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A REPRESENTATION OF EXCESSIVE FUNCTIONS AS EXPECTED SUPREMA

BY

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Dedicated to the memory of Kazimierz Urbanik

Abstract. For a nice Markov process such as Brownian motion on a domain in $\boldsymbol{R}^d$, we prove a representation of excessive functions in terms of expected suprema. This is motivated by recent work of El Karoui [5] and El Karoui and Meziou [8] on the max-plus decomposition for supermartingales. Our results provide a singular analogue to the non-linear Riesz representation in El Karoui and Föllmer [6], and they extend the representation of potentials in Föllmer and Knispel [10] by clarifying the role of the boundary behavior and of the harmonic points of the given excessive function.

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1. INTRODUCTION

Consider a bounded superharmonic function u on the open disk S. Such a function admits a limit $u(y)$ in almost all boundary points $y \in \partial S$ with respect to the *fine topology*, and we have

$$u(x) \geqslant \int u(y)\, \mu_x(dy),$$

where μ_x denotes the harmonic measure on the boundary. The right-hand side defines a harmonic function h on S, and the difference $u-h$ can be represented as the potential of a measure on S. This is the classical Riesz representation of the superharmonic function u.

In probabilistic terms, μ_x may be viewed as the exit distribution of Brownian motion on S starting in x, u is an excessive function of the process, the fine limit can be described as a limit along Brownian paths to the boundary, and the Riesz representation takes the form

$$u(x) = E_x\,[\lim_{t \uparrow \zeta} u(X_t) + A_\zeta],$$

where ζ denotes the first exit time from S and $(A_t)_{t\geqslant 0}$ is the additive functional generating the potential $u-h$; cf., e.g., Blumenthal and Getoor [4].

In this paper we consider an alternative probabilistic representation of the excessive function u in terms of expected suprema. We construct a function f on the closure of S which coincides with the boundary values of u on ∂S and yields the representation

$$u(x) = E_x[\sup_{0<t\leqslant\zeta} f(X_t)], \tag{1}$$

i.e.,

$$u(x) = E_x[\sup_{0<t<\zeta} f(X_t) \vee \lim_{t\uparrow\zeta} u(X_t)]. \tag{2}$$

Instead of Brownian motion on the unit disk, we consider a general Markov process with state space S and life time ζ. Under some regularity conditions we prove in Section 3 that an excessive function u admits a representation of the form (2) in terms of some function f on S. Under additional conditions, the limit in (2) can be identified as a boundary value $f(X_\zeta)$ for some function f on the Martin boundary of the process, and in this case (2) can also be written in the condensed form (1).

The representing function f is in general not unique. In Section 4 we characterize the class of representing functions in terms of a maximal and a minimal representing function. These bounds are described in potential theoretic terms. They coincide at points where the excessive function u is not harmonic, the lower bound is equal to zero on the set H of harmonic points, and the upper bound is constant on the connected components of H.

Our representation (2) of an excessive function is motivated by recent work of El Karoui and Meziou [8] and El Karoui [5] on problems of portfolio insurance. Their results involve a representation of a given supermartingale as the process of conditional expected suprema of another process. This may be viewed as a singular analogue to a general representation for semimartingales in Bank and El Karoui [1], which provides a unified solution to various representation problems arising in connection with optimal consumption choice, optimal stopping, and multi-armed bandit problems. We refer to Bank and Föllmer [2] for a survey and to the references given there, in particular to El Karoui and Karatzas [7] and Bank and Riedel [3]; see also Kaspi and Mandelbaum [11].

In the context of probabilistic potential theory such representation problems take the following form:

For a given function u and a given additive functional $(B_t)_{t\geqslant 0}$ of the underlying Markov process we want to find a function f such that

$$u(x) = E_x\Big[\int_0^\zeta \sup_{0<t\leqslant\zeta} f(X_t)\, dB_t\Big].$$

In El Karoui and Föllmer [6] this potential theoretic problem is discussed for the smooth additive functional $B_t = t\wedge\zeta$ and for the case when u has boundary behavior zero. The results are easily extended to the case where the random measure corresponding to the additive functional satisfies the regularity assumptions required in [1].

Our representation (2) corresponds to the singular case $B_t = 1_{[\zeta,\infty)}(t)$ where the random measure is given by the Dirac measure δ_ζ. This singular representation problem, which does not satisfy the regularity assumptions of [1], is discussed in Föllmer and Knispel [10] for the special case of a potential u. The purpose of the present paper is to consider a general excessive function u and to clarify the impact of the boundary behavior on the representation of u as an expected supremum. We concentrate on those proofs which involve explicitly the boundary behavior of u, and we refer to [10] whenever the argument is the same as in the case of a potential.

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2. PRELIMINARIES

Let $(X_t)_{t\geqslant 0}$ be a strong Markov process with locally compact metric state space (S, d), shift operators $(\theta_t)_{t\geqslant 0}$, and life time ζ, defined on a stochastic base $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, (P_x)_{x\in S})$ and satisfying the assumptions in [6] or [10]. In particular, we assume that the excessive functions of the process are lower-semicontinuous. As a typical example, we could consider a Brownian motion on a domain $S \subset \mathbf{R}^d$.

