

(1.29) (ii)
$$G_0 \overset{(law)}{=} \frac{1}{1+\exp(\pi C)},$$

where C is a standard Cauchy r.v.

(iii) *The Stieltjes transform of (the law of) G_0 is given by*

$$S(f_{G_0})(\lambda) := \int_0^1 \frac{f_{G_0}(u)}{\lambda+u}\,du \tag{1.30}$$

$$= E\left(\frac{1}{\lambda+G_0}\right) = \frac{1}{\lambda(1+\lambda)}\frac{1}{\log((1+\lambda)/\lambda)} \qquad (\lambda \geqslant 0).$$

1.5. The variables G_α, the unilateral stable laws and the Mittag-Leffler distributions. Let $\mu \in]0, 1[$. We denote by T_μ a unilateral ($\boldsymbol{R}_+$-valued) stable r.v. with parameter μ:

$$E[\exp(-\lambda T_\mu)] = \exp(-\lambda^\mu) \qquad (\lambda \geqslant 0). \tag{1.31}$$

Let T'_μ be an independent copy of T_μ, and define

$$Z_\mu :\overset{(\text{law})}{=} (T_\mu/T'_\mu)^\mu. \tag{1.32}$$

On the other hand, we denote by M_μ an r.v. distributed with the Mittag-Leffler law of index μ, that is (see [7], p. 114)

$$E[\exp(\lambda M_\mu)] = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(n\mu+1)} \qquad (\lambda \in \boldsymbol{R}) \tag{1.33}$$

and, consequently,

$$E(M_\mu^n) = \frac{\Gamma(n+1)}{\Gamma(\mu n+1)} \qquad (n > -1) \tag{1.34}$$

from which we deduce

$$M_\mu \overset{(\text{law})}{=} 1/(T_\mu)^\mu. \tag{1.35}$$

There exists a remarkable link between the variables G_α and $Z_{1-\alpha}$.

THEOREM 1.3. Point 1 (Lamperti [16]). *The variable Z_μ has the density*

$$f_{Z_\mu}(x) := \frac{\sin(\pi\mu)}{\pi\mu}\frac{1}{x^2+2x\cos(\pi\mu)+1}1_{(x\geqslant 0)}. \tag{1.36}$$

Point 2. *For any $\alpha \in (0, 1)$:*

$$G_\alpha \overset{(\text{law})}{=} \frac{(Z_{1-\alpha})^{1/\alpha}}{1+(Z_{1-\alpha})^{1/\alpha}} \overset{(\text{law})}{=} \frac{(T_{1-\alpha})^{(1-\alpha)/\alpha}}{(T'_{1-\alpha})^{(1-\alpha)/\alpha}+(T_{1-\alpha})^{(1-\alpha)/\alpha}} \tag{1.37 (i)}$$

(this relation implies obviously that $G_\alpha \overset{(\text{law})}{=} 1-G_\alpha$).

$$G_\alpha \overset{(\text{law})}{=} \frac{(M_{1-\alpha})^{1/\alpha}}{(M_{1-\alpha})^{1/\alpha}+(M'_{1-\alpha})^{1/\alpha}}, \tag{1.38 (ii)}$$

where $M_{1-\alpha}$ and $M'_{1-\alpha}$ on the right-hand side are two independent copies of Mittag-Leffler r.v.'s of index $1-\alpha$.

1.6. The "algebra" of the variables γ_α, G_α, and $X_{a,b}$. It is a classical result that, if γ_a and γ_b denote two independent gamma variables with respective parameters a and b, then

$$\left(\frac{\gamma_a}{\gamma_a+\gamma_b}, \gamma_a+\gamma_b\right) \overset{\text{(law)}}{=} (\beta_{a,b}, \gamma_{a+b}), \tag{1.39}$$

where $\beta_{a,b}$ and γ_{a+b} on the right-hand side are independent and distributed as beta (a, b) and gamma $(a+b)$, respectively. From this relation we deduce, in particular,

$$\gamma_{a+b}\cdot\beta_{a,b} \overset{\text{(law)}}{=} \gamma_a \quad \text{and} \quad \mathrm{e}\cdot\beta_{a,1-a} \overset{\text{(law)}}{=} \gamma_a \text{ if } b = 1-a \text{ and } 0 < a < 1. \tag{1.40}$$

It is the kind of properties such as (1.39) and (1.40) which justifies the usual terminology of "beta-gamma algebra" (see also Dufresne [10] for further developments). Our r.v.'s G_α $(0 \leqslant \alpha \leqslant 1)$ also enjoy – together with the r.v.'s $X_{a,b}$ defined below – some "algebraic properties" akin to those of the beta-gamma algebra. We note the fact that, for $p \geqslant 2$, p an integer, and $\alpha = 1/p$, the density of G_α is a barycentric combination of some beta densities, as asserted by Theorem 1.2.

THEOREM 1.4. Point 1 (Existence of the variables $X_{a,b}$). *For every a, b such that $0 < a \leqslant b \leqslant 1$ there exists an $\boldsymbol{R}_+$-valued variable $X_{a,b}$ such that*

$$E[\exp(-\lambda X_{a,b})] = \frac{b(1+\lambda)^a-1}{a(1+\lambda)^b-1} \quad (\lambda \geqslant 0). \tag{1.41}$$

Point 2. *These variables $X_{a,b}$ are infinitely divisible and satisfy: for any sequence $0 < a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n < 1$*

$$X_{a_1,a_n} \overset{\text{(law)}}{=} \sum_{i=1}^{n-1} X_{a_i,a_{i+1}}, \quad X_{a,a} = 0, \tag{1.42}$$

where the r.v.'s on the right-hand side are assumed independent.

Point 3 (Algebraic properties). *For any α, $0 \leqslant \alpha \leqslant 1$,*

$$\mathrm{e} \overset{\text{(law)}}{=} \mathrm{e}_1 G_\alpha + \mathrm{e}_2 G_{1-\alpha}, \tag{1.43}$$

where e_1, e_2, G_α and $G_{1-\alpha}$ on the right-hand side are independent, and e, e_1, e_2 are standard exponential variables. In other terms, the variables G_α and $G_{1-\alpha}$ yield an affine decomposition of the exponential law.

Point 4. *More generally, for any $\alpha \in [\frac{1}{2}, 1]$,*

$$\mathrm{e}G_\alpha \overset{\text{(law)}}{=} \gamma_{(1-\alpha)} + X_{1-\alpha,\alpha}, \tag{1.44}$$

where, as usual, the r.v.'s which appear on each side of (1.44) *are assumed independent, whereas for $\alpha \in [0, \frac{1}{2}]$:*

$$X_{\alpha,1-\alpha} + \mathrm{e}G_\alpha \overset{\text{(law)}}{=} \gamma_{(1-\alpha)}. \tag{1.45}$$

We note that (1.44) implies that, for $\alpha \geqslant \frac{1}{2}$, $\mathbf{e}G_\alpha$ is infinitely divisible and that the addition term by term of (1.44) and (1.45), where α is replaced by $1-\alpha$, implies (1.43).

1.7. The r.v.'s $G_{\alpha,\beta}$ and their "algebraic" properties $(0 < \alpha,\ \beta < 1)$. Recall that the (laws of) G_α $(0 < \alpha < 1)$ are characterized by

$$(1.46)\qquad E\left(\frac{1}{1+\lambda G_\alpha}\right) = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - 1} \qquad (\lambda \geqslant 0).$$

This relation led us to raise the following questions:

- Do there exist variables $G_{\alpha,\beta}$ such that

$$(1.47)\qquad E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right) = \frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^\alpha - 1}?$$

- If yes, do these variables have "algebraic" properties similar to those described in Theorem 1.4?

The next theorem answers these questions in the affirmative.

Theorem 1.5. *Let α, β be such that $0 < \alpha, \beta < 1$.*

Point 1 (Existence of the variable $G_{\alpha,\beta}$):

(i) *There exists an r.v. $G_{\alpha,\beta}$, taking values in $[0, 1]$, such that*

$$(1.48)\qquad E[\exp(-\lambda \mathbf{e} G_{\alpha,\beta})] = E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right) = \frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^\alpha - 1} \qquad (\lambda \geqslant 0).$$

(ii) *In close relation with (1.48), the Stieltjes transform of $G_{\alpha,\beta}$ is*

$$(1.49)\qquad E\left(\frac{1}{\lambda+G_{\alpha,\beta}}\right) = \frac{\alpha}{1-\beta}\,\frac{(\lambda^{\beta-1}-(1+\lambda)^{\beta-1})\,\lambda^{\alpha-\beta}}{(1+\lambda)^\alpha - \lambda^\alpha} \qquad (\lambda \geqslant 0).$$

(iii) *The density of $G_{\alpha,\beta}$, denoted by $f_{G_{\alpha,\beta}}$, is*

$$(1.50)\qquad f_{G_{\alpha,\beta}}(u) = 1_{[0,1]}(u)\,\frac{\alpha}{\pi(1-\beta)}$$
$$\times\frac{(1-u)u^{\alpha-1}\sin(\pi\alpha) + u^{2\alpha-\beta}(1-u)^{\beta-1}\sin(\pi\beta) + (1-u)^{\alpha+\beta-1}u^{\alpha-\beta}\sin(\pi(\alpha-\beta))}{(1-u)^{2\alpha} - 2(1-u)^\alpha u^\alpha\cos(\pi\alpha) + u^{2\alpha}}$$

(note that it is not quite obvious to verify that $f_{G_{\alpha,\beta}} \geqslant 0$ for $\alpha < \beta$).

$$(1.51)\ \text{(iv)}\qquad G_{\alpha,\alpha} \overset{(law)}{=} G_\alpha.$$

$$(1.52)\ \text{(v)}\qquad G_{\alpha,1-\alpha} \text{ is a beta } (\alpha, 1-\alpha) \text{ r.v.}$$

Point 2 (Algebraic properties). *We have:*

$$(1.53)\ \text{(i)}\qquad \text{if } \alpha+\beta \geqslant 1, \text{ then } \mathbf{e}G_{\alpha,\beta} \overset{(law)}{=} \gamma_{(1-\beta)} + X_{1-\beta,\alpha};$$

(1.54) (ii) *if* $\alpha+\beta \leqslant 1$, *then* $\gamma_{(1-\beta)} \stackrel{(law)}{=} \mathrm{e}G_{\alpha,\beta}+X_{\alpha,1-\beta}$;

(1.55) (iii) *for all* $0<\alpha,\ \beta,\ \gamma<1$, $\mathrm{e}_1 G_{\alpha,\beta}+\mathrm{e}_2 G_{\beta,\gamma} \stackrel{(law)}{=} \mathrm{e}_1 G_{\alpha,\gamma}+\mathrm{e}_2 G_\beta$;

and, if $\alpha+\beta \geqslant 1$, *from* (1.55) *and* (1.53) *we obtain*

$$\gamma_{(1-\beta)}+X_{1-\beta,\alpha}+\mathrm{e}_2 G_{\beta,\gamma} \stackrel{(law)}{=} \mathrm{e}_1 G_{\alpha,\gamma}+\mathrm{e}_2 G_\beta, \tag{1.56}$$

whereas, if $\alpha+\beta \leqslant 1$, *then from* (1.55) *and* (1.54) *we get*

$$\gamma_{(1-\beta)}+\mathrm{e}G_{\beta,\gamma} \stackrel{(law)}{=} \mathrm{e}_1 G_{\alpha,\gamma}+\mathrm{e}_2 G_\beta+X_{\alpha,1-\beta}; \tag{1.57}$$

(iv) *if* $0<\alpha<\beta<1$, *then* $\mathrm{e}(1-G_{\alpha,\beta}) \stackrel{(law)}{=} \gamma_{(\beta-\alpha)}+\mathrm{e}G_{\alpha,\beta}$,

if $0<\beta<\alpha<1$, *then* $\gamma_{(\alpha-\beta)}+\mathrm{e}(1-G_{\alpha,\beta}) \stackrel{(law)}{=} \mathrm{e}G_{\alpha,\beta}$.

Of course, in all the above relations, on each side, the featured r.v.'s are independent. The relations (1.43)–(1.45) *are particular cases of the relations* (1.53)–(1.57).

1.8. On (δ, G) self-decomposable variables. The formula (1.16), where we do not mention the index α:

$$E(e^{-\lambda\Delta}) = \exp\left(-\delta \int_0^\infty (1-e^{-\lambda x}) E(e^{-xG})\frac{dx}{x}\right), \tag{1.58}$$

led us to study the r.v.'s Δ whose laws may be obtained from those of G via the relation (1.58), thus generalizing the relation between Δ_α and G_α.

Remark and definition. Let G be an $\boldsymbol{R}_+$-valued r.v. The following properties are equivalent:

(1.59) (i) $$\int_1^\infty E(e^{-xG})\frac{dx}{x} < \infty,$$

(1.60) (ii) $$E(\log^+(1/G)) < \infty,$$

(1.61) (iii) $$\int_0^\infty (x\wedge 1) E(e^{-xG})\frac{dx}{x} < \infty,$$

i.e. the measure $E(e^{-xG})\frac{dx}{x} 1_{(x\geqslant 0)}$ is the Lévy measure of a subordinator,

(1.62) (iv) $$E\left(\log\left(1+\frac{\lambda}{G}\right)\right) < \infty \quad \text{for some (hence all) } \lambda > 0.$$

Let G satisfy one (hence all) of these conditions and let $\delta > 0$. We say that an r.v. Δ is (δ, G) *self-decomposable* if

$$E(e^{-\lambda\Delta}) = \exp\left(-\delta \int_0^\infty (1-e^{-\lambda x}) E(e^{-xG})\frac{dx}{x}\right) \quad (\lambda \geqslant 0) \tag{1.63}$$

$$= \exp\left(-\delta E(\log(1+\lambda/G))\right) \quad (\lambda \geqslant 0). \tag{1.64}$$

(Note that (1.63) may be considered as a definition of the law of Δ in terms of (δ, G), whereas (1.64) follows from (1.63) via the simple Frullani integral argument; see, e.g., Lebedev [17], p. 6.)

The (δ, G) self-decomposable r.v.'s are closely linked to the standard gamma subordinator; in fact, their laws are the generalized gamma convolutions which have been studied extensively by Bondesson [5], [6].

THEOREM 1.6. *Let $(\gamma_t, t \geqslant 0)$ denote the gamma standard subordinator, i.e. the subordinator such that*

$$E[\exp(-\lambda\gamma_t)] = \frac{1}{(1+\lambda)^t} = \exp(-t\log(1+\lambda)) \qquad (t, \lambda \geqslant 0)$$

and let $h\colon]0, \infty[\to \boldsymbol{R}_+$, a Borel function.

Point 1. *Let*

$$\Delta_h := \int_0^\infty h(u)\, d\gamma_u. \tag{1.65}$$

Then Δ_h is finite a.s. if and only if

$$\int_0^\infty \log(1+h(u))\, du < \infty. \tag{1.66}$$

Point 2. *Under the hypothesis* (1.66), *Δ_h is self-decomposable and*

$$E[\exp(-\lambda\Delta_h)] = \exp\left(-\int_0^\infty (1-e^{-\lambda x}) F_h(x)\frac{dx}{x}\right)$$

with

$$F_h(x) := \int_0^\infty \exp\left(-\frac{x}{h(u)}\right) du.$$

Point 3. *For all positive r.v.'s G satisfying* (1.59) *and all $\delta > 0$, there exists h satisfying* (1.66) *so that*

$$\delta E(e^{-xG}) = F_h(x) = \int_0^\infty \exp\left(-\frac{x}{h(u)}\right) du. \tag{1.67}$$

In other terms, all r.v.'s Δ which are (δ, G) self-decomposable can be written as $\Delta \overset{(\text{law})}{=} \Delta_h$ because, by (1.67),

$$E[\exp(-\lambda\Delta_h)] = \exp\left(-\delta\int_0^\infty (1-e^{-\lambda x}) E(e^{-xG})\frac{dx}{x}\right).$$

Here are some further precisions about this theorem:

- An explicit relation between h and G as in (1.67) is

$$h(u) = \frac{1}{\mathcal{G}^{-1}(u/\delta)} \text{ for } u \in (0, \delta) \quad \text{and} \quad h(u) = 0 \text{ for } u > \delta,$$

where $\mathscr{G}^{-1}$ denotes the inverse (in the sense of the composition of functions) of the distribution function of the r.v. G.

• Moreover, it is known (see [6], and also [23], Theorem 5.24, p. 362) that a positive r.v. Δ is of the form

$$\Delta_h = \int_0^\infty h(u)\,d\gamma(u),$$

i.e. its law is a generalized gamma convolution if and only if its Laplace transform $\psi_\Delta(\lambda) := E(e^{-\lambda\Delta})$ is hyperbolically completely monotone, i.e. it satisfies: for all $u > 0$, the function

$$(v+1/v) \to \psi_\Delta(uv)\,\psi_\Delta(u/v)$$

is completely monotone, as a function of $(v+1/v)$.

