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AN ERDÖS–RÉNYI LAW FOR MIXING PROCESSES

BY

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Abstract. We prove a large deviation type result for ψ-mixing processes and derive an ergodic version of the Erdös–Rényi law. The result applies to expanding and Gibbs–Markov dynamical systems, including Gibbs measures and continued fractions.

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1. INTRODUCTION

Fluctuations of partial sums of dependent random variables have only recently been studied despite the fact that such questions are central to i.i.d. processes for more than 50 years. In this note we shall make some contribution to large deviation and Erdös–Rényi laws [10] in the context of mixing stationary processes and their application to dynamical system theory.

While large deviation theory for dynamical systems and time series has one of its origins in the work of Orey [12] and Takahashi [13] and [14], to our knowledge, the Erdös–Rényi law has not been considered before the work of Grigull [11] for expanding maps and Chazottes and Collet [6] for maps of the interval.

As noticed in [7] the concept of ψ-mixing is well suited to derive large deviation results. It should be noticed that we do not need the differentiability of the free energy function for our result. We also show that 0 is the unique minimum of the information function if the one-dimensional marginal of the process has positive variance. Therefore, the information function is strictly increasing, and so we are able to obtain an Erdös–Rényi law without any differentiability assumption (in case of an expanding system, e.g. for subshifts of finite type, the differentiability follows from general thermodynamic facts).

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In this note we are finally interested in extending the class of dynamical systems where a large deviation and Erdös–Rényi result holds. Gibbs–Markov dynamical systems have been introduced in [2]. These systems play an important role in non-hyperbolic and parabolic dynamics, including many maps of the interval (like continued fraction transformation), countable Markov shifts and Poincaré maps associated with Fuchsian and Kleinian groups. They also include most expanding maps, thus our result extends the ones in [11] and [6].

In Section 2 we recall the definition of a Gibbs–Markov map including a description of ψ-mixing processes generated by these systems. In Section 3 we prove our large deviation result and in Section 4 we give an application in proving the analogous result of the Erdös–Rényi law [10].

Note added in proof. After the paper was submitted, we would like to mention that Bryc' papers [15], [16] contain general results for large deviations of mixing processes, while Kiesel and Stadtmüller [17] prove an Erdös–Rényi law for uniformly mixing processes under hypergeometric mixing rates.

2. GIBBS–MARKOV MAPS

Gibbs–Markov dynamical systems cover a large class of transformations being explored recently. This class contains finite state aperiodic Markov chains and certain recurrent Markov chains with infinite state space. Many Markov maps of the unit interval, including those of Lasota–Yorke type and those of Rychlik, are covered as well. In particular, the continued fraction transformation falls into this class. It is also possible to study parabolic rational maps and their equilibria by this method including the classical Gibbs measures on subshifts of finite type (see [4] or [8]). Applications to geodesic flows can be obtained through Poincaré section maps (see [1]).

In this section we introduce Gibbs–Markov dynamical systems briefly as it has been done in [2].

Let T denote a nonsingular transformation of a standard probability space $(\Omega, \mathscr{B}, P)$, i.e. $T: \Omega \to \Omega$ is measurable and $P(T^{-1}(A)) = 0$ if and only if $P(A) = 0$. The transformation T is called a *Markov* map if there is a measurable partition α such that:

- $T(A) \in \sigma(\alpha) \pmod P$ for all $A \in \alpha$;
- α generates $\mathscr{B}$ under T in the sense that $\sigma(\{T^{-n}\alpha\colon n \geqslant 0\}) = \mathscr{B}$, and
- $T|_A$ is invertible with measurable inverse and nonsingular for $A \in \alpha$.

Note that Markov maps are called *Markov fibred systems* in [3].

Write $\alpha_0^n = \alpha \vee T^{-1}\alpha \vee \ldots \vee T^{-n}\alpha$ $(n \geqslant 0)$ for common refinements of the pull backs of α by T. Fix $r \in (0, 1)$ and define a distance $d = d_r$ on Ω by $d(x, y) = r^{t(x,y)}$, where $t(x, y)$ denotes the least n for which $T^n(x)$ and $T^n(y)$ lie in different atoms of α.