For any measurable function $u \geqslant 0$ on S and for any stopping time T we use the notation

$$P_T u(x) := E_x[u(X_T);\ T < \zeta].$$

Recall that u is *excessive* if $P_t u \leqslant u$ for any $t > 0$ and $\lim_{t\downarrow 0} P_t u(x) = u(x)$ for any $x \in S$. In that case the process $(u(X_t)1_{\{t<\zeta\}})_{t\geqslant 0}$ is a right-continuous P_x-supermartingale for any $x \in S$ such that $u(x) < \infty$, and this implies the existence of

$$u_\zeta := \lim_{t\uparrow\zeta} u(X_t) \quad P_x\text{-a.s.}$$

Let us denote by $\mathscr{T}(x)$ the class of all exit times

$$T_U := \inf\{t \geqslant 0 \mid X_t \notin U\} \wedge \zeta$$

from open neighborhoods U of $x \in S$, and by $\mathscr{T}_0(x)$ the subclass of all exit times from open neighborhoods of x which are relatively compact. Note that $\zeta = T_S \in \mathscr{T}(x)$. For $T \in \mathscr{T}(x)$ and any measurable function $u \geqslant 0$ we introduce the notation

$$u_T := u(X_T)1_{\{T<\zeta\}} + \overline{\lim_{t\uparrow\zeta}}\, u(X_t)1_{\{T=\zeta\}}$$

and

$$\tilde{P}_T u(x) := E_x[u_T] = P_T u(x) + E_x[\overline{\lim_{t\uparrow\zeta}} u(X_t);\ T = \zeta].$$

We say that a function u *belongs to class* (D) if for any $x \in S$ the family $\{u(X_T) \mid T \in \mathcal{T}_0(x)\}$ is uniformly integrable with respect to P_x. Recall that an excessive function u is *harmonic* on S if $P_T u(x) = u(x)$ for any $x \in S$ and any $T \in \mathcal{T}_0(x)$. A harmonic function u of class (D) also satisfies $u(x) = \tilde{P}_T u(x)$ for all $T \in \mathcal{T}(x)$, and u is uniquely determined by its boundary behavior:

$$u(x) = E_x[\lim_{t\uparrow\zeta} u(X_t)] = E_x[u_\zeta] \quad \text{for any } x \in S. \tag{3}$$

PROPOSITION 2.1. *Let $f \geq 0$ be an upper-semicontinuous function on S and let $\phi \geq 0$ be $\mathcal{F}$-measurable such that $\phi = \phi \circ \theta_T$ P_x-a.s. for any $x \in S$ and any $T \in \mathcal{T}_0(x)$. Then the function u on S defined by the expected suprema*

$$u(x) := E_x[\sup_{0<t<\zeta} f(X_t) \vee \phi] \tag{4}$$

is excessive, hence lower-semicontinuous. Moreover, u belongs to class (D) *if and only if u is finite on S. In this case u has the boundary behavior*

$$u_\zeta = \overline{\lim_{t\uparrow\zeta}} f(X_t) \vee \phi = f_\zeta \vee \phi \ \ P_x\text{-a.s.}, \tag{5}$$

and u admits a representation (2), *i.e., a representation* (4) *with $\phi = u_\zeta$.*

Proof. It follows as in [10] that u is an excessive function. If $u(x) < \infty$, then

$$\sup_{0<t<\zeta} f(X_t) \vee \phi \in \mathcal{L}^1(P_x).$$

Thus $\{u(X_T) \mid T \in \mathcal{T}_0(x)\}$ is uniformly integrable with respect to P_x, since

$$\begin{aligned} 0 \leq u(X_T) &= E_x[\sup_{T<t<\zeta} f(X_t) \vee (\phi \circ \theta_T) \mid \mathcal{F}_T] \\ &\leq E_x[\sup_{0<t<\zeta} f(X_t) \vee \phi \mid \mathcal{F}_T] \quad \text{for all } T \in \mathcal{T}_0(x). \end{aligned}$$

Conversely, if u belongs to class (D), then u is finite on S since by lower-semicontinuity

$$u(x) \leq E_x[\underline{\lim}_{n\uparrow\infty} u(X_{T_{\varepsilon_n}})] \leq \underline{\lim}_{n\uparrow\infty} E_x[u(X_{T_{\varepsilon_n}})] < \infty \qquad \text{for } \varepsilon_n \downarrow 0,$$

where $T_{\varepsilon_n} \in \mathcal{T}_0(x)$ denotes the exit time from the open ball $U_{\varepsilon_n}(x)$.

In order to verify (5), we take a sequence $(U_n)_{n\in N}$ of relatively compact open neighborhoods of x increasing to S and denote by T_n the exit time from U_n. Since u is excessive and finite on S, we conclude that

$$\overline{\lim_{t\uparrow\zeta}} f(X_t) \vee \phi = \lim_{n\uparrow\infty} \sup_{T_n<s<\zeta} f(X_s) \vee (\phi \circ \theta_{T_n})$$

$$= \lim_{n\uparrow\infty} E_x[\sup_{T_n<s<\zeta} f(X_s) \vee (\phi\circ\theta_{T_n}) \mid \mathscr{F}_{T_n}]$$
$$= \lim_{n\uparrow\infty} u(X_{T_n}) = u_\zeta \; P_x\text{-a.s.},$$

where the second identity follows from a martingale convergence argument.

In view of (5) we have

$$\{\phi \leqslant \sup_{0<t<\zeta} f(X_t)\} = \{u_\zeta \leqslant \sup_{0<t<\zeta} f(X_t)\} \; P_x\text{-a.s.}$$

and $\phi = u_\zeta$ on $\{\phi > \sup_{0<t<\zeta} f(X_t)\}$ P_x-a.s. Thus we can write

$$u(x) = E_x[\sup_{0<t<\zeta} f(X_t);\, \phi \leqslant \sup_{0<t<\zeta} f(X_t)] + E_x[\phi;\, \phi > \sup_{0<t<\zeta} f(X_t)]$$
$$= E_x[\sup_{0<t<\zeta} f(X_t);\, u_\zeta \leqslant \sup_{0<t<\zeta} f(X_t)] + E_x[u_\zeta;\, u_\zeta > \sup_{0<t<\zeta} f(X_t)]$$
$$= E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta]. \quad \blacksquare$$

In the next section we show that, conversely, any excessive function u of class (D) admits a representation of the form (2), where f is some upper-semicontinuous function on S.

3. CONSTRUCTION OF A REPRESENTING FUNCTION

Let $u \geqslant 0$ be an excessive function of class (D). In order to avoid additional technical difficulties, we also assume that u is continuous. For convenience we introduce the notation $u^c := u \vee c$.