THEOREM 1.7. *Let G satisfy* (1.60) *and let Δ denote an r.v. which is* (δ, G) *self-decomposable.*

Point 1. *There exists a (δ, K) positive compound Poisson process $(Y_t, t \geqslant 0)$ with $K \overset{(\text{law})}{=} \mathrm{e}/G$, such that*

$$\Delta := \int_0^\infty e^{-t}\,dY_t. \tag{1.68}$$

Point 2. *Δ satisfies the affine equation*

$$\Delta \overset{(\text{law})}{=} U^{1/\delta}(\Delta + K), \tag{1.69}$$

where the r.v.'s U, Δ and K on the right-hand side are independent, and U is uniform on $[0, 1]$.

Point 3. *Let $\psi(\lambda) := E(e^{-\lambda\Delta})$. Then the Stieltjes transform of G equals*

$$E\left(\frac{1}{\lambda+G}\right) = -\frac{\psi'}{\psi}(\lambda) = -\frac{\partial}{\partial\lambda}(\log E(e^{-\lambda\Delta})) \quad (\lambda \geqslant 0). \tag{1.70}$$

We note that Theorem 1.7 presents the points 2 and 3 of Theorem 1.1 in a more general set-up. We shall now establish a converse of Theorem 1.7 which, essentially, hinges upon the properties of the inverse Stieltjes transform. This leads to the following:

DEFINITION. A function $F:]0, \infty[\to]0, \infty[$, which is C^1, is said to satisfy the *condition* (ST, δ) (obviously, ST stands for Stieltjes transform) if:

(i) F extends holomorphically to $C\backslash]-\infty, 0[$;

(ii) for any $u \geqslant 0$, the limits

$$F_+(u) := \lim_{\eta\to 0+} F(-u+i\eta), \qquad F_-(u) := \lim_{\eta\to 0+} F(-u-i\eta)$$

exist, are continuous, and satisfy

(1.71) $$\operatorname{Im}\left(F_{-}(u)-F_{+}(u)\right) \geqslant 0 \quad \text{for any } u \geqslant 0;$$

(iii) for $\lambda \in \boldsymbol{R}$, $\lim_{\lambda\to\infty} \lambda F(\lambda) = \delta$.

This definition proves useful in the following:

THEOREM 1.8. *Let Δ denote a positive r.v. with Laplace transform ψ, i.e. $E(e^{-\lambda\Delta}) = \psi(\lambda)$ $(\lambda \geqslant 0)$. Assume that $F := -\psi'/\psi$ satisfies the condition (ST, δ). Then*

$$f(u) := \frac{1}{2\pi\delta} \operatorname{Im}\left(F_{-}(u)-F_{+}(u)\right) \quad (u \geqslant 0)$$

defines a density of probability on $\boldsymbol{R}_{+}$, and Δ is an r.v. which is (δ, G) self-decomposable, when G denotes an r.v. with density $f_G = f$.

Acknowledgments. 1) After making some quite informal computation, we were able to "verify" that the density of G_α, resp. $G_{\alpha,\beta}$, is indeed given by (1.17), resp. (1.50). We thank most sincerely Mrs Y. Yano who indicated to us a more rigorous proof based on Stieltjes transform inversion.

2) We also thank Prof. Z. J. Jurek who provided us with some references, and also helped us to eradicate some misprints.

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2. PROOF OF THEOREM 1.1

2.1. Proof of point 1 of Theorem 1.1. First we recall this point:

(2.1) (i) $$\Delta_\alpha \overset{(law)}{=} \frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}},$$

where $\gamma_{(1-\alpha)}$ and $\beta_{(\alpha,1)}$ on the right-hand side are two independent respective gamma $(1-\alpha)$ and beta $(\alpha, 1)$ variables.

(ii) *The density f_{Δ_α} of Δ_α is given by*

(2.2) $$f_{\Delta_\alpha}(x) := \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}(1-e^{-x})\, 1_{[0,\infty[}(x).$$

(iii) *The Laplace transform of (the law of) Δ_α is*

(2.3) $$E[\exp(-\lambda\Delta_\alpha)] = (1+\lambda)^\alpha - \lambda^\alpha \quad (\lambda \geqslant 0).$$

As indicated in the Introduction, this point is a particular case of the results of Winkel [25]. However, below, we give three proofs of this point. The first and second proofs are very specific to the Bessel process context in which we are working, whereas the third one, of a more general kind, uses arguments close to those of Winkel.

2.1.1. First proof of point 1 of Theorem 1.1.

2.1.1.a. By scaling, we have

$$\Delta_\alpha \overset{\text{(law)}}{=} \mathrm{e}\,(d_1 - g_1) \overset{\text{(law)}}{=} \mathrm{e}\big((1-g_1)+(d_1-1)\big). \tag{2.4}$$

Furthermore, $(1-g_1, d_1-1) \overset{\text{(law)}}{=} (1-g_1, R_1^2\, T_0^{(1)})$, where the pair $(1-g_1, R_1)$ is independent of $T_0^{(1)} \equiv \inf\{t \geqslant 0 \colon R_t^{(1)} = 0\}$, with $(R_u^{(1)}, u \geqslant 0)$ being a Bessel process starting from 1. This is obtained by applying the Markov property to R at time 1, together with the scaling property. It is well known (see, e.g., [8]; [26], p. 14; [13]) that

$$T_0^{(1)} \overset{\text{(law)}}{=} 1/2\gamma_{(\alpha)}, \tag{2.5}$$

where $\gamma_{(\alpha)}$ is gamma (α) distributed. Thus, from (2.4) we get

$$\Delta_\alpha \overset{\text{(law)}}{=} \mathrm{e}\left((1-g_1)+R_1^2 \frac{1}{2\gamma_{(\alpha)}}\right), \tag{2.6}$$

where the pair $\big((1-g_1), R_1\big)$ on the right-hand side is independent of $\gamma_{(\alpha)}$. Moreover, classical properties of the Bessel meander (see, e.g., [8], where these properties are recalled) imply

$$(R_1^2, 1-g_1) \overset{\text{(law)}}{=} \big((1-g_1)\, 2\mathrm{e}_1, (1-g_1)\big), \tag{2.7}$$

where e_1 is a standard exponential variable, independent of g_1, and g_1 is beta $(\alpha, 1-\alpha)$ distributed. Bringing (2.7) in (2.6), we obtain

$$\Delta_\alpha \overset{\text{(law)}}{=} (1-g_1)\,\mathrm{e}\left(1+\frac{\mathrm{e}_1}{\gamma_{(\alpha)}}\right),$$

where the r.v.'s g_1, e, e_1, $\gamma_{(\alpha)}$ on the right-hand side are assumed independent. Furthermore, the classical properties of the "beta-gamma algebra" imply

$$(1-g_1)\,\mathrm{e} \overset{\text{(law)}}{=} \gamma_{(1-\alpha)} \quad \text{and} \quad 1+\frac{\mathrm{e}_1}{\gamma_{(\alpha)}} \overset{\text{(law)}}{=} \frac{1}{\beta_{(\alpha,1)}},$$

and hence, finally,

$$\Delta_\alpha \overset{\text{(law)}}{=} \frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}}.$$

2.1.1.b. The expression of the density (given by (2.2)) of Δ_α follows from (2.1). Furthermore

$$E[\exp(-\lambda\Delta_\alpha)] = E\left[\exp\left(-\lambda\frac{\gamma_{(1-\alpha)}}{\beta_{(\alpha,1)}}\right)\right] = E\left(\frac{1}{1+\lambda/\beta_{(\alpha,1)}}\right)^{1-\alpha} = E\left(\frac{\beta_{(\alpha,1)}}{\lambda+\beta_{(\alpha,1)}}\right)^{1-\alpha}$$

$$= \alpha\int_0^1 \left(\frac{u}{\lambda+u}\right)^{1-\alpha} u^{\alpha-1}\,du = \alpha\int_0^1 (\lambda+u)^{\alpha-1}\,du = (1+\lambda)^\alpha - \lambda^\alpha.$$

2.1.2. Second proof of point 1 of Theorem 1.1. It hinges upon the same arguments as in the preceding proof, but it has a more analytic flavor. We shall show that

$$E[\exp(-\lambda \Delta_\alpha)] = (1+\lambda)^\alpha - \lambda^\alpha \qquad (\lambda \geqslant 0). \tag{2.8}$$

We denote by $P^{(\alpha)}$ the distribution of the Bessel process, starting from 0, with dimension $d = 2(1-\alpha)$ $(0 < \alpha < 1)$, and let $(A_t := t - g_t,\ t \geqslant 0)$ denote the age process of excursions of R away from 0. Then, for fixed $t \geqslant 0$, we have

$$E^{(\alpha)}[\exp(-\lambda(d_t - g_t))] = E^{(\alpha)}[\exp(-\lambda(A_t + T_0 \circ \theta_t))] \tag{2.9}$$

$$= E^{(\alpha)}(\exp(-\lambda A_t) E^{(\alpha)}_{R_t}[\exp(-\lambda T_0)]), \tag{2.10}$$

where T_0 denotes the first hitting time of 0 by $(R_t, t \geqslant 0)$ and $(\theta_t, t \geqslant 0)$ is the usual family of translation operators. The Laplace transform of T_0 featured in (2.10) may be computed explicitly (see, e.g., [13]), in agreement with (2.5):

$$E^{(\alpha)}[\exp(-\lambda(d_t - g_t))] = E^{(\alpha)}[\exp(-\lambda A_t) K_\alpha(R_t\sqrt{2\lambda})(R_t\sqrt{2\lambda})^\alpha] \tag{2.11}$$

$$= E^{(\alpha)}[\exp(-\lambda A_t)(\Phi(1, 1-\alpha, \lambda A_t) - \Gamma(1-\alpha)(\lambda A_t)^\alpha \exp(\lambda A_t))], \tag{2.12}$$

where K_α denotes the Bessel–Mac Donald function with index α, and $\Phi(1, 1-\alpha, \cdot)$ denotes the confluent hypergeometric function with parameter $(1, 1-\alpha)$ (see [17], p. 260). We now replace in (2.12) the fixed time t by a variable $\mathfrak{e}$, exponentially distributed and independent of $(R_u, u \geqslant 0)$. Note that, by scaling,

$$A_{\mathfrak{e}} \overset{\text{(law)}}{=} \mathfrak{e} A_1 \overset{\text{(law)}}{=} \mathfrak{e}\beta_{(1-\alpha,\alpha)} \overset{\text{(law)}}{=} \gamma_{(1-\alpha)}, \tag{2.13}$$

and hence, using the definition of the hypergeometric function $\Phi(1, 1-\alpha, \cdot)$,

$$\begin{aligned} E[\exp(-\lambda \Delta_\alpha)] &= E^{(\alpha)}[\exp(-\lambda(d_{\mathfrak{e}} - g_{\mathfrak{e}}))] \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{-\lambda z - z} z^{-\alpha} \Phi(1, 1-\alpha, \lambda z)\, dz - \int_0^\infty e^{-z} (\lambda z)^\alpha z^{-\alpha}\, dz \\ &= \left[\frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{-(\lambda+1)z} z^{-\alpha} \left(\sum_{k=0}^\infty \frac{k!\,\Gamma(1-\alpha)}{\Gamma(1-\alpha+k)} \frac{(\lambda z)^k}{k!}\right) dz\right] - \lambda^\alpha. \end{aligned}$$

Consequently,

$$\begin{aligned} E[\exp(-\lambda \Delta_\alpha)] &= \left(\sum_{k=0}^\infty \int_0^\infty dz\, e^{-(\lambda+1)z} \frac{\lambda^k z^{k-\alpha}}{\Gamma(1-\alpha+k)}\right) - \lambda^\alpha \\ &= \left(\sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(1-\alpha+k)} \int_0^\infty \frac{e^{-u} u^{k-\alpha}}{(1+\lambda)^{k-\alpha+1}} du\right) - \lambda^\alpha \end{aligned}$$

$$= \sum_{k=0}^{\infty} \left(\frac{\lambda}{1+\lambda}\right)^k \frac{1}{(1+\lambda)^{1-\alpha}} - \lambda^\alpha = (1+\lambda)^{\alpha-1} \frac{1}{1-\lambda/(1+\lambda)} - \lambda^\alpha$$

$$= (1+\lambda)^\alpha - \lambda^\alpha.$$

2.1.3. Third proof of point 1 of Theorem 1. It hinges only – as in the proof of Winkel [25] – upon the fact that the process

$$\tau_l := \inf\{t \geqslant 0;\ L_t > l\}, \qquad l \geqslant 0,$$

is a stable subordinator, without drift term, where $(L_t,\ t \geqslant 0)$ denotes the local time process at 0 of $(R_t,\ t \geqslant 0)$. Thus

$$(2.14) \quad E[\exp(-\lambda\tau_l)] = \exp\left(-l\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}2^{-\alpha}\lambda^\alpha\right) := e^{-l\Phi(\lambda)} \qquad (\lambda \geqslant 0),$$

where $\Phi(\lambda)$ is the characteristic exponent of $(\tau_l,\ l \geqslant 0)$ (cf. [8] for a discussion of the values of normalization constants related to $(L_t,\ t \geqslant 0)$ and $(\tau_l,\ l \geqslant 0)$). Now, let in general $(\tau_l,\ l \geqslant 0)$ denote a subordinator without drift. In other terms:

$$(2.15) \qquad E[\exp(-\lambda\tau_l)] = \exp(-l\Phi(\lambda))$$

with

$$\Phi(\lambda) := \exp\left(-\int_0^\infty (1-e^{-\lambda x})\,\nu(dx)\right),$$

where ν denotes the Lévy measure of $(\tau_l,\ l \geqslant 0)$. Let us define

$$L_t := \inf\{l;\ \tau_l > t\}, \qquad t \geqslant 0,$$

and let $\mathfrak{e}$ denote an exponential variable, with mean 1, independent of $(\tau_l,\ l \geqslant 0)$.

LEMMA 2.1. *Let*

$$(2.16) \qquad \Delta^{(\tau)} := \tau_{(L_{\mathfrak{e}})} - \tau_{(L_{\mathfrak{e}})^-}.$$

Then

$$(2.17) \qquad E[\exp(-\lambda\Delta^{(\tau)})] = \frac{\Phi(1+\lambda)-\Phi(\lambda)}{\Phi(1)}.$$

Clearly, point 1 (iii) of our Theorem 1.1 is an immediate consequence of (2.17), when Lemma 2.1 is applied to the subordinator defined by (2.14), i.e. when

$$\Phi(\lambda) = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}2^{-\alpha}\lambda^\alpha \qquad (\lambda \geqslant 0).$$

Proof of Lemma 2.1. By the definition of $\Delta^{(\tau)}$, we have

$$E[\exp(-\lambda\Delta^{(\tau)})] = E\Big(\int_0^\infty \exp\big(-t-\lambda(\tau_{L_t}-\tau_{(L_t)-})\big)\,dt\Big)$$

$$= E\Big(\sum_{l>0}\int_{\tau_{l-}}^{\tau_l} e^{-t}\exp(-\lambda\delta_l)\,dt\Big) \quad (\text{where } \delta_l := \tau_l - \tau_{l-})$$

$$= E\Big(\sum_{l>0}[\exp(-\tau_{l-})-\exp(-\tau_l)]\exp(-\lambda\delta_l)\Big)$$

$$= E\Big(\int_0^\infty \exp(-\tau_{l-})\,dl \int_0^\infty (1-e^{-v})e^{-\lambda v}\,\nu(dv)\Big)$$

$$= E\Big(\int_0^\infty \exp(-\tau_l)\,dl\Big)\big(\Phi(1+\lambda)-\Phi(\lambda)\big)$$

$$= \big(\Phi(1+\lambda)-\Phi(\lambda)\big)\int_0^\infty e^{-l\Phi(1)}\,dl = \frac{\Phi(1+\lambda)-\Phi(\lambda)}{\Phi(1)}. \quad \blacksquare$$

2.2. Proof of point 2 of Theorem 1.1. We first recall this point:

(i) *Δ_α is self-decomposable, and the Lévy–Khintchine formula takes the form*

$$E[\exp(-\lambda\Delta_\alpha)] = \exp\left(-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\,E[\exp(-xG_\alpha)]\frac{dx}{x}\right), \tag{2.18}$$

where G_α denotes an r.v. taking values in $[0,1]$, *with density*

$$f_{G_\alpha}(u) = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\,\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}}\,1_{[0,1]}(u). \tag{2.19}$$

(ii) *The law of G_α is characterized by its Stieltjes transform*

$$S(f_{G_\alpha})(\lambda) := \int_0^1 \frac{f_{G_\alpha}(u)}{\lambda+u}\,du = E\left(\frac{1}{\lambda+G_\alpha}\right) \tag{2.20}$$

$$= \frac{\alpha}{1-\alpha}\,\frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha} \quad (\lambda \geqslant 0)$$

or, equivalently, by

$$E[\exp(-\lambda e G_\alpha)] = E\left(\frac{1}{1+\lambda G_\alpha}\right) \equiv \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1} \quad (\lambda \geqslant 0). \tag{2.21}$$

2.2.1. We prove that f_{G_α}, *as defined by* (2.19), *is a probability density, which is characterized by* (2.10), *or* (2.21).