For $n \geqslant 1$, there are P-nonsingular inverse branches of T denoted by v_A: $T^n(A) \to A$ $(A \in \alpha_0^{n-1})$ with Radon–Nikodym derivatives

$$v'_A := \frac{dP \circ v_A}{dP}.$$

Since $T\alpha \subset \sigma(\alpha)$, $T^n \alpha_0^{n-1} = T\alpha$, and there exists a (finite or countable) partition β coarser than α so that $\sigma(T\alpha) = \sigma(\beta)$.

A Markov map $(\Omega, \mathscr{B}, P, T, \alpha)$ is *Gibbs* (Gibbs–Markov) if the big image property

$$\inf_{A\in\alpha} P(T(A)) > 0$$

holds, and the Radon–Nikodym derivatives v'_A satisfy the following distortion property:

there exists $M > 0$ such that

$$\left|\frac{v'_A(x)}{v'_A(y)} - 1\right| \leqslant M d(x, y) \quad \text{for all } n \geqslant 1,\ A \in \alpha_0^{n-1},\ x, y \in T^n(A).$$

Let us assume that T is *topologically mixing* in the sense that for all $A, B \in \alpha$ there exists some $n_{A,B} \in N$ such that $T^n(A) \supset B$ for all $n \geqslant n_{A,B}$.

The *Frobenius–Perron* operators $P_{T^n}\colon L^1(P) \to L^1(P)$ are defined by

$$\int_\Omega P_{T^n} f \cdot g dP = \int_\Omega f \cdot g \circ T^n dP$$

and have the form

$$P_{T^n} f = \sum_{B\in\beta} 1_B \sum_{A\in\alpha_0^{n-1}, T^n(A)\supset B} v'_A \cdot f \circ v_A.$$

We now consider P_T acting on the space L which consists of all measurable functions $f\colon \Omega \to C$ with norm

$$\|f\|_L = \|f\|_{L_\infty(P)} + \sup_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{d(x, y)}.$$

THEOREM 2.1 (Aaronson and Denker [2]). *P_T acts on L and has a splitting*

$$P_T = \mu + Q$$

in Hom(L, L), *where* $\mu f = \int_\Omega f dP \cdot h$, $Q\mu = \mu Q = 0$, *and the spectral radius* $r(Q)$ *of* Q *is strictly less than* 1.

Since μ is the projection to an invariant (one-dimensional) subspace generated by h, we may pass to the measure hdP, which is invariant. In the sequel we always assume that P is this invariant measure (so $h = 1$).

The theorem permits to strengthen the exactness part of Rényi's theorem known as "exponential decay of correlations":

By Theorem 2.1 there exist $\theta \in (0, 1)$ and $K > 1$ such that

$$\left\|P_{T^n} f - \int_\Omega f dP\right\|_L \leqslant K\theta^n \|f\|_L \quad \text{for all } n \geqslant 1,\ f \in L.$$

Since for all $n \geqslant 1$ and $A \in \alpha_0^{n-1}$ we have $P_{T^n} 1_A = v'_A$ and

$$\|v'_A\|_L \leqslant (M+1) M' P(A) \quad \text{for some constant } M',$$

it follows that the process defined by (T, α) is *continued fraction mixing*, in particular *ψ-mixing*:

for some $K' > 0$ and for all $n, k \geqslant 1$, $A \in \alpha_0^{k-1}$, and $B \in \mathscr{B}$

$$(1) \qquad |P(A \cap T^{-(n+k)}(B)) - P(A)P(B)| \leqslant K' \theta^n P(A)P(B).$$