Consider the family of optimal stopping problems

$$Ru^c(x) := \sup_{T\in\mathscr{T}_0(x)} E_x[u^c(X_T)] \tag{6}$$

for $c \geqslant 0$ and $x \in S$. It is well known that the value function Ru^c of the optimal stopping problem (6) can be characterized as the smallest excessive function dominating u^c. In particular, Ru^c is lower-semicontinuous. Moreover,

$$Ru^c(x) \geqslant E_x[u^c(X_T);\, T < \zeta] + E_x[\lim_{t\uparrow\zeta} u^c(X_t){:}\, T = \zeta] = \tilde{P}_T u^c(x) \tag{7}$$

for any stopping time $T \leqslant \zeta$, and equality holds for the first entrance time into the closed set $\{Ru^c = u^c\}$; cf. for example the proof of Lemma 4.1 in [6].

The following lemma can be verified by a straightforward modification of the arguments in [10]:

LEMMA 3.1. 1) *For any* $x \in S$, $Ru^c(x)$ *is increasing, convex and Lipschitz-continuous in* c, *and*

$$\lim_{c\uparrow\infty} (Ru^c(x) - c) = 0. \tag{8}$$

2) *For any* $c \geqslant 0$,

(9) $$Ru^c(x) = E_x[u^c_{D^c}] = \tilde{P}_{D^c} u^c(x),$$

where $D^c := \inf\{t \geqslant 0 \mid Ru^c(X_t) = u(X_t)\} \wedge \zeta$ *is the first entrance time into the closed set* $\{Ru^c = u\}$. *Moreover, the map* $c \mapsto D^c$ *is increasing and* P_x*-a.s. left-continuous.*

Since the function $c \mapsto Ru^c(x)$ is convex, it is almost everywhere differentiable. The following identification of the derivatives is similar to Lemma 3.2 of [10].

LEMMA 3.2. *The left-hand derivative* $\partial^- Ru^c(x)$ *of* $Ru^c(x)$ *with respect to* $c > 0$ *is given by*

$$\partial^- Ru^c(x) = P_x[u_\zeta < c,\, D^c = \zeta].$$

Proof. For any $0 \leqslant a < c$, the representation (9) for the parameter c together with the inequality (7) for the parameter a and for the stopping time $T = D^c$ implies

$$Ru^c(x) - Ru^a(x) \leqslant E_x[u^c(X_{D^c}) - u^a(X_{D^c});\, D^c < \zeta] + E_x[u^c_\zeta - u^a_\zeta;\, D^c = \zeta].$$

Since

$$u(X_{D^c}) = Ru^c(X_{D^c}) \geqslant c > a \quad \text{on } \{D^c < \zeta\}$$

and $u^c_\zeta - u^a_\zeta \leqslant (c-a)\, 1_{\{u_\zeta < c\}}$, the previous estimate simplifies to

$$Ru^c(x) - Ru^a(x) \leqslant (c-a)\, P_x[u_\zeta < c,\, D^c = \zeta].$$

This shows that $\partial^- Ru^c(x) \leqslant P_x[u_\zeta < c,\, D^c = \zeta]$. In order to prove the converse inequality, we use the estimate

$$Ru^c(x) - Ru^a(x) \geqslant (c-a)\, P_x[u_\zeta < a,\, D^a = \zeta]$$

obtained by reversing the role of a and c in the preceding argument. This implies

$$\partial^- Ru^c(x) \geqslant \lim_{a \uparrow c} P_x[u_\zeta < a,\, D^a = \zeta] = P_x[u_\zeta < c,\, D^c = \zeta]$$

since $\bigcup_{a<c} \{D^a = \zeta\} = \{D^c = \zeta\}$ on $\{u_\zeta < c\}$, due to the Lipschitz-continuity of $Ru^c(x)$ in c. ∎

Let us now introduce the function f^* defined by

(10) $$f^*(x) := \sup\{c \mid x \in \{Ru^c = u\}\}$$

for any $x \in S$. Note that $f^*(x) \geqslant c$ is equivalent to $Ru^c(x) = u(x)$ due to the continuity of $Ru^c(x)$ in c. As in [10], Lemma 3.3, it follows that the function f^* is upper-semicontinuous and satisfies $0 \leqslant f^* \leqslant u$.

We are now ready to derive a representation of the value functions Ru^c in terms of the function f^*. In the special case of a potential u, where $u_\zeta = 0$ and $u^c_\zeta = c$ P_x-a.s., our representation (11) reduces to Theorem 3.1 of [10].

THEOREM 3.1. *For any $c \geqslant 0$ and any $x \in S$,*

$$Ru^c(x) = E_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta^c] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta^c]. \tag{11}$$

Proof. By Lemma 3.2 and (8) we get

$$Ru^c(x) - c = \int_c^\infty -\frac{\partial}{\partial \alpha}(Ru^\alpha(x) - \alpha)\, d\alpha = \int_c^\infty (1 - P_x[u_\zeta < \alpha,\, D^\alpha = \zeta])\, d\alpha.$$

Since

$$\{D^{c+\varepsilon} < \zeta\} \subseteq \{\sup_{0 \leqslant t < \zeta} f^*(X_t) > c\} \subseteq \{D^c < \zeta\}$$

for any $c \geqslant 0$ and for any $\varepsilon > 0$, we have

$$\begin{aligned} Ru^c(x) - c &= \int_c^\infty (1 - P_x[u_\zeta < \alpha,\, D^\alpha = \zeta])\, d\alpha \\ &\geqslant \int_c^\infty \big(1 - P_x[u_\zeta \leqslant \alpha,\, \sup_{0 \leqslant t < \zeta} f^*(X_t) \leqslant \alpha]\big)\, d\alpha \\ &\geqslant \int_c^\infty (1 - P_x[u_\zeta < \alpha + \varepsilon,\, D^{\alpha+\varepsilon} = \zeta])\, d\alpha = Ru^{c+\varepsilon}(x) - (c + \varepsilon). \end{aligned}$$