2.2.1.a. Let

$$F_\alpha(\lambda) := \frac{\alpha}{1-\alpha}\,\frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha}. \tag{2.22}$$

Since the function f_{G_α} is continuous and integrable on [0, 1], in order to prove (2.19), we may use the inversion formula for the Stieltjes transform. Recall (cf. [24], p. 340) that if f is integrable and if $S(f)$ denotes its Stieltjes transform

$$S(f)(\lambda) = \int_0^\infty \frac{f(u)\,du}{\lambda+u}, \tag{2.23}$$

we have

$$f(u) = \lim_{\eta\to 0+} \frac{Sf(-u-i\eta)-Sf(-u+i\eta)}{2i\pi}. \tag{2.24}$$

Thus, to prove (2.19) amounts, thanks to the injectivity of the Stieltjes transform, to showing that

$$\lim_{\eta\downarrow 0+} \frac{F_\alpha(-u-i\eta)-F_\alpha(-u+i\eta)}{2i\pi} = \begin{cases} 0 & \text{if } u>1, \\ f_{G_\alpha}(u) & \text{if } u\in[0,1]. \end{cases} \tag{2.25}$$

Formula (2.25) follows from an elementary computation; in fact, we shall prove this result later in a more general framework (cf. 5.1.1 below).

2.2.1.b. We prove that f_{G_α} *is a probability density.*

Since $f_{G_\alpha} \geqslant 0$, it suffices to show that

$$\int_0^1 f_{G_\alpha}(u)\,du = 1.$$

Now, from (2.20) we obtain

$$\int_0^1 f_{G_\alpha}(u)\,du = \lim_{\lambda\to\infty} \lambda S(f_{G_\alpha})(\lambda) = \lim_{\lambda\to\infty} \frac{\alpha}{1-\alpha}\lambda\cdot\frac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha}$$

$$= \lim_{\lambda\to\infty} \frac{\alpha}{1-\alpha}\,\frac{1-(1+1/\lambda)^{\alpha-1}}{(1+1/\lambda)^\alpha-1} = 1.$$

We also note that the equivalence of (2.20) and (2.21) follows from

$$E[\exp(-\lambda e G_\alpha)] = E\left(\frac{1}{1+\lambda G_\alpha}\right) = \frac{1}{\lambda}E\left(\frac{1}{1/\lambda+G_\alpha}\right) = \frac{1}{\lambda}S(f_{G_\alpha})\left(\frac{1}{\lambda}\right) \tag{2.26}$$

$$= \frac{1}{\lambda}\frac{\alpha}{1-\alpha}\frac{(1/\lambda)^{\alpha-1}-((1+\lambda)/\lambda)^{\alpha-1}}{((1+\lambda)/\lambda)^\alpha-(1/\lambda)^\alpha} = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1}.$$

2.2.2. Proof of (2.18). With the help of (2.21), and taking logarithmic derivatives on both sides of (2.18), the question amounts to showing

$$\frac{\partial}{\partial\lambda}\log\left((1+\lambda)^\alpha-\lambda^\alpha\right) = -(1-\alpha)\int_0^\infty e^{-\lambda x} E[\exp(-xG_\alpha)]\,dx$$

by (2.3), or

$$\text{(2.27)}\qquad \alpha\frac{(1+\lambda)^{\alpha-1}-\lambda^{\alpha-1}}{(1+\lambda)^{\alpha}-\lambda^{\alpha}} = -(1-\alpha)\int_0^{\infty} e^{-\lambda x}\,dx\int_0^1 e^{-xu} f_{G_\alpha}(u)\,du$$

$$= -(1-\alpha)\int_0^1 \frac{1}{\lambda+u} f_{G_\alpha}(u)\,du \qquad \text{(Fubini)}$$

$$= -(1-\alpha)\,E\left(\frac{1}{\lambda+G_\alpha}\right).$$

However, (2.27) is nothing else but (2.20). ∎

The careful reader may have been surprised by the above proof, in particular by the proof given in 2.2.1.a, which may seem quite unnatural. Clearly, it is not in this manner that we discovered formula (2.18). Here is our original proof, which is more intuitive, but which, unfortunately, contains some non-rigorous features.

2.2.3. Another proof of (2.18).

2.2.3.a. Our aim is to find, from 2.2.1, an r.v. G_α, taking values in $[0, 1]$, such that

$$\text{(2.28)}\qquad E\left(\frac{1}{1+\lambda G_\alpha}\right) = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^{\alpha}-1} \qquad (\lambda \geqslant 0).$$

When $\alpha = 1/2$, choosing for $G_{1/2}$ an r.v. with distribution beta $(\frac{1}{2}, \frac{1}{2})$, we see that the relation (2.28) is satisfied, since from the beta-gamma algebra

$$\text{(2.29)}\qquad \mathrm{e}\cdot\beta_{(\alpha,1-\alpha)} \overset{\text{(law)}}{=} \gamma_{(\alpha)}$$

we deduce

$$\text{(2.30)}\qquad E[\exp(-\lambda \mathrm{e}\beta_{(\alpha,1-\alpha)})] = E\left(\frac{1}{1+\lambda\beta_{(\alpha,1-\alpha)}}\right) = E[\exp(-\lambda\gamma_{(\alpha)})] = \frac{1}{(1+\lambda)^{\alpha}}.$$

Hence, for $\alpha = 1/2$, with $G_{1/2} \overset{\text{(law)}}{=} \beta_{(1/2,1/2)}$

$$\text{(2.31)}\qquad E\left(\frac{1}{1+\lambda G_{1/2}}\right) = \frac{1}{\sqrt{1+\lambda}} = \frac{1/2}{1-1/2}\,\frac{1-(1+\lambda)^{1/2}}{(1+\lambda)^{1/2}-1}.$$

This particular result for $\alpha = 1/2$ invites to look whether the density f_α of the r.v. G_α may be written in the form

$$\text{(2.32)}\qquad f_\alpha(u) = \int h_\gamma(u)\,\mu_\alpha(d\gamma), \qquad u \in [0, 1],$$

where h_γ denotes here the density of a beta $(\gamma, 1-\gamma)$ variable, and $\mu_\alpha(d\gamma)$ a certain positive measure. Since from (2.30) we have

$$\text{(2.33)}\qquad \int_0^1 \frac{1}{1+\lambda u} h_\gamma(u)\,du = E\left(\frac{1}{1+\lambda\beta_{(\gamma,1-\gamma)}}\right) = \frac{1}{(1+\lambda)^{\gamma}},$$

the problem amounts to finding a measure $\mu_\alpha(d\gamma)$ such that

$$\int_0^1 \frac{f_\alpha(u)\,du}{1+\lambda u} = \int \frac{1}{(1+\lambda)^\gamma}\mu_\alpha(d\gamma) = \frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1} \qquad (\lambda \geqslant 0). \tag{2.34}$$

2.2.3.b. Searching for $\mu_\alpha(d\gamma)$ *such that* (2.34) *is satisfied.* We replace in (2.34) $(1+\lambda)$ by e^t $(t \geqslant 0)$, and we obtain

$$\begin{aligned} \frac{\alpha}{1-\alpha}\,\frac{1-e^{t(\alpha-1)}}{e^{t\alpha}-1} &= \frac{\alpha}{1-\alpha}\Big(\sum_{m=0}^{\infty} e^{-t(m+1)\alpha} - \sum_{n=0}^{\infty} e^{-t(n\alpha+1)}\Big) \\ &= \int e^{-\alpha t}\,\mu_\alpha(d\gamma). \end{aligned} \tag{2.35}$$

Consequently, since both sides of (2.35) are Laplace transforms, we obtain

$$\mu_\alpha(d\gamma) = \frac{\alpha}{1-\alpha}\Big\{\sum_{m=0}^{\infty} \delta_{(m+1)\alpha}(d\gamma) - \sum_{n=0}^{\infty} \delta_{(n\alpha+1)}(d\gamma)\Big\}. \tag{2.36}$$

We shall now discuss two cases: (i) and (ii).

(i) $\alpha = 1/p$, *p an integer*, $p \geqslant 2$.

In this case, the following computation is entirely rigorous. In formula (2.36), one finds only $p-1$ terms, since

$$(p-1+1)\alpha = p\cdot\frac{1}{p} = 1 = 0\cdot\alpha+1.$$

Hence

$$\mu_\alpha(d\gamma) = \frac{\alpha}{1-\alpha}\sum_{k=1}^{p-1} \delta_{k\alpha}(d\gamma) = \frac{1}{p-1}\sum_{k=1}^{p-1} \delta_{k/p}(d\gamma), \tag{2.37}$$

so that, plugging this value of μ_α in (2.32), we obtain

$$\begin{aligned} f_\alpha(u) &= \frac{1}{p-1}\sum_{k=1}^{p-1} h_{k/p}(u) = \frac{1}{p-1}\sum_{k=1}^{p-1} \frac{u^{k/p-1}(1-u)^{-k/p}}{\Gamma(k/p)\,\Gamma(1-k/p)} \\ &= \frac{1}{p-1}\sum_{k=1}^{p-1} \frac{\sin(\pi k\cdot p^{-1})}{\pi}\,u^{k/p-1}(1-u)^{-k/p} \end{aligned}$$

from the formula of complements for the gamma function

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

(cf. [17], p. 3). Hence

$$f_\alpha(u) = \frac{1}{(p-1)\pi u}\operatorname{Im}\left(\sum_{k=1}^{p-1} e^{i\pi k/p}\left(\frac{u}{1-u}\right)^{k/p}\right)$$

$$= \frac{1}{(p-1)\pi u}\operatorname{Im}\left(\frac{\exp(i\pi\cdot p^{-1})(u/(1-u))^{1/p}+u/(1-u)}{1-\exp(i\pi\cdot p^{-1})(u/(1-u))^{1/p}}\right)$$

$$= \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}} \quad \text{with } \alpha=\frac{1}{p}.$$

(ii) *α is not of the form $1/p$, p an integer, $p \geqslant 2$.*

Plugging (2.36) in (2.32), we get

$$(2.38)\qquad f_\alpha(u) = \frac{\alpha}{1-\alpha}\left(\sum_{m=0}^{\infty}\frac{u^{(m+1)\alpha-1}(1-u)^{-(m+1)\alpha}}{\Gamma((m+1)\alpha)\Gamma(1-(m+1)\alpha)} - \sum_{n=0}^{\infty}\frac{u^{n\alpha}(1-u)^{-n\alpha-1}}{\Gamma(n\alpha+1)\Gamma(-n\alpha)}\right)$$

$$(2.39)\qquad = \frac{\alpha}{1-\alpha}\left\{\frac{1}{u\pi}\sum_{m=0}^{\infty}\sin(\pi\alpha(m+1))\left(\frac{u}{1-u}\right)^{(m+1)\alpha} - \frac{1}{(1-u)\pi}\sum_{n=0}^{\infty}\sin(\pi\alpha n)\left(\frac{u}{1-u}\right)^{n\alpha}\right\}$$

again from the formula of complements. Hence

$$f_\alpha(u) = \frac{\alpha}{1-\alpha}\left\{\frac{1}{u\pi}\operatorname{Im}\left(\frac{e^{i\pi\alpha}(u/(1-u))^\alpha}{1-e^{i\pi\alpha}(u/(1-u))^\alpha}\right)+\frac{1}{(1-u)\pi}\operatorname{Im}\left(\frac{1}{1-e^{i\pi\alpha}(u/(1-u))^\alpha}\right)\right\}$$

$$= \frac{\alpha}{1-\alpha}\frac{1}{\pi}\left\{\frac{1}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}} \times (\sin(\pi\alpha))u^{\alpha-1}(1-u)^{\alpha-1}(1-u+u)\right\}$$

$$= \frac{\alpha}{1-\alpha}\frac{\sin(\pi\alpha)}{\pi}\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}}, \quad u\in[0,1].$$

In fact, this computation may be made quite rigorous with the help of the following two arguments:

• Although the function $h_\gamma(u)$ is a density only for $\gamma\in[0,1]$, we may replace everywhere in this computation h_γ by its holomorphic prolongation (with respect to the γ variable).

• The two series which appear in this computation may be "reduced" to

$$\sum_{n=0}^{\infty} e^{i\pi n\alpha}\left(\frac{u}{1-u}\right)^{n\alpha}$$

which only converges for $u/(1-u) < 1$, i.e. for $u < \frac{1}{2}$. But it is not difficult to see that the density f_α, which we are trying to obtain, is such that $f_\alpha(u) = f_\alpha(1-u)$ for $u \in [0, 1]$ (see, e.g., (1.19) and point 3 of Theorem 1.2). Thus, it suffices to consider $u \in [0, 1/2]$, and it is precisely for these values of u for which the previous series converges.

2.2.4. We prove that Δ_α *is self-decomposable.* From Lukacs [19], p. 164, this is equivalent to the property that $x \to x v_\alpha(x)$ is a decreasing function of x, where v_α denotes the density of the Lévy measure of Δ_α. This is satisfied, since

$$v_\alpha(x) = \frac{1-\alpha}{x} E[\exp(-xG_\alpha)].$$

In fact, all generalized gamma convolutions are self-decomposable.

2.2.5. Remark 2.2. It is well known that a self-decomposable distribution σ is the invariant measure of a generalized Ornstein–Uhlenbeck process $(Y_t, t \geqslant 0)$, i.e. a process which solves

$$dY_t = -Y_t\,dt + dZ_t, \tag{2.40}$$

where $(Z_t, t \geqslant 0)$ is a Lévy process (cf. [21] and [22], p. 49). Furthermore, if Φ_Z (resp. Φ_σ) denotes the characteristic exponent of Z (resp. σ), we have

$$\Phi_Z(\lambda) = \lambda \frac{\Phi'_\sigma(\lambda)}{\Phi_\sigma(\lambda)} \qquad (\lambda \geqslant 0). \tag{2.41}$$

We deduce from this formula that if w (resp. u) denotes the density of the Lévy measure of Z (resp. σ), then

$$w(x) = -u(x) - xu'(x). \tag{2.42}$$

We apply this in the case where σ_α is the law of Δ_α, that is, from (1.14) we obtain

$$\sigma_\alpha(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}(1-e^{-x})\, 1_{[0,\infty[}(x)\, dx.$$

Then there exists a Lévy process $(Z_t^{(\alpha)}, t \geqslant 0)$ with Lévy exponent Φ_α and Lévy density w_α such that the process $(Y_t^{(\alpha)}, t \geqslant 0)$, which solves

$$dY_t^{(\alpha)} = -Y_t^{(\alpha)}\, dt + dZ_t^{(\alpha)}, \tag{2.43}$$

admits σ_α as its invariant probability measure. Formulae (2.41) and (2.42) now become

$$\Phi_\alpha(\lambda) = \alpha\lambda \frac{\lambda^{\alpha-1} - (1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha - \lambda^\alpha}, \qquad w_\alpha(x) = (1-\alpha) E\big[G_\alpha(\exp(-xG_\alpha))\big]. \tag{2.44}$$

2.3. Proof of point 3 of Theorem 1.1. First we recall this point: *Let* $K_\alpha \overset{(\text{law})}{=} \mathrm{e}/G_\alpha$ *with* e *and* G_α *independent. Then*:

(i) *There exists a* $(1-\alpha, K_\alpha)$ *positive compound Poisson process* $(Y_t, t \geqslant 0)$ *such that*

$$\Delta_\alpha \overset{(law)}{=} \int_0^\infty e^{-t}\, dY_t. \tag{2.45}$$

(ii) Δ_α *satisfies the following affine equation*:

$$\Delta_\alpha \overset{(law)}{=} U^{1/(1-\alpha)}(\Delta_\alpha + K_\alpha), \tag{2.46}$$

where U, Δ_α *and* K_α *on the right-hand side are independent, and* U *is uniformly distributed on* $[0, 1]$.

2.3.1. Proof of (2.45) and (2.46). It hinges upon the following proposition:

PROPOSITION 2.3. *Let* $(Y_t, t \geqslant 0)$ *denote a subordinator, without drift, and with Lévy measure* μ. *Let*

$$X := \int_0^\infty e^{-t}\, dY_t. \tag{2.47}$$

We assume that $X < \infty$ *a.s. which* (from Jurek and Vervaat [15]; see also Erickson and Maller [11]) *is equivalent to*

$$\int_{[1,\infty[} (\log x)\, \mu(dx) < \infty. \tag{2.48}$$

Then:

$$E(e^{-\lambda X}) = \exp\left(-\int_0^\infty (1-e^{-\lambda v})\, \mu([v, \infty[)\frac{dv}{v}\right). \tag{2.49 (i)}$$

In particular, X *is self-decomposable.*

(ii) *If, in addition,* $(Y_t, t \geqslant 0)$ *is a* (γ, K) *compound Poisson process* (*i.e.* $\gamma := \mu(\boldsymbol{R}_+) < \infty$), *then*

$$X \overset{(law)}{=} U^{1/\gamma}(X+K), \tag{2.50}$$

where U, X *and* K *on the right-hand side are independent, and* U *is uniform on* $[0, 1]$.