3. A LARGE DEVIATION THEOREM FOR MIXING PROCESSES

Rewriting (1) in general terms of σ-algebras one obtains the notion of ψ-mixing. Let $\mathscr{F} = \{\mathscr{F}_n^m\colon 1 \leqslant n \leqslant m \leqslant \infty\}$ be a ψ-mixing family of σ-fields, i.e. $\mathscr{F}_n^m \subset \mathscr{F}_k^l$ for $k \leqslant n \leqslant m \leqslant l$ and

$$\sup_{A \in \mathscr{F}_1^k; B \in \mathscr{F}_{k+n}^\infty} \left| \frac{P(A \cap B)}{P(A)P(B)} - 1 \right| =: \psi(n) \to 0 \quad \text{as } n \to \infty.$$

Let $\{X_i\colon i \geqslant 1\}$ be a strictly stationary process such that X_i is $\mathscr{F}_i^i$-measurable ($i \in N$). Define $S_n = X_1 + \ldots + X_n$.

For the calculations in the sequel the following weaker condition suffices (but which turns out to be equivalent to ψ-mixing by Bradley's result in [5]): there are $q \in N$ and $C_1, C_2 \in R$ such that

$$(2) \qquad 0 < C_1 < \frac{P(A \cap B)}{P(A)P(B)} < C_2 < \infty$$

for all $k \in N$, $A \in \mathscr{F}_1^k$, $B \in \mathscr{F}_{k+q}^\infty$.

If W is a non-degenerate bounded random variable, we set

$$f_W(t) = \log E[\exp(tW)] \quad \text{for all } t \in R.$$

This function is analytic, its derivative is increasing, and

$$\lim_{t \to -\infty} f_W'(t) = \operatorname{ess\,inf} W, \qquad \lim_{t \to +\infty} f_W'(t) = \operatorname{ess\,sup} W.$$

PROPOSITION 3.1. *If $\{X_n\colon n \geqslant 1\}$ is a ψ-mixing, bounded process, then*

$$(3) \qquad f(t) = \lim_{n \to \infty} \frac{1}{n} f_{S_n}(t) \quad (t \in R)$$

exists. Moreover, the convergence is uniform on compacts.

Proof. Choose q such that (2) holds. Define $S_{n,m} = S_n - S_m$ for $m \leqslant n$. Since $\|tX_1\|_\infty < \infty$, we have

$$0 < m_1(t) \leqslant \exp[tX_n] \leqslant m_2(t) < \infty, \quad n \geqslant 1,$$

and, consequently,

$$E[\exp(tS_{n+m})]$$
$$= E[\exp(tS_n)\exp(tS_{n+q,n})\exp(tS_{n+q+m,n+q})\exp(-tS_{n+m+q,n+m})]$$

$$\leqslant m_2(t)^q m_1(t)^{-q} E[\exp(tS_n)\exp(tS_{n+q+m,n+q})]$$

$$\leqslant m_2(t)^q m_1(t)^{-q} C_2 E[\exp(tS_n)] E[\exp(tS_{n+q+m,n+q})]$$

$$= CE[\exp(tS_n)] E[\exp(tS_m)]$$

for come constant $C = C(t) < \infty$.

It follows that the sequence $f_{S_n}(t)$ is almost subadditive, and hence $\lim_{n\to\infty} n^{-1} f_{S_n}(t)$ exists.

Note that one can also prove that

$$E[\exp(tS_{n+m})] \geqslant C' E[\exp(tS_n)] E[\exp(tS_m)]$$

for some constant $C' = C'(t) > -\infty$. The uniform convergence follows then from

$$f(t) = \lim_{r\to\infty} \frac{1}{rm} f_{S_{rm}}(t) \leqslant \frac{1}{m} f_{S_m}(t) + \frac{1}{m} \log C(t)$$

and

$$f(t) = \lim_{r\to\infty} \frac{1}{rm} f_{S_{rm}}(t) \geqslant \frac{1}{m} f_{S_m}(t) + \frac{1}{m} \log C'(t). \ \blacksquare$$

COROLLARY 3.2. *Let $(X, \mathcal{F}, m, T, \alpha)$ be a mixing Gibbs–Markov system with invariant probability measure m. Let $g\colon X \to \mathbf{R}$ be a function such that*

$$\omega_n(g) := \sup\{|g(x) - g(y)|\colon x \in \alpha_n(y)\} \to 0.$$

Define

$$f_n(t) = \log \int \exp(tS_n g)\, dm \qquad (t \in \mathbf{R},\ n \geqslant 1), \tag{4}$$

where $S_n g = \sum_{i=0}^{n-1} g \circ T^i$. Then the free energy function

$$f(t) = \lim_{n\to\infty} \frac{1}{n} f_n(t) \qquad (t \in \mathbf{R}) \tag{5}$$

exists.