By continuity of $c \mapsto Ru^c$ we obtain

$$\begin{aligned} Ru^c(x) - c &\geqslant \int_c^\infty \big(1 - P_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta \leqslant \alpha]\big)\, d\alpha \\ &\geqslant \lim_{\varepsilon \downarrow 0} \big(Ru^{c+\varepsilon}(x) - (c + \varepsilon)\big) = Ru^c(x) - c, \end{aligned}$$

hence

$$\begin{aligned} Ru^c(x) &= \int_c^\infty P_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta > \alpha]\, d\alpha + c \\ &= E_x\big[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta - (\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta) \wedge c + c\big] \\ &= E_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta^c]. \end{aligned}$$

Moreover, we can conclude that

$$\begin{aligned} Ru^c(x) &= \lim_{t \downarrow 0} P_t(Ru^c)(x) \\ &= \lim_{t \downarrow 0} E_x[\sup_{t \leqslant s < \zeta} f^*(X_s) \vee u_\zeta^c;\, t < \zeta] = E_x[\sup_{0 < s < \zeta} f^*(X_s) \vee u_\zeta^c] \end{aligned}$$

since Ru^c is excessive, i.e., $Ru^c(x)$ also admits the second representation in equation (11). ∎

As a corollary we see that f^* is a representing function for u.

COROLLARY 3.1. *The excessive function u admits the representations*

$$u(x) = E_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta] \tag{12}$$

in terms of the upper-semicontinuous function $f^* \geqslant 0$ defined by (10). *Moreover,*

$$f^*(x) \leqslant \sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta \;\; P_x\text{-a.s.} \quad \text{for any } x \in S.$$

Proof. Note that $u = Ru^0$ since u is excessive. Applying Theorem 3.1 with $c = 0$ we obtain

$$u(x) = Ru^0(x) = E_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta].$$

In particular, we get

$$\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta = \sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta \;\; P_x\text{-a.s.},$$

and this implies $f^*(x) \leqslant \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta$ P_x-a.s. for any $x \in S$. ∎

Remark 3.1. Under additional regularity conditions, the underlying Markov process admits a *Martin boundary* ∂S, i.e., a compactification of the state space such that $\lim_{t \uparrow \zeta} u(X_t)$ can be identified with the values $f(X_\zeta)$ for a suitable continuation of the function f to the Martin boundary; cf., e.g., [9], (4.12) and (5.7). In such a situation the general representation (12) may be written in the condensed form (1).

Corollary 3.1 shows that u admits a representing function which is regular in the following sense:

DEFINITION 3.1. Let us say that a nonnegative function f on S is *regular* with respect to u if it is upper-semicontinuous and satisfies the condition

$$f(x) \leqslant \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta \;\; P_x\text{-a.s.} \tag{13}$$

for any $x \in S$.

Note that a regular function f also satisfies the inequality

$$f(X_T) \leqslant \sup_{T < t < \zeta} f(X_t) \vee u_\zeta \;\; P_x\text{-a.s. on } \{T < \zeta\} \tag{14}$$

for any stopping time T, due to the strong Markov property.

4. THE MINIMAL AND THE MAXIMAL REPRESENTATION

Let us first derive an alternative description of the representing function f^* in terms of the given excessive function u. To this end, we introduce the superadditive operator

$$\underline{D}u(x) := \inf\{c \geqslant 0 \mid \exists T \in \mathscr{T}(x) : \tilde{P}_T u^c(x) > u(x)\}.$$

PROPOSITION 4.1. *The functions f^* and $\underline{D}u$ coincide. In particular, $x \mapsto \underline{D}u(x)$ is regular with respect to u.*

Proof. Recall that $f^*(x) \geqslant c$ is equivalent to $Ru^c(x) = u(x)$. Thus $f^*(x) \geqslant c$ yields, by (7),

$$u(x) = Ru^c(x) \geqslant \tilde{P}_T u^c(x) \quad \text{for any } T \in \mathcal{T}(x).$$

This amounts to $\underline{D}u(x) \geqslant c$, and so we obtain $f^*(x) \leqslant \underline{D}u(x)$. In order to prove the converse inequality, we take $c > f^*(x)$ and define $T_c \in \mathcal{T}(x)$ as the first exit time from the open neighborhood $\{f^* < c\}$ of x. Then

$$\begin{aligned} u(x) < Ru^c(x) &= E_x[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta^c] \\ &= E_x[\sup_{T_c \leqslant t < \zeta} f^*(X_t) \vee u_\zeta;\ T_c < \zeta] + E_x[u_\zeta^c;\ T_c = \zeta] \\ &= E_x[E_{X_{T_c}}[\sup_{0 \leqslant t < \zeta} f^*(X_t) \vee u_\zeta] \vee c;\ T_c < \zeta] + E_x[u_\zeta^c;\ T_c = \zeta] \\ &= E_x[u^c(X_{T_c});\ T_c < \zeta] + E_x[u_\zeta^c;\ T_c = \zeta] = \tilde{P}_{T_c} u^c(x), \end{aligned}$$

and hence $\underline{D}u(x) \leqslant c$. This shows that $\underline{D}u(x) \leqslant f^*(x)$. ∎

Remark 4.1. A closer look at the proof of Proposition 4.1 shows that

$$\underline{D}u(x) = \inf\{c \geqslant 0 \mid \exists T \in \mathcal{T}(x):\ u(x) - P_T u(x) < E_x[u_\zeta^c;\ T = \zeta]\}.$$

For any potential u of class (D) we have $u_\zeta = 0$ P_x-a.s., and so we get

$$\underline{D}u(x) = \inf \frac{u(x) - P_T u(x)}{P_x[T = \zeta]},$$

where the infimum is taken over all exit times T from open neighborhoods of x such that $P_x[T = \zeta] > 0$. Thus our general representation in Corollary 3.1 contains, as a special case, the representation of a potential of class (D) given in [10].