2.3.2. We prove that *Proposition* 2.3 *implies* (2.45) *and* (2.46). We know from (1.16) that

$$E[\exp(-\lambda\Delta_\alpha)] = \exp\left(-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\, E[\exp(-xG_\alpha)]\frac{dx}{x}\right). \tag{2.51}$$

On the other hand, from the definition of K_α we have

$$P(K_\alpha \geqslant x) = P(\mathrm{e}/G_\alpha \geqslant x) = P(\mathrm{e} > xG_\alpha) = E[\exp(-xG_\alpha)]. \tag{2.52}$$

We denote by μ_α the law of K_α. Then, replacing $E[\exp(-xG_\alpha)]$ in (2.51) by its value as obtained in (2.52), we get

$$E[\exp(-\lambda\Delta_\alpha)] = \exp\left(-(1-\alpha)\int_0^\infty (1-e^{-\lambda x})\,\mu_\alpha([x,\infty[)\frac{dx}{x}\right). \tag{2.53}$$

It then suffices to compare (2.51) and (2.49). Then we apply Proposition 2.3 to obtain (2.45) and (2.46), with $\gamma = 1-\alpha$.

2.3.3. Proof of Proposition 2.3 (see also [15] for the original proof).

2.3.3.a. Approximating $X = \int_0^\infty e^{-t}\,dY_t$ by the Riemann sums $\sum_i \exp(-t_i)\times(Y_{t_{i+1}} - Y_{t_i})$ we obtain

$$\begin{aligned} E(e^{-\lambda X}) &= \exp\left(-\int_0^\infty dt \int_0^\infty (1-\exp(-\lambda e^{-t}x))\,\mu(dx)\right) \\ &= \exp\left(-\int_0^\infty \mu(dx)\left(\int_0^x (1-e^{-\lambda v})\frac{dv}{v}\right)\right) \\ &= \exp\left(-\int_0^\infty (1-e^{-\lambda v})\,\mu([v,\infty])\frac{dv}{v}\right) \end{aligned} \tag{2.54}$$

by Fubini's theorem (after making the change of variables $e^{-t}x = v$ in the last equality).

2.3.3.b. Proof of point (ii) of Proposition 2.3. Recall that $(Y_t,\, t \geqslant 0)$ may be represented as

$$Y_t = \sum_{i=1}^{N_t} K_i,$$

where $(N_t,\, t \geqslant 0)$ denotes a Poisson process with parameter γ, independent of the sequence of i.i.d. variables (K_i). Let T_1 be the first jump time of $(N_t,\, t \geqslant 0)$. Then we have

$$\begin{aligned} X = \int_0^\infty e^{-t}\,dY_t &= \int_0^{T_1} e^{-t}\,dY_t + \int_{T_1}^\infty e^{-t}\,dY_t \\ &= \exp(-T_1)K_1 + \exp(-T_1)\tilde{X}, \end{aligned}$$

where $\tilde{X}$ is independent of (T_1, K_1), and is distributed as X. This proves (2.46), since, as T_1 is exponentially distributed, with parameter γ, we obtain

$$\exp(-T_1) \overset{\text{(law)}}{=} U^{1/\gamma}.$$

2.3.3.c. Another proof of (2.50). If we denote by θ (resp. φ) the Laplace transform of X (resp. K), then, the relation (2.50) is equivalent to

$$\begin{aligned} \theta(\lambda) &= E\left(\int_0^1 \exp[-\lambda u^{1/\gamma}(X+K)]\,du\right) = \gamma E\left(\int_0^1 \exp[-\lambda v(X+K)]\,v^{\gamma-1}\,dv\right) \\ &= \frac{\gamma}{\lambda^\gamma}\int_0^\lambda \theta(u)\,\varphi(u)\,u^{\gamma-1}\,du, \end{aligned}$$

i.e.

(2.55) $$\lambda^\gamma \theta(\lambda) = \gamma \int_0^\lambda \theta(u)\, \varphi(u)\, u^{\gamma-1}\, du,$$

which, taking derivatives, is equivalent to

$$-\theta'(\lambda)/\theta(\lambda) = \gamma(1-\varphi(\lambda)) \qquad (\lambda \geqslant 0),$$

and hence

(2.56) $$\theta(\lambda) = E(e^{-\lambda X}) = \exp\left(-\int_0^\infty (1-e^{-\lambda v})\, \gamma \mu_K([v, \infty[)\frac{dv}{v}\right),$$

where μ_K denotes the law of K. It now remains to observe that the Lévy measure of subordinator $(Y_t, t \geqslant 0)$ is equal to $\gamma \cdot \mu_K$, and then to compare (2.54) and (2.56).

2.4. Remark 2.4. We come back to the result of Winkel (cf. Subsection 1.2). Let $(\tau_l, l \geqslant 0)$ be a subordinator, without drift and with Lévy exponent Φ. Let

$$\Delta^{(\tau)} := \Delta_e$$

with the notation of Subsection 1.2. Hence, by (1.9),

(2.57) $$E[\exp(-\lambda \Delta^{(\tau)})] = \frac{\Phi(1+\lambda) - \Phi(\lambda)}{\Phi(1)}.$$

A natural question is the following: which are the positive r.v.'s Δ such that $\Delta \overset{(\text{law})}{=} \Delta^{(\tau)}$ for some subordinator $(\tau_l, l \geqslant 0)$? The answer to this question is elementary; for any positive r.v. Δ there exists a unique subordinator $(\tau_l, l \geqslant 0)$ without drift and with Lévy exponent Φ, with $\Phi(1) = 1$, such that

(2.58) $$\Delta \overset{(\text{law})}{=} \Delta^{(\tau)}.$$

2.4.1. Proof of Remark 2.4. Let ψ be the Laplace transform of Δ and denote by μ_Δ the law of Δ. Then

$$\psi(\lambda) = E(e^{-\lambda \Delta}) = \int_0^\infty e^{-\lambda x}\, d\mu_\Delta(x) = \int_0^\infty e^{-\lambda x}(1-e^{-x})\frac{\mu_\Delta(dx)}{1-e^{-x}}.$$

Let $\tilde{\nu}_\Delta$ be defined by

(2.59) $$\tilde{\nu}_\Delta(dx) = \frac{1}{1-e^{-x}}\mu_\Delta(dx).$$

There is no difficulty in showing that $\tilde{\nu}_\Delta$ is a Lévy measure, i.e.

$$\int_0^\infty (x \wedge 1)\, \tilde{\nu}_\Delta(dx) < \infty.$$

Let Φ denote the associated Bernstein function and $(\tau_l, l \geqslant 0)$ the correspon-ding subordinator

$$\Phi(\lambda) = \int_0^\infty (1-e^{-\lambda x})\tilde{\nu}_\Delta(dx).$$

We have

$$(2.60)\qquad \psi(\lambda) = \int_0^\infty e^{-\lambda x}(1-e^{-x})\tilde{\nu}_\Delta(dx)$$

$$= \int_0^\infty (1-e^{-(\lambda+1)x})\tilde{\nu}_\Delta(dx) - \int_0^\infty (1-e^{-\lambda x})\tilde{\nu}_\Delta(dx) = \Psi(1+\lambda)-\Psi(\lambda).$$

It is clear that $\Psi(0) = 1 = \Phi(1)-\Phi(0) = \Phi(1)$. Then, from (2.57) and (2.60) we obtain

$$E[\exp(-\lambda\Delta^{(\tau)})] = \Phi(1+\lambda)-\Phi(\lambda) = \Psi(\lambda) = E(e^{-\lambda\Delta}),$$

that is

$$\Delta^{(\tau)} \overset{\text{(law)}}{=} \Delta.$$

The uniqueness of $(\tau_l, l \geqslant 0)$ may be proved by using similar arguments.

3. PROPERTIES OF THE VARIABLES G_α $(0 < \alpha < 1)$ PROOFS OF THEOREMS 1.2 AND 1.3

3.1. Proof of point 1 of Theorem 1.2. First we recall this point: *$G_{1/2}$ is arc-sine distributed; i.e. it is distributed as beta $(\frac{1}{2}, \frac{1}{2})$:*

$$(3.1)\qquad f_{G_{1/2}}(u) = \frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}1_{[0,1]}(u).$$

Proof of 3.1. It suffices to take $\alpha = 1/2$ in (1.17) or to note that

$$E[\exp(-\lambda e\beta(1/2, 1/2))] = E[\exp(-\lambda\gamma_{1/2})] = \frac{1}{\sqrt{1+\lambda}} = \frac{1-(1+\lambda)^{-1/2}}{(1+\lambda)^{1/2}-1}$$

$$= E\left(\frac{1}{1+\lambda\beta_{(1/2,1/2)}}\right) = E\left(\frac{1}{1+\lambda G_{1/2}}\right),$$

where the last equality follows from (1.19).

3.2. Proof of point 2 of Theorem 1.2:
If $\alpha = 1/p$, with p an integer, $p \geqslant 2$, then

$$f_{G_\alpha}(u) = f_{G_{1/p}}(u) = \frac{1}{\pi(p-1)}\sum_{i=1}^{p-1}\sin\left(\frac{\pi i}{p}\right)u^{i/p-1}(1-u)^{-i/p}1_{[0,1]}(u).$$

In fact, this is the formula following (2.37), which was proved above in 2.2.3.b. We obtain (1.26) from (1.25) after the change of index $j=p-i$.

3.3. Proof of point 3 of Theorem 1.2:

$$G_\alpha \overset{(law)}{=} 1-G_\alpha. \tag{3.2}$$

Thanks to (1.17), or (1.18), this relation is obvious.

3.4. Proof of point 4 of Theorem 1.2:

As $\alpha \to 1$, G_α converges in law to a uniformly distributed r.v. on [0, 1]. To prove this assertion it is sufficient to observe that

$$\lim_{\alpha\to 1} E\left(\frac{1}{1+\lambda G_\alpha}\right)=\lim_{\alpha\to 1}\frac{\alpha}{1-\alpha}\,\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1}=\frac{\log(1+\lambda)}{\lambda},$$

where the last equality follows from (1.19). Moreover, if U is a uniform r.v. on [0, 1], we have

$$E\left(\frac{1}{1+\lambda U}\right)=\int_0^1 \frac{1}{1+\lambda u}\,du=\frac{1}{\lambda}[\log(1+\lambda)-\log(1)]=\frac{\log(1+\lambda)}{\lambda}.$$

3.5. Proof of point 5 of Theorem 1.2:

As $\alpha\to 0$, G_α converges in law to an r.v. G_0 which satisfies

$$\text{(i)}\quad f_{G_0}(u)=\frac{1}{\pi}\Big(\int_0^1(\sin(\pi\beta))\,u^{\beta-1}(1-u)^{-\beta}\,d\beta\Big)\,1_{[0,1]}(u) \tag{3.3}$$

$$=\frac{1}{u(1-u)}\,\frac{1}{\pi^2+(\log(1-u)/u)}\,1_{[0,1]}(u); \tag{3.4}$$

$$\text{(ii)}\quad G_0 \overset{(law)}{=} \frac{1}{1+\exp(\pi C)}, \tag{3.5}$$

where C is a standard Cauchy variable.

3.5.1. Proof of (3.3) and (3.4).

3.5.1.a. We first note that formula (3.3) indicates, with the notation used in formula (2.32), that the measure $\nu_0(d\gamma)$ is Lebesgue measure on the interval [0, 1]. On the other hand, from (1.25) we obtain

$$f_{G_{1/p}}(u)=\left(\frac{1}{\pi(p-1)}\sum_{i=1}^{p-1}\sin\left(\frac{\pi i}{p}\right)u^{i/p-1}(1-u)^{-i/p}\right)1_{[0,1]}(u)$$

$$\xrightarrow{p\to\infty}\frac{1}{\pi}\Big(\int_0^1(\sin(\pi\beta))\,u^{\beta-1}(1-u)^{-\beta}\,d\beta\Big)\,1_{[0,1]}(u),$$

which proves (3.3). In fact, we have only studied the limit, as $p\to\infty$, of $f_{G_{1/p}}$.

But the explicit formula (1.17), which gives f_{G_α}, easily shows that, as $\alpha \downarrow 0$, f_{G_α} converges (to f_{G_0}).

3.5.1.b. The relation (3.4) follows from

$$\frac{1}{\pi}\int_0^1 (\sin(\pi\beta))\, u^{\beta-1}(1-u)^{-\beta}\, d\beta = \frac{1}{u\pi}\operatorname{Im}\int_0^1 \exp\left(\beta\left(i\pi+\log\frac{u}{1-u}\right)\right) d\beta$$

$$= \frac{1}{u(1-u)}\frac{1}{\pi^2+(\log[u/(1-u)])^2}.$$

3.5.1.c. Proof of (3.5). We observe from (3.4) that

$$E\left(\frac{1}{1+\lambda G_0}\right) = \int_0^1 \frac{1}{1+\lambda u}\frac{1}{u(1-u)}\frac{1}{\pi^2+(\log[u/(1-u)])^2}\, du$$

and after the change of variable $u/(1-u) = v$ and then $\log v = \pi w$ we get

$$E\left(\frac{1}{1+\lambda G_0}\right) = \int_0^\infty \frac{1+v}{1+v+\lambda v}\frac{1}{\pi^2+\log^2 v}\, dv = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1+e^{\pi w}}{1+(\lambda+1)e^{\pi w}}\frac{dw}{1+w^2},$$

whereas

$$E\left(\frac{1}{1+\lambda/(1+e^{\pi C})}\right) = E\left(\frac{1+e^{\pi C}}{1+\lambda+e^{\pi C}}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1+e^{-\pi x}}{1+\lambda+e^{-\pi x}}\frac{dx}{1+x^2}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1+e^{\pi w}}{1+(1+\lambda)e^{\pi w}}\frac{dw}{1+w^2},$$

which yields (3.4). Below (cf. Remark 3.2), we shall give another proof of the convergence in law of G_α, as $\alpha \to 0$, towards $(1+\exp(\pi C))^{-1}$. This completes the proof of Theorem 1.2. ∎

3.6. Remark 3.1 (A relation between G_0 and the gamma subordinator).

3.6.1. For any λ and μ positive reals, we write, using (1.15) and (1.16),

$$(3.6)\quad \frac{E[\exp(-(\lambda+\mu)\Delta_\alpha)]}{E[\exp(-\mu\Delta_\alpha)]} = \frac{(1+\lambda+\mu)^\alpha-(\lambda+\mu)^\alpha}{(1+\mu)^\alpha-\mu^\alpha}$$

$$= \exp\left(-(1-\alpha)\int_0^\infty (1-e^{-(\lambda+\mu)x}-1+e^{-\mu x})\, E[\exp(-xG_\alpha)]\frac{dx}{x}\right).$$

Letting $\alpha \to 0$ on both sides of (3.6), and using the already proved fact that $G_\alpha \overset{\text{(law)}}{\to} G_0$ as $\alpha \to 0$, we obtain

$$(3.7)\quad \frac{\log(1+\lambda+\mu)-\log(\lambda+\mu)}{\log(1+\mu)-\log\mu}$$

$$= \exp\left(-\int_0^\infty e^{-\mu x}(1-e^{-\lambda x})\, E[\exp(-xG_0)]\frac{dx}{x}\right).$$

3.6.2. We denote by Φ_μ the Lévy exponent of the subordinator $(\mu^{-1}\gamma_t, t \geqslant 0)$, where $(\gamma_t, t \geqslant 0)$ denotes the standard gamma subordinator. Thus

$$(3.8) \qquad E\left[\exp\left(-\frac{\lambda}{\mu}\gamma_t\right)\right] = \frac{1}{(1+\lambda/\mu)^t} = \exp\{-t(\log(\lambda+\mu)-\log\mu)\},$$

i.e.

$$\Phi_\mu(\lambda) = \log(\lambda+\mu) - \log\mu.$$

Hence, formula (3.7) takes the form

$$(3.9) \qquad \frac{\Phi_\mu(1+\lambda)-\Phi_\mu(\lambda)}{\Phi_\mu(1)} = \exp\left(-\int_0^\infty (1-e^{-\lambda x})\, e^{-\mu x} E[\exp(-xG_0)]\frac{dx}{x}\right).$$

3.6.3. Let $(X_t^{(\mu)}, t \geqslant 0)$ denote a diffusion process whose inverse local time $(\tau_l^{(\mu)}, l \geqslant 0)$ at 0 is distributed as $(\mu^{-1}\gamma_l, l \geqslant 0)$. Such a diffusion $(X_t^{(\mu)}, t \geqslant 0)$ has been described explicitly by Donati-Martin and Yor (cf. [9]) as an illustration of Krein's representation of subordinators. Furthermore, we define, for $t \geqslant 0$,

$$(3.10) \qquad \begin{gathered} g_t^{(\mu)} := \sup\{s \leqslant t, X_s^{(\mu)} = 0\}, \qquad d_t^{(\mu)} := \inf\{s \geqslant t; X_s^{(\mu)} = 0\}, \\ \Delta_{(\mu)} := d_{\mathrm{e}}^{(\mu)} - g_{\mathrm{e}}^{(\mu)}, \end{gathered}$$

where e denotes a standard exponential variable independent of $(X_t^{(\mu)}, t \geqslant 0)$. Then, as we apply Lemma 2.1, formula (3.9) becomes

$$(3.11) \quad E[\exp(-\lambda\Delta_{(\mu)})] = \exp\left(-\int_0^\infty (1-e^{-\lambda x})\, E[\exp(-x(\mu+G_0))]\frac{dx}{x}\right).$$

It follows from (3.11) that $\Delta_{(\mu)}$ is self-decomposable.