Proof. We can suppose $t = 1$. Assume first that g depends only on a finite number of coordinates. Then the result follows from Proposition 3.1 and the previous section (see (1)).

The sequence of (non-linear) functionals $E_n\colon C(X) \to \mathbf{R}$ defined by

$$E_n(g) = n^{-1} \log \int \exp(S_n g)$$

is uniformly continuous and convergent for g depending on the finite number of coordinates. Using a simple approximation argument it is easy to prove that this implies that the sequence is convergent for all g with the property $\omega_n(g) \to 0$. ■

The crucial property for applying a general result on large deviation ([9], Theorem II.6.1) is the differentiability of the free energy function f. The following large deviation result for mixing processes can be obtained without this assumption.

Note that the sequence $\operatorname{ess\,sup} S_n$ is subadditive, and hence

$$c^+ := \lim_{n\to\infty} \operatorname{ess\,sup} n^{-1} S_n$$

exists and is equal to $\inf_{n\in N} \operatorname{ess\,sup} n^{-1} S_n$. Define c^- in an analogous way using $\operatorname{ess\,inf} S_n$.

THEOREM 3.3. *If* $\{X_n: n \geq 1\}$ *is a* ψ*-mixing, bounded stationary process satisfying* $EX_1 = 0$, *then*

$$\lim_{n\to\infty} n^{-1} \log P(S_n \geq nc) = \inf\{-tc+f(t): t > 0\}, \quad c \in (0, c^+), \tag{6}$$

$$\lim_{n\to\infty} n^{-1} \log P(S_n \leq nc) = \inf\{-tc+f(t): t < 0\}, \quad c \in (c^-, 0). \tag{7}$$

Proof. We only prove (6), since (7) is similar.

By Markov's inequality we have for all $c, t > 0$

$$\limsup_{n\to\infty} n^{-1} \log P(S_n \geq nc) \leq \lim_{n\to\infty} n^{-1} \log e^{-ntc} E\exp(tS_n) = -tc+f(t),$$

whence

$$\limsup_{n\to\infty} n^{-1} \log P(S_n \geq nc) \leq \inf\{-tc+f(t): t > 0\}.$$

We show the converse inequality for $c \in (0, c^+)$. Fix q such that (2) holds and let $p \in N$. Take $n \in N$ and write $n = r(p+q)+w$ for suitable $r, w \in N_0$ with $w < p+q$. Define

$$Y_j = \sum_{l=j(p+q)+1}^{j(p+q)+p} X_l, \quad Z_j = \sum_{l=j(p+q)+p+1}^{(j+1)(p+q)} X_l \quad (j = 0, \ldots, r-1),$$

$$Z_r = S_n - \sum_{j=0}^{r-1} Y_j - \sum_{j=0}^{r-1} Z_j.$$

Since X_1 is bounded (by M say), we have

$$\Big|\sum_{j=0}^{r} Z_j\Big| \leq (rq+p+q)M.$$

The variables Y_j are separated by time q. If $W = (W_0, \ldots, W_{r-1})$ are i.i.d. with the same distribution as Y_0, then by ψ-mixing we obtain

$$P\big((Y_0, \ldots, Y_{r-1}) \in A\big) \geq C_1^r P(W \in A)$$

for any measurable set $A \subset \mathbb{R}^r$. Putting everything together gives for n large enough

$$P(S_n \geqslant nc) \geqslant P\Big(\sum_{j=0}^{r-1} Y_j \geqslant nc+(rq+p+q)M\Big)$$

$$\geqslant P\left(\sum_{j=0}^{r-1} Y_j \geqslant r\left[\frac{nc}{r}+\left(q+\frac{p+q}{r}\right)M\right]\right)$$

$$\geqslant C_1^r P\Big(\sum_{j=0}^{r-1} W_j \geqslant r\big((p+q+1)c+(q+1)M\big)\Big).$$