We are now going to identify the maximal and the minimal representing function for the given excessive function u.

THEOREM 4.1. *Suppose that u admits the representation*

$$u(x) = E_x[\sup_{0 < t < \zeta} f(X_t) \vee u_\zeta] \quad \textit{for any } x \in S,$$

where f is regular with respect to u on S. Then f satisfies the bounds

$$f_* \leqslant f \leqslant f^* = \underline{D}u,$$

where the function f_* is defined by

$$f_*(x) := \inf\{c \geqslant 0 \mid \exists T \in \mathcal{T}(x):\ \tilde{P}_T u^c(x) \geqslant u(x)\} \quad \textit{for any } x \in S.$$

Proof. Let us first show that $f \leqslant f^* = \underline{D}u$. If $f(x) \geqslant c$, then we get for any $T \in \mathscr{T}(x)$

$$u(x) = E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta^c] \geqslant E_x[\sup_{T<t<\zeta} f(X_t) \vee u_\zeta^c;\ T < \zeta] + E_x[u_\zeta^c;\ T = \zeta]$$

$$\geqslant E_x\big[E_x[\sup_{T<t<\zeta} f(X_t) \vee u_\zeta \mid \mathscr{F}_T] \vee c;\ T < \zeta\big] + E_x[u_\zeta^c;\ T = \zeta] = \tilde{P}_T u^c(x)$$

due to our assumption (13) on f and Jensen's inequality. Thus $\underline{D}u(x) \geqslant c$, and this yields $f(x) \leqslant \underline{D}u(x)$. In order to verify the lower bound, take $c > f(x)$, and let $T_c \in \mathscr{T}(x)$ denote the first exit time from $\{f < c\}$. Since by property (14) of f

$$c \leqslant f(X_{T_c}) \leqslant \sup_{T_c<t<\zeta} f(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f(X_t) \vee u_\zeta \quad P_x\text{-a.s. on } \{T_c < \zeta\},$$

we obtain

$$\tilde{P}_{T_c} u^c(x) = E_x[u^c(X_{T_c});\ T_c < \zeta] + E_x[u_\zeta^c;\ T_c = \zeta]$$

$$= E_x\big[E_x[\sup_{T_c<t<\zeta} f(X_t) \vee u_\zeta \mid \mathscr{F}_{T_c}] \vee c;\ T_c < \zeta\big] + E_x[u_\zeta^c;\ T_c = \zeta]$$

$$= E_x[\sup_{T_c<t<\zeta} f(X_t) \vee u_\zeta;\ T_c < \zeta] + E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta^c;\ T_c = \zeta]$$

$$\geqslant E_x[\sup_{0<t<\zeta} f(X_t) \vee u_\zeta] = u(x),$$

and hence $c \geqslant f_*(x)$. This implies $f_*(x) \leqslant f(x)$. ■

The following example shows that the representing function may not be unique, and that it is in general not possible to drop the limit u_ζ in the representation (2).

Example 4.1. Let $(X_t)_{t \geqslant 0}$ be a Brownian motion on the interval $S = (0, 3)$. Then the function u defined by

$$u(x) = \begin{cases} x, & x \in (0, 1), \\ \frac{1}{2}x + \frac{1}{2}, & x \in [1, 2], \\ \frac{1}{4}x + 1, & x \in (2, 3), \end{cases}$$

is concave on S, and hence excessive. Here the maximal representing function f^* takes the form

$$f^*(x) = \tfrac{1}{2} 1_{[1,2)}(x) + 1_{[2,3)}(x),$$

and f_* is given by

$$f_*(x) = \tfrac{1}{2} 1_{\{1\}}(x) + 1_{\{2\}}(x).$$

In particular, we get

$$u(x) > E_x[\sup_{0<t<\zeta} f^*(X_t)] \quad \text{for any } x \in (2, 3).$$

This shows that we have to include u_ζ into the representation of u. Moreover, for any $x \in S$

$$\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta \geqslant f^*(x) \geqslant f_*(x) \quad P_x\text{-a.s.},$$

and so f_* is a regular representing function for u. In particular, the representing function is not unique.

We are now going to derive an alternative description of f_* which will allow us to identify f_* as the minimal representing function for u.

DEFINITION 4.1. Let us say that a point $x_0 \in S$ is *harmonic* for u if the mean-value property

$$u(x_0) = E_{x_0}[u(X_{T_\varepsilon})]$$

holds for x_0 and for some $\varepsilon > 0$, where T_ε denotes the first exit time from the ball $U_\varepsilon(x_0)$. We denote by H the set of all points in S which are harmonic with respect to u.

Under the regularity assumptions of [10], the set H coincides with the set of all points $x_0 \in S$ such that u is harmonic in some open neighborhood G of x_0, i.e., the mean-value property

$$u(x) = E_x[u(X_{T_{U_\varepsilon(x)}})]$$

holds for all $x \in G$ and all $\varepsilon > 0$ such that $\overline{U_\varepsilon(x)} \subset G$; cf. Lemma 4.1 in [10]. In particular, H is an open set.

The following proposition extends Proposition 4.1 in [10] from potentials to general excessive functions.