We note that this formula (3.11) is quite similar to (1.16), when we replace:

- the stable (α) process $(\tau_l^{(\alpha)}, l \geqslant 0)$ by the gamma process $(\mu^{-1}\gamma_l, l \geqslant 0)$;
- the r.v. G_α by the r.v. $\mu+G_0$ (and also replace the coëfficient $(1-\alpha)$ by 1 in (1.16)).

It is tempting to let μ tend to 0 in (3.11). However, this is not possible for two reasons:

(i) the process $(\mu^{-1}\gamma_l, l \geqslant 0)$ does not converge as $\mu \to 0$;

(ii) the measure $x^{-1}E[\exp(-xG_0)]\,dx$ is not integrable near ∞ (as $E(\log(1/G_0)) = \infty$), and hence it does not define a Lévy measure.

3.7. Proof of Theorem 1.3 (Links between the r.v.'s G_α, the unilateral stable variables, and the Mittag-Leffler distribution). We refer the reader to the Introduction (Subsection 1.5) for the definitions of T_μ, T'_μ, Z_μ and M_μ $(\mu \in]0, 1[)$.

3.7.1. Proof of point 1 of Theorem 1.3:
Z_μ admits the density:

$$f_{Z_\mu}(x) = \frac{\sin(\pi\mu)}{\pi\mu} \frac{1}{x^2+2x\cos(\pi\mu)+1} 1_{[0,\infty[}(x). \tag{3.12}$$

In fact, formula (3.12) is due to Lamperti [16]. A proof of (3.12) can be also found in Chaumont and Yor (cf. [7], ex. 4.21, p. 116). We refer the interested reader to this proof.

3.7.2. Proof of point 2 (i) of Theorem 1.3:

$$G_\alpha \overset{(law)}{=} \frac{(Z_{1-\alpha})^{1/\alpha}}{1+(Z_{1-\alpha})^{1/\alpha}} \overset{(law)}{=} \frac{(T_{1-\alpha})^{(1-\alpha)/\alpha}}{(T'_{1-\alpha})^{(1-\alpha)/\alpha}+(T_{1-\alpha})^{(1-\alpha)/\alpha}}. \tag{3.13}$$

To prove this formula we shall show that $(G_\alpha/(1-G_\alpha))^\alpha$ is distributed as $Z_{1-\alpha}$, which implies (3.13). Indeed, for any h: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$, Borel, we have

$$\begin{aligned} E\left[h\left(\left(\frac{G_\alpha}{1-G_\alpha}\right)^\alpha\right)\right] &= \frac{\alpha}{1-\alpha}\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty h\left(\left(\frac{u}{1-u}\right)^\alpha\right)\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{(1-u)^{2\alpha}-2(1-u)^\alpha u^\alpha\cos(\pi\alpha)+u^{2\alpha}}du \\ &= \frac{\sin(\pi\alpha)}{\pi(1-\alpha)}\int_0^\infty h(x)\frac{dx}{x^2-2x\cos(\pi\alpha)+1} \end{aligned}$$

(after making the change of variables $(u/(1-u))^\alpha = x$). Consequently, by (3.12),

$$\begin{aligned} E\left[h\left(\left(\frac{G_\alpha}{1-G_\alpha}\right)^\alpha\right)\right] &= \frac{\sin(\pi(1-\alpha))}{\pi(1-\alpha)}\int_0^\infty h(x)\frac{dx}{x^2+2x\cos(\pi(1-\alpha))+1} \\ &= E[h(Z_{1-\alpha})]. \end{aligned}$$

3.7.3. Proof of point 2 (ii) of Theorem 1.3:

$$G_\alpha \overset{(law)}{=} \frac{(M_{1-\alpha})^{1/\alpha}}{(M_{1-\alpha})^{1/\alpha}+(M'_{1-\alpha})^{1/\alpha}}, \tag{3.14}$$

where $M_{1-\alpha}$ and $M'_{1-\alpha}$ on the right-hand side denote two independent r.v.'s, with the Mittag-Leffler distribution with parameter $1-\alpha$.

To prove (3.14), we use (cf. Introduction, Subsection 1.5)

$$E(M_\mu^n) = \frac{\Gamma(n+1)}{\Gamma(\mu n+1)} \quad (n > -1). \tag{3.15}$$

On the other hand, using the elementary formula we get

$$(3.16)\qquad E\left(\frac{1}{T_\mu^{\mu n}}\right)=\frac{1}{\Gamma(\mu n)}\int_0^\infty u^{\mu n-1}E[\exp(-uT_\mu)]\,du$$
$$=\frac{1}{\Gamma(\mu n)}\int_0^\infty u^{\mu n-1}\exp(-u^\mu)\,du=\frac{\Gamma(1+n)}{\Gamma(\mu n+1)}\qquad (n>-1).$$

Now, comparing (3.16) and (3.15), we deduce that

$$M_\mu \stackrel{\text{(law)}}{=} 1/(T_\mu)^\mu,$$

and (3.14) now follows from (3.13).

3.8. Remark 3.2. We present here another proof of the convergence in law of G_α, as $\alpha\to 0$, to $1/(1+e^{\pi C})$, where C is a standard Cauchy r.v. It suffices to prove that

$$\log(1-G_\alpha)-\log(G_\alpha)\stackrel{\text{(law)}}{\to}\pi C\qquad\text{as } \alpha\to 0$$

or, by (3.13), that

$$(3.17)\qquad \frac{1}{\alpha}(\log(T'_{1-\alpha})-\log(T_{1-\alpha}))\stackrel{\text{(law)}}{\to}\pi C\qquad\text{as } \alpha\to 0,$$

where $T_{1-\alpha}$ and $T'_{1-\alpha}$ are two independent copies of a one-sided stable $(1-\alpha)$ r.v. But $T_{1-\alpha}\to 1$, as $\alpha\to 0$, in probability. Hence (3.17) is equivalent to

$$(3.18)\qquad \frac{1}{\alpha}(T'_{1-\alpha}-T_{1-\alpha})\stackrel{\text{(law)}}{\to}\pi C\qquad\text{as } \alpha\to 0.$$

We prove (3.18):

$$E\left[\exp\left(i\frac{\lambda}{\alpha}T_{1-\alpha}\right)\right]=E\left[\exp\left(\frac{\lambda}{\alpha}\exp\left(i\frac{\pi}{2}\right)T_{1-\alpha}\right)\right]$$
$$=\exp\left\{-\frac{|\lambda|^{1-\alpha}}{\alpha^{1-\alpha}}\exp\left(i\frac{\pi}{2}(1-\alpha)\right)\right\}$$
$$=\exp\left\{-\frac{|\lambda|^{1-\alpha}}{\alpha^{1-\alpha}}\left[\cos\left(\frac{\pi}{2}(1-\alpha)\right)+i\sin\left(\frac{\pi}{2}(1-\alpha)\right)\right]\right\}.$$

Hence

$$E\left[\exp\left(i\frac{\lambda}{\alpha}(T_{1-\alpha}-T'_{1-\alpha})\right)\right]=\left|E\left[\exp\left(i\frac{\lambda}{\alpha}T_{1-\alpha}\right)\right]\right|^2$$
$$=\exp\left\{-2|\lambda|^{1-\alpha}\alpha^\alpha\frac{1}{\alpha}\cos\left(\frac{\pi}{2}(1-\alpha)\right)\right\}$$
$$=\exp\left(-2|\lambda|^{1-\alpha}\alpha^\alpha\frac{\sin((\pi/2)\alpha)}{\alpha}\right)$$
$$\to\exp(-\pi|\lambda|)=E(\exp(i\lambda\pi C))\qquad\text{as } \alpha\to 0.$$

4. PROOF OF THEOREM 1.4 (ON THE ALGEBRA OF VARIABLES G, X, γ)

4.1. Proof of points 1 and 2 of Theorem 1.4. First we recall these points.

Point 1. *For every a, b such that $0 < a \leqslant b < 1$ there exists an r.v. $X_{a,b}$ such that*

$$E[\exp(-\lambda X_{a,b})] = \frac{b(1+\lambda)^a - 1}{a(1+\lambda)^b - 1} \quad (\lambda \geqslant 0). \tag{4.1}$$

Point 2. *For every $0 < a_1 < \ldots < a_n < 1$,*

$$X_{a_1,a_n} \stackrel{(law)}{=} \sum_{i=1}^{n-1} X_{a_i,a_{i+1}}, \tag{4.2}$$

where the variables on the right-hand side are assumed to be independent.

The r.v.'s $X_{a,b}$ are infinitely divisible.

4.1.1. Proof of (4.1).

4.1.1.a. For this purpose, we shall work in a slightly more general framework than what we strictly need. We first recall that we use the term Bernstein function for a function $\Phi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ of the form

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\,\nu(dx) \tag{4.3}$$

for $\nu(dx) \geqslant 0$ such that $\int_0^\infty (1 \wedge x)\,\nu(dx) < \infty$.

In other terms, Φ is the Lévy exponent of a subordinator $(T_y,\ y \geqslant 0)$ with Lévy measure $\nu(dx)$, and without drift term, i.e.

$$E[\exp(-\lambda T_y)] = \exp(-y\Phi(\lambda)). \tag{4.4}$$

Lemma 4.1. *Let Φ_1, Φ_2, Φ_3 denote three Bernstein functions which satisfy*

(i) $\Phi_1 = \Phi_3 \circ \Phi_2$;

(ii) $\int_0^\infty x\nu_3(dx) < \infty$, *where ν_3 denotes the Lévy measure associated with Φ_3.*

Then there exists a positive r.v. X such that

$$E(e^{-\lambda X}) = \frac{1}{C_3}\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} \quad \textit{with } C_3 = \int_0^\infty x\nu_3(dx).$$

Moreover:

$$E(e^{-\lambda X}) = \frac{1}{C_3}\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = \frac{1}{C_3}E\Big(\int_0^\infty \exp(-\lambda T_y^{(2)})\,\bar{\nu}_3(y)\,dy\Big) \quad (\lambda \geqslant 0), \tag{4.5}$$

where

(4.6) $(T_y^{(2)},\ y \geqslant 0)$ *denotes the subordinator associated with Φ_2*;

(4.7) $\bar{\nu}_3(y) = \nu_3([y, \infty))$ *is the tail of ν_3.*

Proof of Lemma 4.1. We have

$$\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}=\frac{\Phi_3(\Phi_2(\lambda))}{\Phi_2(\lambda)}=\int_0^\infty\left(\frac{1-\exp(-\Phi_2(\lambda)x)}{\Phi_2(\lambda)}\right)\nu_3(dx)$$

$$=\int_0^\infty \nu_3(dx)\int_0^x \exp(-\Phi_2(\lambda)y)\,dy$$

$$=\int_0^\infty \exp(-\Phi_2(\lambda)y)\,\bar\nu_3(y)\,dy \qquad \text{(Fubini)}.$$

Hence

$$\frac{1}{C_3}\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}=\frac{1}{C_3}E\Big(\int_0^\infty \exp(-\lambda T_y^{(2)})\,\bar\nu_3(y)\,dy\Big),$$

which proves Lemma 4.1, once we have observed that

$$\int_0^\infty \bar\nu_3(y)\,dy=\int_0^\infty dy\int_y^\infty \nu_3(dx)=\int_0^\infty x\nu_3(dx)=C_3.$$

4.1.1.b. We now prove (4.1). For any $\delta\in]0,1[$ we write

$$\Phi_\delta(\lambda)=(1+\lambda)^\delta-1. \tag{4.8}$$

Φ_δ is a Bernstein function since

$$\Phi_\delta(\lambda)=\frac{\delta}{\Gamma(1-\delta)}\int_0^\infty (1-e^{-\lambda x})\frac{e^{-x}\,dx}{x^{\delta+1}} \tag{4.9}$$

(in fact, Φ_δ is the Lévy exponent of the Esscher transform (cf. [21]) of the stable (δ) subordinator) with associated Lévy measure

$$\nu_\delta(dx)=\frac{\delta}{\Gamma(1-\delta)}\frac{e^{-x}}{x^{\delta+1}}1_{[0,\infty[}(x)\,dx. \tag{4.10}$$

In particular,

$$\int_0^\infty x\nu_\delta(dx)=\delta. \tag{4.11}$$

In the sequel, δ denotes either a, b, or $c:=a/b<1$, where $0<a<b<1$. Note that

$$\begin{aligned}\Phi_c(\Phi_b(\lambda))&=(\Phi_b(\lambda)+1)^c-1=((1+\lambda)^b-1+1)^c-1\\&=(1+\lambda)^{bc}-1=(1+\lambda)^a-1=\Phi_a(\lambda)\end{aligned} \tag{4.12}$$

and that

$$\int_0^\infty x\nu_c(dx)=c=a/b<\infty. \tag{4.13}$$

We may then use Lemma 4.1 with $\Phi_1 = \Phi_a$, $\Phi_2 = \Phi_b$ and $\Phi_3 = \Phi_c$; given (4.12) and (4.13), we deduce the existence of an $\boldsymbol{R}_+$-valued r.v. such that

$$E[\exp(-\lambda X_{a,b})] = \frac{1}{C_3}\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = \frac{b(1+\lambda)^a - 1}{a(1+\lambda)^b - 1} \qquad (\lambda \geqslant 0).$$

4.1.1.c. We now prove (4.2). This follows immediately from the definition of $X_{a,b}$ and from the obvious formula

$$\frac{a_n}{a_1}\frac{(1+\lambda)^{a_1}-1}{(1+\lambda)^{a_n}-1} = \prod_{i=1}^{n-1} \frac{a_{i+1}}{a_i}\frac{(1+\lambda)^{a_i}-1}{(1+\lambda)^{a_{i+1}}-1} \qquad (\lambda \geqslant 0). \tag{4.14}$$

4.1.1.d. We now prove the *infinite divisibility of* $X_{a,b}$. We may write, from (4.2):

$$X_{a,b} \overset{(\text{law})}{=} \sum_{i=0}^{n-1} X_{a+i\frac{b-a}{n},\, a+(i+1)\frac{b-a}{n}}. \tag{4.15}$$

We know (cf. [18], pp. 314–321) that $X_{a,b}$ is infinitely divisible as soon as the following condition (called "uan") is satisfied:

$$\forall \varepsilon > 0, \quad \sup_{i=0,1,2,\dots,n-1} P(X_{a+i\frac{b-a}{n},\, a+(i+1)\frac{b-a}{n}} > \varepsilon) \to 0 \quad \text{as } n \to \infty. \tag{4.16}$$

But, by differentiation of (4.1), we obtain

$$E(X_{a,b}) = \frac{b-a}{2}. \tag{4.17}$$

Thus,

$$\delta_i^{(n)} := P(X_{a+\frac{i(b-a)}{n},\, a+\frac{(i+1)(b-a)}{n}} > \varepsilon) \leqslant \frac{b-a}{2n\varepsilon},$$

and hence

$$\sup_{i=0,1,2,\dots,n-1} \delta_i^{(n)} \leqslant \frac{b-a}{2n\varepsilon} \to 0 \quad \text{as } n \to \infty.$$

4.2. Remark 4.2

4.2.1 (Self-decomposability of $X_{c,1}$, $0 < c < 1$):

Let $X_{c,1}$ denote an r.v. whose law is characterized by

$$E[\exp(-\lambda X_{c,1})] = \frac{1}{c}\frac{(1+\lambda)^c - 1}{\lambda}. \tag{4.18}$$

Then $X_{c,1}$ is infinitely divisible, and its Lévy measure $\mu_{c,1}$ is given by

$$\mu_{c,1}(dx) = (1-c)E[\exp(-x/G_c)]\frac{dx}{x}. \tag{4.19}$$

Proof of (4.19). In order to prove that

$$\frac{1}{c}\frac{(1+\lambda)^c-1}{\lambda}=\exp\Big(-\int_0^\infty(1-e^{-\lambda x})\,\mu_{c,1}(dx)\Big) \tag{4.20}$$

we take logarithmic derivatives of both sides. Thus

$$\frac{1}{\lambda}-c\frac{(1+\lambda)^{c-1}}{(1+\lambda)^c-1}=\int_0^\infty e^{-\lambda x}\,x\mu_{c,1}(dx). \tag{4.21}$$

Denoting by (L) the left-hand side of (4.21), we get, using (1.19),

$$\begin{aligned}(L)&=\frac{1-c}{\lambda}+c\left(\frac{1}{\lambda}-\frac{(1+\lambda)^{c-1}}{(1+\lambda)^c-1}\right)\\ &=\frac{1-c}{\lambda}+\frac{c}{\lambda}\frac{(1+\lambda)^{c-1}-1}{(1+\lambda)^c-1}=(1-c)\left(\frac{1}{\lambda}-\frac{1}{\lambda}\frac{c}{1-c}\frac{1-(1+\lambda)^{c-1}}{(1+\lambda)^c-1}\right)\\ &=(1-c)\Big[\int_0^\infty e^{-\lambda x}\,dx-\Big(\int_0^\infty e^{-\lambda x}\,dx\Big)E[\exp(-\lambda \mathrm{e}G_c)]\Big].\end{aligned}$$

We then deduce from (4.21) that

$$\mu_{c,1}(dx)=(1-c)\frac{1}{x}[(\delta_0-\mu_{\mathrm{e}G_c})*l_+]\,dx, \tag{4.22}$$

where l_+ denotes Lebesgue measure on $\boldsymbol{R}_+$, and $\mu_{\mathrm{e}G_c}$ the law of $\mathrm{e}G_c$. The explicit computation of the convolution in (4.22) easily leads to (4.19). We note that the obtained formula:

$$E[\exp(-\lambda X_{c,1})]=\exp\left(-(1-c)\int_0^\infty(1-e^{-\lambda x})\,E[\exp(-x/G_c)]\frac{dx}{x}\right), \tag{4.23}$$

may be compared with the "dual" formula (1.16)

$$E[\exp(-\lambda\Delta_c)]=\exp\left(-(1-c)\int_0^\infty(1-e^{-\lambda x})\,E[\exp(-xG_c)]\frac{dx}{x}\right).$$

On the other hand, formula (4.23) implies that $X_{c,1}$ is self-decomposable.