Recall that $f_{S_p}(t) = \log E[\exp(tS_p)]$. Since W_i are i.i.d. variables, it follows from the classical large deviations principle that

$$\lim_{r\to\infty} \frac{1}{r} \log P\Big(\sum_{j=0}^{r-1} W_j \geqslant rf'_{S_p}(s)\Big) = -sf'_{S_p}(s)+f_{S_p}(s), \qquad s \geqslant 0.$$

We have

$$\frac{(p+q+1)c+(q+1)M}{p} \in (0, c^+) \quad \text{for } p \text{ large.}$$

Since $(c^-, c^+) \subset p^{-1} f'_{S_p}(\mathbb{R})$, we may choose $s = s(p, q, c)$ satisfying

$$(f_{S_p})'(s) = (p+q+1)c+(q+1)M,$$

and it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log P(S_n \geqslant nc)$$

$$\geqslant \frac{1}{p+q+1} \log C_1 + \frac{1}{p+q+1}\big(-s(f_{S_p})'(s)+f_{S_p}(s)\big)$$

$$= \frac{1}{p+q+1} \log C_1 - sc - sM\frac{q+1}{p+q+1} + \frac{1}{p+q+1} f_{S_p}(s)$$

$$\geqslant \frac{1}{p+q+1} \log C_1 - sc - sM\frac{q+1}{p+q+1} + \frac{p}{p+q+1} f(s) - O\left(\frac{1}{p}\right).$$

The last estimation holds for p large enough. Letting $p \to \infty$ we arrive at

$$\liminf_{n\to\infty} \frac{1}{n} \log P(S_n \varphi \geqslant nc) \geqslant \inf\{-tc+f(t)\colon t > 0\}. \ \blacksquare$$

PROPOSITION 3.4. *Let $(X_n\colon n \geqslant 1)$ be a ψ-mixing, bounded process with $EX_1 = 0$ and* $\mathrm{Var}(X_1) > 0$. *Then for any $\varepsilon > 0$ there exist $\Lambda = \Lambda(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in N$ such that*

$$P(|S_n| \geqslant n\varepsilon) \leqslant \exp[-n\Lambda] \qquad (n \geqslant n_0).$$

Proof. Suppose that $W_1, W_2, \ldots$ is an i.i.d. process with the same distribution as X_1. Since the variance of W_1 is positive, the information function $I(\varepsilon)$ of W_1 is strictly positive. Choose p so large that

$$I(\varepsilon)-\log(1+\psi(p)) > 0.$$

Next choose r_0 so large that

$$I(\varepsilon)-\log(1+\psi(p))-\frac{\log p}{r_0} > 0.$$

Let $n = rp$, $r \geqslant r_0$. Write $S_n = Z_1 + \ldots + Z_p$, where

$$Z_j = \sum_{l=1}^{r} X_{(l-1)p+j} \quad (j = 1, \ldots, p),$$

and put

$$Y_j = \sum_{l=1}^{r} W_{(l-1)p+j} \quad (j = 1, \ldots, p).$$

It follows that $P(S_n \geqslant \varepsilon n) \leqslant pP(Z_1 \geqslant r\varepsilon)$. By ψ-mixing we conclude that

$$\begin{aligned} P(S_n \geqslant \varepsilon n) &\leqslant p(1+\psi(p))^r P(Y_1 \geqslant \varepsilon r) \\ &\leqslant p\exp\left[-r\big(I(\varepsilon)-\log(1+\psi(p))\big)\right] \\ &= \exp\left[-n\frac{I(\varepsilon)-\log(1+\psi(p))-r_0^{-1}\log p}{p}\right]. \end{aligned}$$