PROPOSITION 4.2. *For any* $x \in S$,

$$f_*(x) = f^*(x) 1_{H^c}(x). \tag{15}$$

In particular, f_* *is upper-semicontinuous.*

Proof. For $x \in H$ there exists $\varepsilon > 0$ such that

$$\overline{U_\varepsilon(x)} \subset S \quad \text{and} \quad u(x) = E_x[u(X_{T_{U_\varepsilon(x)}})] = \tilde{P}_{T_{U_\varepsilon(x)}} u^0(x),$$

and this implies $f_*(x) = 0$. Now suppose that $x \in H^c$, i.e., u is not harmonic in x. Let us first prove that

$$\tilde{P}_T u(x) < u(x) \quad \text{for all } T \in \mathcal{T}(x). \tag{16}$$

Indeed, if T is the first exit time from some open neighborhood G of x, then

$$\begin{aligned} \tilde{P}_T u(x) &= E_x[E_{X_{T_{U_\varepsilon(x)}}}[u(X_T);\ T < \zeta] + E_{X_{T_{U_\varepsilon(x)}}}[u_\zeta;\ T = \zeta]] \\ &\leqslant E_x[Ru^0(X_{T_{U_\varepsilon(x)}})] = E_x[u(X_{T_{U_\varepsilon(x)}})] < u(x) \end{aligned}$$

for any $\varepsilon > 0$ such that $\overline{U_\varepsilon(x)} \subseteq G$. In view of Theorem 4.1 we have to show that $f_*(x) \geqslant f^*(x)$, and we may assume $f^*(x) > 0$. Choose $c > 0$ such that $f^*(x) > c$. Then there exists $\varepsilon > 0$ such that $Ru^{c+\varepsilon}(x) = u(x)$, i.e., by (7),

$$\tilde{P}_T u^{c+\varepsilon}(x) \leqslant u(x) \tag{17}$$

for any $T \in \mathscr{T}(x)$. Fix $\delta \in (0, \varepsilon)$ and $T \in \mathscr{T}(x)$. If

$$P_x[u(X_T) \leqslant c+\delta;\ T < \zeta] + P_x[u_\zeta \leqslant c+\delta;\ T = \zeta] > 0,$$

we get the estimate

$$\tilde{P}_T u^{c+\delta}(x) = E_x[u^{c+\delta}(X_T);\ T < \zeta] + E_x[u_\zeta^{c+\delta};\ T = \zeta] < \tilde{P}_T u^{c+\varepsilon}(x) \leqslant u(x).$$

On the other hand, if

$$P_x[u(X_T) \leqslant c+\delta;\ T < \zeta] = P_x[u_\zeta \leqslant c+\delta;\ T = \zeta] = 0,$$

then, by (16),

$$\tilde{P}_T u^{c+\delta}(x) = E_x[u(X_T);\ T < \zeta] + E_x[u_\zeta;\ T = \zeta] = \tilde{P}_T u(x) < u(x).$$

Thus we obtain $u(x) > \tilde{P}_T u^{c+\delta}(x)$ for any $T \in \mathscr{T}(x)$, and hence $f_*(x) \geqslant c+\delta$. This concludes the proof of (15). Upper-semicontinuity of f_* follows immediately since f^* is upper-semicontinuous and H^c is closed. ■

Our next purpose is to show that f^* is constant on connected components of H.

PROPOSITION 4.3. *For any* $x \in H$,

$$f^*(x) = \operatorname*{ess\,inf}_{P_x} f_T^*, \tag{18}$$

where T denotes the first exit time from the maximal connected neighborhood $H(x) \subseteq H$ of x. In particular, f^* is constant on $H(x)$.

Proof. 1) Let us first show that, for a connected open set $U \subset S$ and for any $x, y \in U$, the measures P_x and P_y are equivalent on the σ-field describing the exit behavior from U:

$$P_x \approx P_y \quad \text{on } \hat{\mathscr{F}}_U := \sigma(\{g_{T_U} \mid g \text{ measurable on } S\}). \tag{19}$$

Indeed, any $A \in \hat{\mathscr{F}}_U$ satisfies $1_A \circ \theta_{T_\varepsilon} = 1_A$ if T_ε denotes the exit time from some neighborhood $U_\varepsilon(x)$ such that $\overline{U_\varepsilon(x)} \subset U$. Thus

$$P_x[A] = E_x[1_A \circ \theta_{T_\varepsilon}] = \int P_z[A]\, \mu_{x,\varepsilon}(dz),$$

where $\mu_{x,\varepsilon}$ is the exit distribution from $U_\varepsilon(x)$. Since $\mu_{x,\varepsilon} \approx \mu_{y,\varepsilon}$ by assumption A3) of [10], we obtain $P_x \approx P_y$ on $\hat{\mathscr{F}}_U$ for any $y \in U_\varepsilon(x)$. For arbitrary $y \in U$ we can choose $x_0, \ldots, x_n$ and $\varepsilon_1, \ldots, \varepsilon_n$ such that $x_0 = x$, $x_n = y$, $x_k \in U_{\varepsilon_k}(x_{k-1})$ and $\overline{U_{\varepsilon_k}(x_{k-1})} \subset U$. Hence $P_{x_k} \approx P_{x_{k-1}}$ on $\hat{\mathscr{F}}_U$, and this yields (19).

2) For $x \in H$ let $c(x)$ be the right-hand side of equation (18). In order to verify $f^*(x) \leqslant c(x)$, we take a sequence of relatively compact open neighborhoods $(U_n(x))_{n \in N}$ of x increasing to $H(x)$ and denote by T_n the first exit time from $U_n(x)$. Since f^* is upper-semicontinuous on S, we get the estimate

$$\overline{\lim_{n \uparrow \infty}} f^*(X_{T_n}) \leqslant f^*(X_T) 1_{\{T<\zeta\}} + \overline{\lim_{t \uparrow \zeta}} f^*(X_t) 1_{\{T=\zeta\}} = f_T^* \quad P_x\text{-a.s.},$$

and hence $P_x[\overline{\lim}_{n \uparrow \infty} f^*(X_{T_n}) < c] > 0$ for any $c > c(x)$. Thus, there exists n_0 such that

$$P_x[Ru^c(X_{T_{n_0}}) > u(X_{T_{n_0}})] = P_x[f^*(X_{T_{n_0}}) < c] > 0,$$

and this implies

$$u(x) = E_x[u(X_{T_{n_0}})] < E_x[Ru^c(X_{T_{n_0}})] \leqslant Ru^c(x)$$

since Ru^c is excessive. But this amounts to $f^*(x) < c$, and taking the limit $c \searrow c(x)$ yields $f^*(x) \leqslant c(x)$.