4.2.2 (Self-decomposability of $X_{a,b}$, $0<a<b<1$). Writing

$$X_{a,b}+X_{b,1}\overset{(\text{law})}{=}X_{a,1}$$

we deduce that the Lévy measure $\nu_{a,b}$ of $X_{a,b}$ equals

$$\nu_{a,b}(dx)=\frac{1}{x}\{(1-a)\,E[\exp(-x/G_a)]-(1-b)\,E[\exp(-x/G_b)]\}\,dx. \tag{4.24}$$

We prove now that $X_{a,b}$ is self-decomposable. From (4.24), this assertion, equivalent to

$$\varphi_{a,b}(x) := (1-a)\,E[\exp(-x/G_a)] - (1-b)\,E[\exp(-x/G_b)],$$

is a decreasing function (of x), or, by derivation,

$$(1-a)\,E\left(\frac{1}{G_a}\exp(-x/G_a)\right) - (1-b)\,E\left(\frac{1}{G_b}\exp(-x/G_b)\right) \geqslant 0$$

or, taking the Laplace transform in x of this expression,

$$\psi(\lambda) := (1-a)\,E\left(\frac{1}{1+\lambda G_a}\right) - (1-b)\,E\left(\frac{1}{1+\lambda G_b}\right)$$

is the Laplace transform of a positive function. But this assertion is an easy consequence of the following

LEMMA 4.3. *For any* $0 < a < b < 1$ *and any* $u \in [0, 1]$

$$(1-a)\,f_{G_a}(u) \geqslant (1-b)\,f_{G_b}(u). \tag{4.25}$$

Indeed, with $h(x) := (1-a)\,f_{G_a}(x) - (1-b)\,f_{G_b}(x)$ we have

$$\psi(\lambda) = \int_0^1 \frac{dx}{1+\lambda x}\,h(x) = \int_0^1 h(x)\,dx \int_0^\infty \frac{1}{x}\exp\left(-\lambda u - \frac{u}{x}\right)du$$

$$= \int_0^\infty e^{-\lambda u}\,du \int_0^1 \frac{1}{x}\exp\left(-\frac{u}{x}\right)h(x)\,dx.$$

We now prove (4.25). By (1.17), we need to show that

$$a\sin(\pi a)\frac{(1-u)^{a-1}u^{a-1}}{(1-u)^{2a} - 2(1-u)^a u^a \cos(\pi a) + u^{2a}}$$

$$= a\sin(\pi a)\frac{((1-u)/u)^{a-1}u^2}{((1-u)/u)^{2a} - 2((1-u)/u)^a \cos(\pi a) + 1}$$

is greater than the same expression where we replace a by b (with $a < b$). Then, putting $(1-u)/u = x$, we have to prove that

$$\frac{a\sin(\pi a)}{b\sin(\pi b)} \geqslant \frac{x^{a+b} - 2x^b\cos(\pi a) + x^{b-a}}{x^{2b} - 2x^b\cos(\pi b) + 1} := \theta(x).$$

But it is easy to verify that $\theta(x) \to 0$ as $x \to +\infty$, $\theta(x) \to 0$ as $x \to 0$, and that $\theta(x)$ reaches its maximum for $x = 1$. The value of this maximum equals

$$\frac{1-\cos(\pi a)}{1-\cos(\pi b)}.$$

Hence, Lemma 4.2 will be proved if we show that

$$\frac{a\sin(\pi a)}{b\sin(\pi b)} \geqslant \frac{1-\cos(\pi a)}{1-\cos(\pi b)} \qquad (0<a<b<1).$$

But, this relation is equivalent to

$$\frac{1}{a}\operatorname{tg}\left(\frac{\pi a}{2}\right) \leqslant \frac{1}{b}\operatorname{tg}\left(\frac{\pi b}{2}\right),$$

i.e. the function $x \to x^{-1}\operatorname{tg}(x)$ is increasing on $[0, \pi/2[$. We have

$$\frac{1}{a}\operatorname{tg}\left(\frac{\pi a}{2}\right) = \frac{1}{\pi}\sum_{n\geqslant 1}\frac{1}{(n-1/2)^2-a^2/4}$$

$$\leqslant \frac{1}{\pi}\sum_{n\geqslant 1}\frac{1}{(n-1/2)^2-b^2/4} = \frac{1}{b}\operatorname{tg}\left(\frac{\pi b}{2}\right). \blacksquare$$

We note that for $0<b<1$ we also have (we define $X_{0,b}$ as the limit in law of $X_{a,b}$ for $a\downarrow 0$):

$$E[\exp(-\lambda X_{0,b})] = b\frac{\log(1+\lambda)}{(1+\lambda)^b-1} \quad \text{and} \quad E[\exp(-\lambda X_{0,1})] = \frac{\log(1+\lambda)}{\lambda}.$$

From the latter relation we easily deduce

$$X_{0,1} \overset{\text{(law)}}{=} \mathfrak{e}\cdot U$$

with $\mathfrak{e}$ and U independent, $\mathfrak{e}$ a standard exponential variable, and U uniform on $[0, 1]$. The density of $X_{0,1}$ equals

$$f_{X_{0,1}}(x) = \left(\int_x^\infty \frac{e^{-t}}{t}\,dt\right) 1_{(x\geqslant 0)}$$

and its Lévy measure, from (1.29), is equal to

$$\nu_{0,1}(dx) = \frac{1}{x}E[\exp(-x(1+e^{\pi C}))]\,dx$$

with C a standard Cauchy r.v., i.e.

$$E[\exp(-\lambda X_{0,1})] = \frac{\log(1+\lambda)}{\lambda} = \exp\left(-\int_0^\infty (1-e^{-\lambda x})\frac{E[\exp(-x(1+e^{\pi C}))]}{x}\,dx\right).$$

4.3. Remark 4.4. Let us come back to Lemma 4.1. Under the hypotheses of this lemma, there exists an $\boldsymbol{R}_+$-valued r.v. X such that

$$E(e^{-\lambda X}) = \frac{1}{C_3}\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}. \tag{4.26}$$

It is natural to look for some criterion which ensures that X is infinitely divisible. Some further hypothesis on the Bernstein functions Φ_1, Φ_2, and Φ_3 is needed. Here is a framework which yields a positive answer to our question. For the sequel of the discussion in this remark, we refer the reader to Bertoin and Le Gall [4]. Let us assume that the functions Φ_1 and Φ_2 are related to a continuous branching process. More precisely, let $(Z(t, x);\ t, x \geqslant 0)$ denote a continuous branching process, where t indicates the time parameter, and $x = Z(0, x)$ is the initial size of the population. Then

$$E[\exp(-\lambda Z(t, x))] = \exp(-xu(t, \lambda)), \tag{4.27}$$

where $u(t, \lambda)$ solves the differential equation

$$\frac{\partial}{\partial t} u(t, \lambda) = -\psi(u(t, \lambda)) \tag{4.28}$$

with ψ denoting the branching mechanism of Z.

For each $t \geqslant 0$, $\lambda \to u(t, \lambda)$ is a Bernstein function and

$$u(t+s, \lambda) = u(t, u(s, \lambda)). \tag{4.29}$$

The relation (4.29) plays here the role of the relation $\Phi_1 = \Phi_3 \circ \Phi_2$ with

$$\Phi_1(\lambda) = u(t+s, \lambda), \quad \Phi_2(\lambda) = u(s, \lambda) \quad \text{and} \quad \Phi_3(\lambda) = u(t, \lambda).$$

In this new set-up, we copy again the relation (4.14), which now takes the form

$$u(t+s, \lambda) = \prod_{i=0}^{n-1} \frac{u(t+is/n, \lambda)}{u(t+(i+1)s/n, \lambda)} \tag{4.30}$$

and we notice, as in point 4.1.1.d above, the infinite divisibility of the r.v. whose Laplace transform (in λ) equals

$$\frac{u(t+s, \lambda)}{u(t, \lambda)}.$$

We also note that the Bernstein function $\Phi_a(\lambda) = (1+\lambda)^a - 1$ $(0 < a < 1)$ coincides with $u(t, \lambda)$, for $a = e^{-t}$, and ψ the branching mechanism:

$$\psi(q) = (1+q)\log(1+q)$$

(see [4]). Point 1 of Theorem 1.4 is a particular case of the situation that we just described in Remark 4.4.

4.4. Remark 4.5. The relation (4.2):

$$X_{a_1, a_n} \overset{\text{(law)}}{=} \sum_{i=1}^{n-1} X_{a_i, a_{i+1}} \quad (0 < a_1 < \ldots < a_n < 1), \tag{4.31}$$

where the variables on the right-hand side are independent, invites to raise the following question: does there exist a homogeneous Markov process without positive jumps $(Z_t,\, t \geqslant 0)$ such that $X_{a,b}$ may be distributed as T_b under P_a, where P_a denotes the law of $(Z_t,\, t \geqslant 0)$, starting from a, and $T_b = \inf\{t \geqslant 0 : Z_t > b\}$ $(a < b)$? The purpose of this Remark 4.5 is to show that such a process $(Z_t,\, t \geqslant 0)$ does not exist; of course, it is also of interest to compare the present Remark 4.5 with the preceding Remark 4.4.

4.4.1. Proof of the non-existence of $(Z_t,\, t \geqslant 0)$. Assume that such a process exists. Since

$$E[\exp(-\lambda X_{a,b})] = \frac{b\,(1+\lambda)^a - 1}{a\,(1+\lambda)^b - 1} \quad (\lambda \geqslant 0,\, a < b), \tag{4.32}$$

we would have

$$E_a\left[f(Z_{T_b})\exp\left(-\int_0^{T_b} \frac{\mathscr{L}f}{f}(Z_s)\right)ds\right] = f(a)$$

for any regular function f, i.e.

$$E_a\left[\exp\left(-\int_0^{T_b} \frac{\mathscr{L}f}{f}(Z_s)\,ds\right)\right] = \frac{f(a)}{f(b)},$$

where P_a denotes the law of Z starting from a, $\mathscr{L}$ is the infinitesimal generator of Z, and f belongs to the (extended) domain of $\mathscr{L}$. Thus, we should infer, for any $\lambda \geqslant 0$, that

$$\left(M_\lambda(t) := \frac{(1+\lambda)^{Z_t}}{Z_t}\exp(-\lambda t),\ t \geqslant 0\right) \tag{4.33}$$

is a martingale. Hence

$$E_a\left(\frac{(1+\lambda)^{Z_t} - 1}{Z_t}\right) = \frac{(1+\lambda)^a - 1}{a}e^{\lambda t}. \tag{4.34}$$

Writing $l = \log(1+\lambda)$, i.e. $\lambda = e^l - 1$, we obtain (4.34) in the form

$$E_a\left(\frac{\exp(lZ_t) - 1}{Z_t}\right) = \exp((e^l - 1)\,t)\left(\frac{e^{al} - 1}{a}\right) \tag{4.35}$$

$$= \frac{1}{a}E\left[\exp(l(N_t + a)) - \exp(lN_t)\right], \tag{4.36}$$

where $(N_t,\, t \geqslant 0)$ denotes a standard Poisson process. Taking derivatives on both sides of (4.36) with respect to l, we obtain

$$E_a[\exp(lZ_t)] = \frac{1}{a}E\left[(N_t + a)\exp(l(N_t + a)) - N_t\exp(lN_t)\right];$$

hence, by the Laplace inversion, the law of Z_t is identified as

$$(4.37)\quad P_a(Z_t \in dx) = \frac{1}{a}\left(\sum_{n=0}^{\infty} \delta_{n+a}(dx)(n+a)e^{-t}\frac{t^n}{n!} - \sum_{n=0}^{\infty} \delta_n(dx)\, ne^{-t}\frac{t^n}{n!}\right).$$

But the measure featured on the right-hand side of (4.37) is signed; hence (Z_t) does not exist.

4.4.2. Looking for signed measures on the path space. Write, for $l \geqslant 0$,

$$\varphi_l(a) = \frac{\exp(la)-1}{a} \qquad (a > 0);$$

then define, for any $t \geqslant 0$,

$$(4.38)\qquad P_t\,\varphi_l(a) = \exp\big((e^l-1)\,t\big)\,\varphi_l(a).$$

Our search for a process $(Z_t,\, t \geqslant 0)$ in 4.4.1 led us to the relation (4.35), which we now write as

$$(4.39)\qquad E_a[\varphi_l(Z_t)] = P_t\,\varphi_l(a).$$

On the other hand, the relation (4.38) leads to the semigroup property for $(P_t)_{t \geqslant 0}$, since

$$(4.40)\qquad \begin{aligned} P_s(P_t\,\varphi_l)(a) &= \exp\big((e^l-1)\,t\big)\,P_s(\varphi_l)(a) \\ &= \exp\big((e^l-1)(t+s)\big)\,\varphi_l(a) = P_{t+s}\,\varphi_l(a). \end{aligned}$$

Of course, by the relation (4.37), the semigroup (P_t) is not positive. Nonetheless, it is tempting to ask the question: does there exist a Markov "process" $\big(\Omega,\, (Z_t,\, t \geqslant 0),\, (P_a,\, a \geqslant 0)\big)$ with signed measures (P_a) on path space, such that the r.v.'s T_b, under P_a, are distributed as $X_{a,b}$?

4.5. Proofs of points 3 and 4 of Theorem 1.4. Let us recall:

Point 3. *For any* $\alpha \in [0, 1]$,

$$(4.41)\qquad \mathfrak{e} \overset{(law)}{=} \mathfrak{e}_1\, G_\alpha + \mathfrak{e}_2\, G_{1-\alpha}.$$

Point 4. *For any* $\alpha \in [\frac{1}{2}, 1]$,

$$(4.42)\qquad \mathfrak{e} G_\alpha \overset{(law)}{=} \gamma_{(1-\alpha)} + X_{1-\alpha,\alpha}.$$

For any $\alpha \in [0, \frac{1}{2}]$,

$$(4.43)\qquad X_{\alpha,1-\alpha} + \mathfrak{e} G_\alpha \overset{(law)}{=} \gamma_{(1-\alpha)}.$$

As usual, it is understood that in these relations, whenever several r.v.'s are featured on one side, they are assumed independent. In the sequel of this work, this convention shall always be in force, without being stated each time. Moreover, $\mathfrak{e}$, with or without an index, indicates a standard exponential r.v.; G_0 and G_1 denote the r.v.'s defined in Theorem 1.2.

4.5.1. Proofs of (4.42) and (4.43). From (1.19) we get

$$E[\exp(-\lambda \mathbf{e} G_\alpha)] = E\left(\frac{1}{1+\lambda G_\alpha}\right) = \frac{\alpha}{1-\alpha}\frac{1-(1-\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1} \quad (\lambda \geqslant 0) \tag{4.44}$$

$$= \frac{\alpha}{1-\alpha}\frac{(1+\lambda)^{1-\alpha}-1}{(1+\lambda)^\alpha-1}\frac{1}{(1+\lambda)^{1-\alpha}}. \tag{4.45}$$

If $1-\alpha \leqslant \frac{1}{2}$, i.e. $\alpha \geqslant \frac{1}{2}$, then (4.45) implies, from the definition (1.41) of the r.v.'s $X_{a,b}$:

$$E[\exp(-\lambda \mathbf{e} G_\alpha)] = E[\exp(-\lambda X_{\alpha,1-\alpha})]\, E[\exp(-\lambda \gamma_{(1-\alpha)})],$$

which yields (4.42).

If $\alpha \leqslant \frac{1}{2}$, (4.45) takes the form

$$\frac{1}{(1+\lambda)^{1-\alpha}} = E[\exp(-\lambda \mathbf{e} G_\alpha)]\frac{1-\alpha}{\alpha}\frac{(1+\lambda)^\alpha-1}{(1+\lambda)^{1-\alpha}-1};$$

hence

$$\frac{1}{(1+\lambda)^{1-\alpha}} = E[\exp(-\lambda \mathbf{e} G_\alpha)]\, E[\exp(-\lambda X_{\alpha,1-\alpha})],$$

which yields (4.43).

We note that, if $\alpha > \frac{1}{2}$, (4.42) implies that $\mathbf{e} G_\alpha$ is infinitely divisible.

4.5.2. Proof of (4.41). It is not difficult to show that (4.42) and (4.43) imply (4.41). However, we may also prove (4.41) directly, since

$$E[\exp(-\lambda \mathbf{e} G_\alpha)] \cdot E[\exp(-\lambda \mathbf{e} G_{1-\alpha})] = E\left(\frac{1}{1+\lambda G_\alpha}\right) \cdot E\left(\frac{1}{1+\lambda G_{1-\alpha}}\right)$$

$$= \frac{\alpha}{1-\alpha} \cdot \frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1} \cdot \frac{1-\alpha}{\alpha} \cdot \frac{1-(1+\lambda)^{-\alpha}}{(1+\lambda)^{\alpha-1}-1} = \frac{1}{1+\lambda} = E(e^{-\lambda \mathbf{e}}).$$

5. PROOF OF THEOREM 1.5 (THE ALGEBRA OF THE R.V.'S $X_{a,b}$, $G_{\alpha,\beta}$ AND GAMMA)

We begin with the existence of the r.v.'s $G_{\alpha,\beta}$.

5.1. Proof of point 1 of Theorem 1.5. Let us recall parts (i), (ii) and (iii) of this point:

For any α, β, $0<\alpha$, $\beta<1$, *there exists an r.v.* $G_{\alpha,\beta}$ *taking values in* $[0, 1]$, *such that*

$$E[\exp(-\lambda \mathbf{e} G_{\alpha,\beta})] = E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right) = \frac{\alpha}{1-\beta}\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^\alpha-1} \quad (\lambda \geqslant 0), \tag{5.1}$$

$$E\left(\frac{1}{1+G_{\alpha,\beta}}\right) = \frac{\alpha}{1-\beta}\frac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^\alpha-\lambda^\alpha} \quad (\lambda \geqslant 0). \tag{5.2}$$

The density of $G_{\alpha,\beta}$ *is*

$$(5.3)\qquad f_{G_{\alpha,\beta}}(u) = 1_{[0,1]}(u)\frac{\alpha}{\pi(1-\beta)}$$

$$\times\frac{(1-u)^{\alpha}u^{\alpha-1}\sin(\pi\alpha)+u^{2\alpha-\beta}(1-u)^{\beta-1}\sin(\pi\beta)+(1-u)^{\alpha+\beta-1}u^{\alpha-\beta}\sin(\pi(\alpha-\beta))}{(1-u)^{2\alpha}-2(1-u)^{\alpha}u^{\alpha}\cos(\pi\alpha)+u^{2\alpha}}.$$

5.1.1. Let us define

$$(5.4)\qquad F_{\alpha,\beta}(\lambda) = \frac{\alpha}{1-\beta}\frac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^{\alpha}-\lambda^{\alpha}}\qquad (\lambda > 0).$$

We shall show that $F_{\alpha,\beta}$ is the Stieltjes transform of the function $f_{G_{\alpha,\beta}}(u)$ defined by (5.3). To prove this, it suffices, with the help of the inverse Stieltjes transform, to show that

$$(5.5)\qquad \frac{F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)}{2i\pi}\xrightarrow{\eta\to 0+} f_{G_{\alpha,\beta}}(u)\qquad (u\geqslant 0).$$

However, for $u\in[0, 1]$, the function

$$\frac{1}{2i\pi}[F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)]$$

$$=\frac{1}{2i\pi}\frac{\alpha}{1-\beta}\left(\frac{(-u-i\eta)^{\alpha-1}-(1-u-i\eta)^{\beta-1}(-u-i\eta)^{\alpha-\beta}}{(1-u-i\eta)^{\alpha}-(-u-i\eta)^{\alpha}}\right.$$

$$\left.-\frac{(-u+i\eta)^{\alpha-1}-(1-u+i\eta)^{\beta-1}(-u+i\eta)^{\alpha-\beta}}{(1-u+i\eta)^{\alpha}-(-u+i\eta)^{\alpha}}\right)$$

converges, as $\eta\downarrow 0$, to

$$\frac{1}{2i\pi}\frac{\alpha}{1-\beta}$$

$$\times\left(\frac{-u^{\alpha-1}e^{-i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{-i\pi(\alpha-\beta)}}{(1-u)^{\alpha}-u^{\alpha}e^{i\pi\alpha}}-\frac{-u^{\alpha-1}e^{i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{i\pi(\alpha-\beta)}}{(1-u)^{\alpha}-u^{\alpha}e^{i\pi\alpha}}\right)$$

$$=\frac{\alpha}{2i\pi(1-\beta)}\frac{N}{(1-u)^{2\alpha}-2(1-u)^{\alpha}u^{\alpha}\cos(\pi\alpha)+u^{2\alpha}},$$

where N is given by

$$N := \left(-u^{\alpha-1}e^{-i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{-i\pi(\alpha-\beta)}\right)\left((1-u)^{\alpha}-u^{\alpha}e^{i\pi\alpha}\right)$$

$$-\left(-u^{\alpha-1}e^{i\pi\alpha}-(1-u)^{\beta-1}u^{\alpha-\beta}e^{i\pi(\alpha-\beta)}\right)\left((1-u)^{\alpha}-u^{\alpha}e^{-i\pi\alpha}\right).$$

Hence, for $u \in]0, 1[$,

$$\frac{1}{2i\pi}[F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)]$$

$$\xrightarrow{\eta\to 0+} \frac{\alpha}{\pi(1-\beta)}$$

$$\times \frac{u^{\alpha-1}(1-u)^{\alpha}\sin(\pi\alpha)+u^{2\alpha-\beta}(1-u)^{\beta-1}\sin(\pi\beta)+(1-u)^{\alpha+\beta-1}u^{\alpha-\beta}\sin(\pi(\alpha-\beta))}{(1-u)^{2\alpha}-2(1-u)^{\alpha}u^{\alpha}\cos(\pi\alpha)+u^{2\alpha}}$$

$$= f_{G_{\alpha,\beta}}(u),$$

and it is not difficult to see that if $u > 1$, we have

$$\frac{1}{2i\pi}[F_{\alpha,\beta}(-u-i\eta)-F_{\alpha,\beta}(-u+i\eta)] \xrightarrow{\eta\to 0+} 0.$$

5.1.2. We now prove that $f_{G_{\alpha,\beta}}$ *is a probability density.*

It is obvious that, for $\alpha \geqslant \beta$, $f_{G_{\alpha,\beta}}(u) \geqslant 0$, and this follows from elementary manipulation if $\alpha \leqslant \beta$. Moreover, $\int_0^1 f_{G_{\alpha,\beta}}(u)\,du = 1$, since from (5.2) we get

$$\int_0^1 f_{G_{\alpha,\beta}}(u)\,du = \lim_{\lambda\to\infty}\frac{\alpha}{1-\beta}\lambda\frac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^{\alpha}-\lambda^{\alpha}}$$

$$= \lim_{\lambda\to\infty}\frac{\alpha}{1-\beta}\frac{1-(1+1/\lambda)^{\beta-1}}{(1+1/\lambda)^{\alpha}-1} = 1.$$

5.1.3. We now prove (5.1).

It follows immediately from (5.2), since

$$E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right) = \frac{1}{\lambda}E\left(\frac{1}{1/\lambda+G_{\alpha,\beta}}\right) = \frac{\alpha}{1-\beta}\frac{1}{\lambda}\frac{(1/\lambda)^{\alpha-1}-((1+\lambda)/\lambda)^{\beta-1}(1/\lambda)^{\alpha-\beta}}{((1+\lambda)/\lambda)^{\alpha}-(1/\lambda)^{\alpha}}$$

$$= \frac{\alpha}{1-\beta}\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1}.$$

5.1.4. We prove that, for any $\alpha \in [0, 1]$, $G_{\alpha,\alpha} \overset{(\text{law})}{=} G_\alpha$ (part (iv) of point 1).

This follows immediately from the explicit value of the density $f_{G_{\alpha,\alpha}}$, as given by (5.3), or again from (5.1):

$$E\left(\frac{1}{1+\lambda G_{\alpha,\alpha}}\right) = \frac{\alpha}{1-\alpha}\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^{\alpha}-1} = E\left(\frac{1}{1+\lambda G_{\alpha}}\right).$$

5.1.5. We prove that, for $\alpha \in]0, 1[$, $G_{\alpha,1-\alpha}$ *is beta* $(\alpha, 1-\alpha)$ *distributed* (part (v) of point 1).

This follows immediately from the explicit value of the density $f_{G_{\alpha,1-\alpha}}$, or again from

$$E\left(\frac{1}{1+\lambda G_{\alpha,1-\alpha}}\right)=\frac{1-(1+\lambda)^{-\alpha}}{(1+\lambda)^{\alpha}-1}=\frac{1}{(1+\lambda)^{\alpha}}=E\left(\frac{1}{1+\lambda\beta_{(\alpha,1-\alpha)}}\right).$$

5.2. Proof of point 2 of Theorem 1.5 (Algebraic properties). We recall:

(5.6) (i) *If* $\alpha+\beta\geqslant 1$, *then* $\mathrm{e}G_{\alpha,\beta}\overset{(law)}{=}\gamma_{(1-\beta)}+X_{1-\beta,\alpha}$.

(5.7) (ii) *If* $\alpha+\beta\leqslant 1$, *then* $\gamma_{(1-\beta)}\overset{(law)}{=}\mathrm{e}G_{\alpha,\beta}+X_{\alpha,1-\beta}$.

(iii) *For any* $0<\alpha,\ \beta,\ \gamma<1$:

$$\mathrm{e}_1\,G_{\alpha,\beta}+\mathrm{e}_2\,G_{\beta,\gamma}\overset{(law)}{=}\mathrm{e}_1\,G_{\alpha,\gamma}+\mathrm{e}_2\,G_{\beta}. \tag{5.8}$$

(5.9) *If* $\alpha+\beta\geqslant 1$, *then* $\gamma_{(1-\beta)}+X_{1-\beta,\alpha}+\mathrm{e}_2\,G_{\beta,\gamma}\overset{(law)}{=}\mathrm{e}_1\,G_{\alpha,\gamma}+\mathrm{e}_2\,G_{\beta}$.

(5.10) *If* $\alpha+\beta\leqslant 1$, *then* $\gamma_{(1-\beta)}+\mathrm{e}G_{\beta,\gamma}\overset{(law)}{=}\mathrm{e}_1\,G_{\alpha,\gamma}+\mathrm{e}_2\,G_{\beta}+X_{\alpha,1-\beta}$.

5.2.1. Proofs of (5.6) and (5.7). From the relation (5.1):

$$E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right)=\frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1},$$

once both the numerator and denominator have been multiplied by $(1+\lambda)^{1-\beta}$, we obtain

$$E[\exp(-\lambda\mathrm{e}G_{\alpha,\beta})]=E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right)=\left(\frac{\alpha}{1-\beta}\,\frac{(1+\lambda)^{1-\beta}-1}{(1+\lambda)^{\alpha}-1}\right)\cdot\frac{1}{(1+\lambda)^{1-\beta}}. \tag{5.11}$$

If $\alpha+\beta\geqslant 1$, i.e. $1-\beta\leqslant\alpha$, this relation takes the form

$$E[\exp(-\lambda\mathrm{e}G_{\alpha,\beta})]=E[\exp(-\lambda X_{1-\beta,\alpha})]\,E[\exp(-\lambda\gamma_{(1-\beta)})],$$

i.e. (5.6).

If $\alpha+\beta\leqslant 1$, i.e. $\alpha\leqslant 1-\beta$, we write (5.11) in the form

$$\frac{1}{(1+\lambda)^{1-\beta}}=E[\exp(-\lambda\mathrm{e}G_{\alpha,\beta})]\cdot\frac{1-\beta}{\alpha}\,\frac{(1+\lambda)^{\alpha}-1}{(1+\lambda)^{1-\beta}-1},$$

i.e.

$$\frac{1}{(1+\lambda)^{1-\beta}}=E[\exp(-\lambda\mathrm{e}G_{\alpha,\beta})]\cdot E[\exp(-\lambda X_{\alpha,1-\beta})].$$

We have obtained (5.7).

5.2.2. Proofs of (5.8), (5.9) and (5.10). From (5.1) we get

$$E[\exp(-\lambda e G_{\alpha,\beta})]\,E[\exp(-\lambda e G_{\beta,\gamma})] = E\left(\frac{1}{1+\lambda G_{\alpha,\beta}}\right)\cdot E\left(\frac{1}{1+\lambda G_{\beta,\gamma}}\right)$$

$$= \frac{\alpha}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\alpha}-1}\cdot\frac{\beta}{1-\gamma}\,\frac{1-(1+\lambda)^{\gamma-1}}{(1+\lambda)^{\beta}-1}$$

$$= \frac{\alpha}{1-\gamma}\,\frac{1-(1+\lambda)^{\gamma-1}}{(1+\lambda)^{\alpha}-1}\cdot\frac{\beta}{1-\beta}\,\frac{1-(1+\lambda)^{\beta-1}}{(1+\lambda)^{\beta}-1}$$

$$= E[\exp(-\lambda e G_{\alpha,\gamma})]\cdot E[\exp(-\lambda e G_{\beta})],$$

i.e. we obtain (5.8).

Finally, the relations (5.9) and (5.10) follow easily from (5.8), (5.6) and (5.7).

The proof of point 2 (iv) of Theorem 1.5 is obtained by similar arguments.

5.3. Remark 5.1

5.3.1. If we take $\gamma=\alpha$ in (5.8), we obtain

$$e_1 G_{\alpha,\beta} + e_2 G_{\beta,\alpha} \overset{(\mathrm{law})}{=} e_1 G_{\alpha} + e_2 G_{\beta}. \tag{5.12}$$

In particular, taking $\beta = 1-\alpha$ in (5.12), we obtain

$$e_1 G_{\alpha} + e_2 G_{1-\alpha} \overset{(\mathrm{law})}{=} e_1 G_{\alpha,1-\alpha} + e_2 G_{1-\alpha,\alpha}$$

$$\overset{(\mathrm{law})}{=} e_1 \beta_{\alpha,1-\alpha} + e_2 \beta_{1-\alpha,\alpha} \overset{(\mathrm{law})}{=} \gamma_{\alpha} + \gamma_{(1-\alpha)} \overset{(\mathrm{law})}{=} e.$$

This is our relation (4.41).

It is not difficult to show, after making some manipulations which are quite similar to the preceding ones, that (4.42) and (4.43) are particular cases of (5.9) and (5.10).

5.3.2. Of course, we did not find directly the explicit value of $f_{G_{\alpha,\beta}}$, as given by (5.3), with the help of the proof described in the above points 5.1.1 and 5.1.2. Prior to that proof, we developed a heuristic computation which was quite similar to the one made in Subsection 2.2.3.

6. THE (δ, G) SELF-DECOMPOSABLE VARIABLES. PROOFS OF THEOREMS 1.6 AND 1.7

6.1. Let G be a positive r.v. such that

$$E[\log^{+}(1/G)] < \infty. \tag{6.1}$$

It is not difficult to show that (6.1) is equivalent to either of the following assertions:

$$\int_1^{\infty} E[\exp(-xG)]\,\frac{dx}{x} < \infty; \tag{6.2}$$

$$\int_0^\infty (x \wedge 1) E(e^{-xG}) \frac{dx}{x} < \infty, \tag{6.3}$$

i.e. $E(e^{-xG})\frac{dx}{x}1_{(x \geqslant 0)}$ is the Lévy measure of a subordinator;

$$E(\log(1+\lambda/G)) < \infty \quad \text{for one (hence any) value of } \lambda > 0. \tag{6.4}$$

We may then formulate, thanks to the Lévy–Khintchine formula, the following

DEFINITION 6.1. Let $\delta > 0$, and G be an $\boldsymbol{R}_+$-valued r.v. which satisfies (6.1). We shall say that an $\boldsymbol{R}_+$-valued r.v. Δ is (δ, G) *self-decomposable* if, for every $\lambda \geqslant 0$,

$$E(e^{-\lambda\Delta}) = \exp\left(-\delta \int_0^\infty (1-e^{-\lambda x}) E(e^{-xG}) \frac{dx}{x}\right). \tag{6.5}$$

The equality (6.5) may also be written as

$$E(e^{-\lambda\Delta}) = \exp\{-\delta E(\log(1+\lambda/G))\}, \tag{6.6}$$

the latter formula (6.6) being obtained, e.g., as an application of the Frullani integral (see [17], p. 6). In fact, we thought of Definition 6.1 after considering formula (1.16), which, in our terminology, may be stated as: the r.v. Δ_α is $(1-\alpha, G_\alpha)$ self-decomposable.