3) In order to prove the converse inequality, we use the fact that for any $c < c(x)$

$$(20) \qquad E_x[u^c(X_{\tilde{T}})] \leqslant u(x) \quad \text{for all } \tilde{T} \in \mathcal{T}_0(x),$$

which is equivalent to $Ru^c(x) = u(x)$. Thus $f^*(x) \geqslant c$ for all $c < c(x)$, and in view of 2) we get $f^*(x) = c(x)$. Since, by (19), $c(x) = c(y)$ for any $y \in H(x)$, we see that f^* is constant on $H(x)$.

It remains to verify (20). To this end, note that, by (19), for any $y \in H(x)$ we have $c < c(x) = c(y) \leqslant f_T^*$ P_y-a.s. Thus, $f^*(X_T) > c$ P_y-a.s. on $\{T < \zeta\}$ for any $y \in H(x)$, and this yields

$$u^c(X_T) \leqslant Ru^c(X_T) = u(X_T) \quad P_y\text{-a.s. on } \{T < \zeta\}.$$

Moreover, we get $c < f_\zeta^* \leqslant u_\zeta$ P_y-a.s. on $\{T = \zeta\}$. Let us now fix $\tilde{T} \in \mathcal{T}_0(x)$. Since $X_{\tilde{T}} \in H(x)$ on $\{\tilde{T} < T\}$, we can conclude that

$$(21) \quad \begin{aligned} E_x[u^c(X_{\tilde{T}}); \tilde{T} < T] &= E_x[\tilde{P}_T u(X_{\tilde{T}}) \vee c; \tilde{T} < T] \\ &\leqslant E_x\big[E_{X_{\tilde{T}}}[u^c(X_T); T < \zeta] + E_{X_{\tilde{T}}}[u_\zeta^c; T = \zeta]; \tilde{T} < T\big] \\ &= E_x\big[E_{X_{\tilde{T}}}[u(X_T); T < \zeta] + E_{X_{\tilde{T}}}[u_\zeta; T = \zeta]; \tilde{T} < T\big] \\ &= E_x[u_T; \tilde{T} < T]. \end{aligned}$$

On the other hand, we have $\{T \leqslant \tilde{T}\} \subseteq \{T < \zeta\}$, and by the P_x-supermartingale property of $(Ru^c(X_t) 1_{\{t<\zeta\}})_{t \geqslant 0}$ we get the estimate

$$\begin{aligned} E_x[u^c(X_{\tilde{T}}); \tilde{T} \geqslant T] &\leqslant E_x[Ru^c(X_{\tilde{T}}); \tilde{T} \geqslant T] \leqslant E_x[Ru^c(X_T); \tilde{T} \geqslant T] \\ &= E_x[u(X_T); \tilde{T} \geqslant T] = E_x[u_T; \tilde{T} \geqslant T], \end{aligned}$$

where the first equality follows from $f^*(X_T) \geqslant c(x) > c$ P_x-a.s. on $\{T < \zeta\}$. This together with (21) yields

$$E_x[u^c(X_{\tilde{T}})] \leqslant E_x[u_T] = u(x). \quad \blacksquare$$

Remark 4.2. A point $x \in S$ is harmonic with respect to u if and only if there exists $\varepsilon > 0$ such that f^* is constant on $U_\varepsilon(x) \subset S$.

Indeed, Proposition 4.3 shows that this condition is necessary. Conversely, take $x \in H^c$ and assume that there exists $\varepsilon > 0$ such that f^* is constant on $U_{2\varepsilon}(x) \subset S$. Then the exit time $T := T_{U_\varepsilon(x)}$ satisfies

$$\tilde{P}_T u(x) = E_x[u(X_T)] = E_x[\sup_{T<t<\zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta] = u(x),$$

in contradiction to (16).

Our next goal is to show that f_* is the minimal representing function for u.

THEOREM 4.2. *Let f be an upper-semicontinuous function on S such that $f_* \leqslant f \leqslant f^*$. Then f is a regular representing function for u. In particular, we obtain the representation*

$$u(x) = E_x[\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta],$$

and f_* is the minimal regular function yielding a representation of u.

Proof. Let us show that

$$\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta \quad P_x\text{-a.s.} \tag{22}$$

for any $x \in S$. Suppose first that $x \in H$. We denote by T_c the exit time from the open set $\{f^* < c\}$. Since $0 \leqslant f_* \leqslant f \leqslant f^*$, it is enough to prove that for fixed $c \geqslant f^*(x)$

$$\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta \geqslant c \quad P_x\text{-a.s. on } \{T_c < \zeta\}. \tag{23}$$

By (15) we see that

$$\sup_{0<t<\zeta} f_*(X_t) \geqslant f_*(X_{T_c}) = f^*(X_{T_c}) \geqslant c \quad P_x\text{-a.s. on } \{T_c < \zeta, X_{T_c} \in H^c\}.$$

On the set $A := \{T_c < \zeta, X_{T_c} \in H\}$ we use the inequality

$$f^*(X_{T_c}) \leqslant f_T^* \quad P_x\text{-a.s. on } A \tag{24}$$

for $T := T_c + T_H \circ \theta_{T_c}$, which follows from Proposition 4.3 and the strong Markov property. Using (15) and (24) we obtain

$$\sup_{0<t<\zeta} f_*(X_t) \geqslant f_*(X_T) = f^*(X_T) \geqslant f^*(X_{T_c}) \geqslant c \quad P_x\text{-a.s. on } A \cap \{T < \zeta\}$$

and

$$u_\zeta \geqslant f_\zeta^* \geqslant f^*(X_{T_c}) \geqslant c \quad P_x\text{-a.s. on } A \cap \{T = \zeta\}.$$

Hence $\sup_{0<t<\zeta} f_*(X_t) \vee u_\zeta \geqslant c$ P_x-a.s. on A. This concludes the proof of (23) for $x \in H$, and so (22) holds for any $x \in H$. In particular, we have

$$(25) \quad \sup_{\tilde{T}<t<\zeta} f_*(X_t) \vee u_\zeta = \sup_{\tilde{T}<t<\zeta} f^*(X_t) \vee u_\zeta \;\; P_x\text{-a.s. on } \{\tilde{T} < \zeta,\, X_{\tilde{T}} \in H\}$$

for any stopping time $\tilde{T}$, due to the strong Markov property.

Let us now fix $x \in H^c$ and denote by $\hat{T}$ the first exit time from H^c. Since the functions f_* and f^* coincide on H^c due to Proposition 4.2, the identity (22) follows immediately on the set $\{\hat{T} = \zeta\}$. On the other hand, using again Proposition 4.2, we get

$$(26) \quad \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta = \sup_{0<t\leqslant \hat{T}} f^*(X_t) \vee \sup_{\hat{T}<t<\zeta} f^*(X_t) \vee u_\zeta$$
$$= \sup_{0<t\leqslant \hat{T}} f_*(X_t) \vee \sup_{\hat{T}<t<\zeta} f^*(X_t) \vee u_\zeta \text{ on } \{\hat{T} < \zeta\}.$$

By the definition of $\hat{T}$, on $\{\hat{T} < \zeta\}$ there exists a sequence of stopping times $\hat{T} < T_n < \zeta$, $n \in N$, decreasing to $\hat{T}$ such that $X_{T_n} \in H$. Thus,

$$\sup_{\hat{T}<t<\zeta} f^*(X_t) \vee u_\zeta = \lim_{n\uparrow\infty} \sup_{T_n<t<\zeta} f^*(X_t) \vee u_\zeta = \lim_{n\uparrow\infty} \sup_{T_n<t<\zeta} f_*(X_t) \vee u_\zeta$$
$$= \sup_{\hat{T}<t<\zeta} f_*(X_t) \vee u_\zeta \quad P_x\text{-a.s. on } \{\hat{T} < \zeta\}$$

due to (25). Combined with (26) this yields (22) on $\{\hat{T} < \zeta\}$. Thus we have shown that (22) holds as well for any $x \in H^c$.

In particular, f is a representing function for u. Moreover, by (22),

$$f(x) \leqslant f^*(x) \leqslant \sup_{0<t<\zeta} f^*(X_t) \vee u_\zeta = \sup_{0<t<\zeta} f(X_t) \vee u_\zeta \;\; P_x\text{-a.s.}$$

for any $x \in S$, and so f is a regular function on S with respect to u. In view of Theorem 4.1 we see that f_* is the minimal regular representing function for u. ∎

Remark 4.3. Suppose that u admits a representation of the form

$$(27) \qquad u(x) = E_x[\sup_{0<t<\zeta} f(X_t)]$$

for all $x \in S$ and for some regular function f on S. Then f satisfies the bounds $f_* \leqslant f \leqslant f^*$.

This follows from Theorem 4.1 combined with Proposition 2.1 for $\phi = 0$. Clearly, such a reduced representation, which does not involve explicitly the boundary behavior of u, holds if and only if $u_\zeta \leqslant \sup_{0<t<\zeta} f(X_t)$ P_x-a.s. In particular, this is the case for a potential u where $u_\zeta = 0$, in accordance with the results in [10]. Example 4.1 shows that a reduced representation (27) is not possible in general. If u is harmonic on S, (27) would in fact imply that u is constant on S. Indeed, by Proposition 4.3, harmonicity of u on S implies that

$f^* = c$ on S for some constant c. Using $f \leqslant f^* \leqslant u$ and (3) we get

$$E_x[\sup_{0<t<\zeta} f(X_t)] \leqslant c \leqslant E_x[u_\zeta] = u(x),$$

and so (27) would imply $u(x) = c$ for all $x \in S$.

REFERENCES

[1] P. Bank and N. El Karoui, *A stochastic representation theorem with applications to optimization and obstacle problems*, Ann. Probab. 32 (1B) (2004), pp. 1030–1067.

[2] P. Bank and H. Föllmer, *American options, multi-armed bandits, and optimal consumption plans: A unifying view*, in: *Paris–Princeton Lectures on Mathematical Finance 2002*, Lecture Notes in Math. No 1814, Springer, Berlin 2003, pp. 1–42.

[3] P. Bank and F. Riedel, *Optimal consumption choice with intertemporal substitution*, Ann. Appl. Probab. 11 (3) (2001), pp. 750–788.

[4] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Pure Appl. Math., Academic Press, New York and London 1968.

[5] N. El Karoui, *Max-plus decomposition of supermartingale — Application to portfolio insurance*, http://www.ima.umn.edu/talks/workshops/4-12-16.2004/el_karoui/IMA2004.pdf (2004).

[6] N. El Karoui and H. Föllmer, *A non-linear Riesz representation in probabilistic potential theory*, Ann. Inst. H. Poincaré Probab. Statist. 41 (3) (2005), pp. 269–283.

[7] N. El Karoui and I. Karatzas, *Synchronisation and optimality for multi-armed bandit problems in continuous time*, Comput. Appl. Math. 16 (2) (1997), pp. 117–152.

[8] N. El Karoui and A. Meziou, *Constrained optimization with respect to stochastic dominance — Application to portfolio insurance*, Math. Finance 16 (1) (2006), pp. 103–117.

[9] H. Föllmer, *Feine Topologie am Martinrand eines Standardprozesses*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 12 (1969), pp. 127–144.

[10] H. Föllmer and T. Knispel, *Potentials of a Markov process are expected suprema*, to appear in: ESAIM Probab. Stat. (Volume in honor of Nicole El Karoui).

[11] H. Kaspi and A. Mandelbaum, *Multi-armed bandits in discrete and continuous time*, Ann. Appl. Probab. 8 (4) (1998), pp. 1270–1290.

[12] K. Urbanik, *Lectures on prediction theory*, Lecture Notes in Math. No 44, Springer, Berlin–Heidelberg–New York 1967, 50 pp.

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