6.2. The notion of (δ, G) self-decomposability is related quite naturally to the standard gamma subordinator.

Statement and proof of Theorem 1.6 (A link between the standard gamma subordinator and the (δ, G) self-decomposability).

Let $(\gamma_t, t \geqslant 0)$ denote the standard gamma subordinator whose Lévy–Khintchine representation is of the form

$$E[\exp(-\lambda\gamma_t)] = \frac{1}{(1+\lambda)^t} = \exp(-t\log(1+\lambda)) \quad (\lambda, t \geqslant 0), \tag{6.7}$$

and let $h\colon [0, \infty[\to [0, \infty[$ Borel.

Point 1. *Define*

$$\Delta_h := \int_0^\infty h(u)\, d\gamma_u. \tag{6.8}$$

Then Δ_h is a.s. finite if and only if

$$\int_0^\infty \log(1+h(u))\, du < \infty. \tag{6.9}$$

Point 2. *Under the hypothesis* (6.9), Δ_h *is self-decomposable, with Lévy–Khintchine representation*

$$(6.10)\qquad E[\exp(-\lambda\Delta_h)] = \exp\left(-\int_0^\infty (1-e^{-\lambda x})\,F_h(x)\frac{dx}{x}\right)$$

with

$$(6.11)\qquad F_h(x) = \int_0^\infty \exp\left(-\frac{x}{h(u)}\right)du.$$

Point 3. *For any r.v.* $G > 0$ *satisfying* (6.1), *there exists* h *satisfying* (6.9) *such that*

$$\delta E(e^{-xG}) = F_h(x) = \int_0^\infty \exp\left(-\frac{x}{h(u)}\right)du.$$

In other terms, every (δ, G) *self-decomposable r.v. may be written in the form* (6.8) *for a well-chosen function* h.

Recall (cf. the remark following the statement of Theorem 1.6 in Subsection 1.8) that:

• The function h, whose existence is asserted in the above point 3 is explicitly given in terms of δ and G via the formula

$$h(u) = \frac{1}{\mathcal{G}^{-1}(u/\delta)} \text{ for } u\in(0, \delta) \quad\text{and}\quad h(u) = 0 \text{ for } u > \delta.$$

• The Laplace transform ψ_{Δ_h} of the r.v. Δ_h is hyperbolically completely monotone.

6.2.1. Proof of (6.9) and (6.10). By a density argument, it suffices to consider h continuous, with compact support. Then we have

$$(6.12)\qquad E\left[\exp\left(-\lambda\int_0^\infty h(u)\,d\gamma_u\right)\right] = \lim E\left[\exp\left(-\lambda\sum h(t_i)(\gamma_{t_{i+1}}-\gamma_{t_i})\right)\right]$$

$$= \lim\exp\left\{-\sum(t_{i+1}-t_i)\log(1+\lambda h(t_i))\right\} \qquad \text{(by (6.7))}$$

$$= \exp\left(-\int_0^\infty \log(1+\lambda h(t))\,dt\right) = \exp\left(-\int_0^\infty dt\int_0^\infty e^{-x}(1-e^{-\lambda h(t)x})\frac{dx}{x}\right)$$

since, for every $v \geqslant 0$, the Frullani integral (cf. [17], p. 6) gives

$$\log(1+v) = \int_0^\infty e^{-x}\frac{dx}{x}(1-e^{-vx}).$$

Hence, making the change of variables $h(t)\,x = y$, and then applying Fubini's theorem, we obtain

$$E\left(\exp\left(-\lambda\int_0^\infty h(u)\,d\gamma_u\right)\right) = \exp\left(-\int_0^\infty dt\int_0^\infty e^{-x}(1-e^{-\lambda h(t)x})\frac{dx}{x}\right)$$

$$= \exp\left(-\int_0^\infty dt\int_0^\infty \exp\left(-\frac{y}{h(t)}\right)(1-e^{-\lambda y})\frac{dy}{y}\right)$$

$$= \exp\left(-\int_0^\infty (1-e^{-\lambda y})\frac{dy}{y}\left(\int_0^\infty \exp\left(-\frac{y}{h(t)}\right)dt\right)\right),$$

which proves both (6.9) and (6.10).

6.2.2. Proof of point 3 of Theorem 1.6. Assume now that G satisfies (6.1), or (6.4), and $\delta > 0$. Let us consider the probability space obtained from the unit interval $[0, 1]$, fitted with Lebesgue measure, and realize G in the form

$$G(w) = (1/h)(\delta w), \quad w \in [0, 1], \tag{6.13}$$

for a well-chosen function h, with support in $[0, \delta]$. Then we obtain

$$\delta E(e^{-xG}) = \delta\int_0^1 \exp\left(-\frac{x}{h(\delta u)}\right)du \tag{6.14}$$

$$= \int_0^\delta \exp\left(-\frac{x}{h(v)}\right)dv = \int_0^\infty \exp\left(-\frac{x}{h(v)}\right)dv.$$

Thus

$$E\left[\exp\left(-\lambda\int_0^\infty h(u)\,d\gamma_u\right)\right] = \exp\left\{-\int_0^\infty (1-e^{-\lambda y})\frac{dy}{y}\left(\int_0^\infty \exp\left(-\frac{y}{h(v)}\right)dv\right)\right\}$$

$$= \exp\left\{-\delta\int_0^\infty (1-e^{-\lambda y})E(e^{-xG})\frac{dy}{y}\right\}.$$

Finally, it is clear, as a consequence of the definition (6.13), that

$$E\left(\log^+\frac{1}{G}\right) < \infty \Leftrightarrow E\left(\log\left(1+\frac{1}{G}\right)\right) < \infty \Leftrightarrow \int_0^\infty \log(1+h(u))\,du < \infty.$$

6.2.3. Proof of Theorem 1.7. *Mutatis mutandis*, it is exactly the same as the proof of point 3 of Theorem 1 (cf. Proposition 2.3 and Subsection 2.3.3 above).

6.2.4. Proof of Theorem 1.8.

DEFINITION 6.2. A function $F:]0, \infty[\to \boldsymbol{R}_+$, which belongs to C^1, *satisfies* (ST, δ) if the following conditions (6.15)–(6.17) hold:

(6.15) F admits a holomorphic extension to $\boldsymbol{C}\backslash]-\infty, 0]$;

(6.16) for every $u > 0$, $\lim_{\eta\to 0_+} F(-u+i\eta) := F_+(u)$ exists and is continuous (resp., $\lim_{\eta\to 0_+} F(-u-i\eta) := F_-(u)$ exists and is continuous), and $\mathrm{Im}\big(F_-(u) - F_+(u)\big) \geqslant 0$ for every $u > 0$;

(6.17) for λ real, $\lim_{\lambda\to\infty} \lambda F(\lambda) = \delta > 0$.

Let Δ denote a positive r.v. with Laplace transform ψ:

$$E(e^{-\lambda\Delta}) = \psi(\lambda), \qquad \lambda \geqslant 0.$$

We assume that $F := \psi'/\psi$ satisfies (ST, δ).

6.2.4.a. We show that: $f(u) := (2\pi\delta)^{-1}\big(\mathrm{Im}\big(F_-(u) - F_+(u)\big)\big)$ *defines a probability density on $\boldsymbol{R}_+$, and Δ is (δ, G) self-decomposable, where G is an r.v. with density f.*

In fact, we have already proved this when we showed the existence of the r.v.'s G_α (Subsection 2.2.1) and of the r.v.'s $G_{\alpha,\beta}$ (Subsection 5.1). We now summarize the important points of this proof:

- By inversion of the Stieltjes transform, we have

$$Sf(\lambda) = \int_0^\infty \frac{f(u)\,du}{\lambda+u} = -\frac{1}{\delta}\frac{\psi'}{\psi}(\lambda).$$

- f is positive (from (6.16)) and has integral 1 (from (6.17)).
- Let G denote an r.v. with density f. Then

$$\delta E\left(\frac{1}{\lambda+G}\right) = -\frac{\psi'}{\psi}(\lambda);$$

hence

$$-\delta \int_0^\infty e^{-\lambda x} E(e^{-xG})\,dx = -\frac{\psi'}{\psi}(\lambda).$$

Consequently, by integration,

$$E(e^{-\lambda\Delta}) = \psi(\lambda) = \exp\left(-\delta \int_0^\infty (1-e^{-\lambda x}) E(e^{-xG}) \frac{dx}{x}\right).$$

The results of the paper are gathered in the table which follows.

Random variable	Density (* only the density is given, not the *LT*)	Laplace transform
Δ_α $(0<\alpha<1)$	$\dfrac{\alpha}{\Gamma(1-\alpha)}x^{-(\alpha+1)}(1-e^{-x})\,1_{(x\geqslant 0)}$	$(1+\lambda)^\alpha-\lambda^\alpha$
G_α $(0<\alpha<1)$	$*\ \dfrac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\dfrac{x^{\alpha-1}(1-x)^{\alpha-1}}{(1-x)^{2\alpha}-2(1-x)^\alpha x^\alpha\cos(\pi\alpha)+x^{2\alpha}}1_{[0,1]}(x)$	
$G_{1/2}$	$\dfrac{1}{\pi}\dfrac{1}{\sqrt{x(1-x)}}1_{[0,1]}(x)$	$\exp\left(-\dfrac{\lambda}{2}\right)I_0\left(\dfrac{\lambda}{2}\right)$
$G_{1/p},\ p\in N,\ p\geqslant 2$	$*\ \dfrac{1}{\pi(p-1)}\sum_{k=1}^{p-1}\sin\left(\dfrac{\pi k}{p}\right)x^{k/p-1}(1-x)^{-k/p}1_{[0,1]}(x)$	
G_1	$1_{[0,1]}(x)$	$\lambda^{-1}(1-e^{-\lambda})$
$G_0\overset{\text{(law)}}{=}\dfrac{1}{1+\exp(\pi C)}$ C standard Cauchy	$*\ \dfrac{1}{x(1-x)}\dfrac{1}{\pi^2+\left(\log\dfrac{1-x}{x}\right)^2}1_{[0,1]}(x)$	
$G_{\alpha,\beta}$ $(0<\alpha,\ \beta<1)$	$*\ \dfrac{\alpha}{\pi(1-\beta)}$ $\dfrac{(1-x)x^{\alpha-1}\sin(\pi\alpha)+x^{2\alpha-\beta}(1-x)^{\beta-1}\sin(\pi\beta)+(1-x)^{\alpha+\beta-1}x^{\alpha-\beta}\sin(\pi(\alpha-\beta))}{(1-x)^{2\alpha}-2(1-x)^\alpha x^\alpha\cos(\pi\alpha)+x^{2\alpha}}1_{[0,1]}(x)$	
$G_{\alpha,\alpha}\overset{\text{(law)}}{=}G_\alpha$		
$G_{\alpha,1-\alpha}\overset{\text{(law)}}{=}\beta_{\alpha,1-\alpha}$	$\dfrac{\sin(\pi\alpha)}{\pi}x^{\alpha-1}(1-x)^{-\alpha}1_{[0,1]}(x)$	
$X_{a,b}$ $(0<a\leqslant b<1)$		$\dfrac{b}{a}\dfrac{(1+\lambda)^a-1}{(1+\lambda)^b-1}$
$X_{a,1}$ $(0<a<1)$		$\dfrac{1}{a}\dfrac{(1+\lambda)^a-1}{\lambda}$
$X_{0,1}\overset{\text{(law)}}{=}\mathrm{e}\cdot U$	$\left(\int_x^\infty\dfrac{e^{-t}}{t}dt\right)1_{(x\geqslant 0)}$	$\dfrac{\log(1+\lambda)}{\lambda}$
$X_{0,b}$ $(0<b<1)$		$b\dfrac{\log(1+\lambda)}{(1+\lambda)^b-1}$

Main properties:

$\Delta_\alpha\overset{\text{(law)}}{=}\int_0^\infty e^{-t}\,dY_t$ $(Y_t,\ t\geqslant 0)$ is an $(\alpha,\ K_\alpha)$ compound Poisson process, with $K_\alpha\overset{\text{(law)}}{=}\mathrm{e}/G_\alpha$.

$\Delta_\alpha\overset{\text{(law)}}{=}U^{1/\alpha}(\Delta_\alpha+K_\alpha)$ $(U,\ \Delta_\alpha,\ K_\alpha$ independent, and U uniform on $[0,\ 1])$.

Random variable	Stieltjes transform	Lévy measure
Δ_α $(0<\alpha<1)$		$(1-\alpha)\dfrac{E[\exp(-xG_\alpha)]}{x}dx$
G_α $(0<\alpha<1)$	$\dfrac{\alpha}{1-\alpha}\dfrac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha}$	non infinitely divisible
$G_{1/2}$	$\dfrac{1}{\sqrt{\lambda(1+\lambda)}}$	non infinitely divisible
$G_{1/p}$, $p\in N$, $p\geqslant 2$		non infinitely divisible
G_1	$\log(1+\lambda)$	non infinitely divisible
$G_0\overset{\text{(law)}}{=}\dfrac{1}{1+\exp(\pi C)}$ C standard Cauchy	$\dfrac{1}{\lambda(1+\lambda)}\dfrac{1}{\log\left(\dfrac{1+\lambda}{\lambda}\right)}$	non infinitely divisible
$G_{\alpha,\beta}$ $(0<\alpha,\beta<1)$	$\dfrac{\alpha}{1-\beta}\dfrac{\lambda^{\alpha-1}-(1+\lambda)^{\beta-1}\lambda^{\alpha-\beta}}{(1+\lambda)^\alpha-\lambda^\alpha}$	non infinitely divisible
$G_{\alpha,\alpha}\overset{\text{(law)}}{=}G_\alpha$	$\dfrac{\alpha}{1-\alpha}\dfrac{\lambda^{\alpha-1}-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-\lambda^\alpha}$	
$G_{\alpha,1-\alpha}\overset{\text{(law)}}{=}\beta_{\alpha,1-\alpha}$	$\dfrac{\lambda^{\alpha-1}}{(1+\lambda)^\alpha}$	non infinitely divisible
$X_{a,b}$ $(0<a\leqslant b<1)$		$\dfrac{1}{x}\left[(1-a)E\left[\exp\left(-\dfrac{x}{G_a}\right)\right]-(1-b)E\left(\exp\left(-\dfrac{x}{G_b}\right)\right)\right]dx$
$X_{a,1}$ $(0<a<1)$		$(1-a)\dfrac{E[\exp(-x/G_a)]}{x}dx$
$X_{0,1}\overset{\text{(law)}}{=}\mathrm{e}\cdot U$		$\dfrac{1}{x}E[\exp(-x(1+e^{\pi C}))]\,dx$ C standard Cauchy
$X_{0,b}$ $(0<b<1)$		$\dfrac{1}{x}\left\{E[\exp(-x(1+e^{\pi C}))]-(1-b)E\left[\exp\left(-\dfrac{x}{G_b}\right)\right]\right\}dx$

Main properties:

$\mathrm{e}\overset{\text{(law)}}{=}\mathrm{e}_1G_\alpha+\mathrm{e}_2G_{1-\alpha}$; if $\alpha\in[1/2,1]$, $\mathrm{e}G_\alpha\overset{\text{(law)}}{=}\gamma_{1-\alpha}+X_{1-\alpha,\alpha}$; if $\alpha\in[0,1/2]$, $X_{\alpha,1-\alpha}+\mathrm{e}G_\alpha\overset{\text{(law)}}{=}\gamma_{1-\alpha}$.
$\mathrm{e}_1G_{\alpha,\beta}+\mathrm{e}_2G_{\beta,\gamma}\overset{\text{(law)}}{=}\mathrm{e}_1G_{\alpha,\gamma}+\mathrm{e}_2G_\beta$; if $\alpha+\beta\geqslant 1$, $\mathrm{e}G_{\alpha,\beta}\overset{\text{(law)}}{=}\gamma_{1-\alpha}+X_{1-\beta,\alpha}$; if $\alpha+\beta\leqslant 1$, $\gamma_{1-\beta}\overset{\text{(law)}}{=}\mathrm{e}G_{\alpha,\beta}+X_{\alpha,1-\beta}$.

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