The proposition follows easily from this estimate. ∎

4. ERDÖS–RÉNYI LAW FOR ψ-MIXING PROCESSES

Let f be the free energy function. The information function is defined by

$$I(\alpha) = \begin{cases} \sup\{t\alpha - f(t)\colon t > 0\} & \text{if } \alpha > 0, \\ \sup\{t\alpha - f(t)\colon t < 0\} & \text{if } \alpha < 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

THEOREM 4.1. *Let $(X_n)_{n\in N}$ be a strictly stationary ψ-mixing, bounded process with bounded random variables X_n, $n \geqslant 1$, $EX_1 = 0$, $\mathrm{Var}(X_1) > 0$, and free energy function f. Suppose that the mixing coefficients $\psi(n)$ decrease exponentially fast. Let c^+ be as in Theorem 3.3. Then for any $\alpha \in (0, c^+)$ the Erdös–Rényi law holds*:

$$\lim_{n\to\infty} \max_{0\leqslant m\leqslant n-[(\log n)/I(\alpha)]} I(\alpha)\frac{S_{m+[(\log n)/I(\alpha)]}-S_m}{\log n} = \alpha \quad a.e.$$

Proof. First note that, by Proposition 3.4, S_n/n converges to 0 exponentially; hence, by Theorem II.6.3 in [9], f is differentiable at 0 and $f(t) > 0$ for every $t \neq 0$. This implies that I is strictly increasing on $[0, c^+)$.

Fix $\alpha \in [0, c^+)$. Let $l_n = [(\log n)/I(\alpha)]$, where $[z]$ denotes the Gauss bracket of z.

Let $\varepsilon > 0$ and define the event

$$A_n(\varepsilon) = \{\max_{0 \leqslant m \leqslant n-l_n} S_{m+l_n} - S_m \geqslant (\alpha+\varepsilon) l_n\}.$$

Choose $0 < 2\delta < I(\alpha+\varepsilon) - I(\alpha)$. We obtain, by stationarity for n large enough,

$$\begin{aligned} P(A_n(\varepsilon)) &\leqslant nP(S_{l_n} \geqslant (\alpha+\varepsilon) l_n) \leqslant n\exp[-l_n(I(\alpha+\varepsilon)-\delta)] \\ &\leqslant n\exp[-l_n(I(\alpha)+\delta)] \\ &\leqslant n\exp\left[-\left(\frac{\log n}{I(\alpha)}-1\right)(I(\alpha)+\delta)\right] = e^{I(\alpha)+\delta} n^{-\delta/I(\alpha)}. \end{aligned}$$

If $d > I(\alpha)/\delta$, then $\sum_{n \in N} n^{-d\delta/I(\alpha)} < \infty$ and the Borel–Cantelli theorem shows that $A_{n^d}(\varepsilon)$ occurs only finitely often. It follows that

$$\limsup_{n\to\infty} \max_{0 \leqslant m \leqslant n^d - l_{n^d}} \frac{S_{m+l_{n^d}} - S_m}{l_{n^d}} \leqslant \alpha+\varepsilon.$$

Since for $n^d < r \leqslant (n+1)^d$ large enough the difference $l_{n^d} - l_r$ is bounded by 1, we have

$$\begin{aligned} &\limsup_{r\to\infty} \max_{0 \leqslant m \leqslant r-l_r} \frac{S_{m+l_r} - S_m}{l_r} \\ &\qquad \leqslant \limsup_{n\to\infty} \max_{0 \leqslant m \leqslant (n+1)^d - l_{(n+1)^d}} \frac{S_{m+l_{(n+1)^d}} - S_m + \|X_1\|_\infty}{l_{n^d}} \leqslant \alpha+\varepsilon. \end{aligned}$$

In order to show the converse inequality, choose $\varepsilon > 0$ so that $\alpha - \varepsilon > 0$ and define

$$B_n(\varepsilon) = \{\max_{0 \leqslant m \leqslant n-l_n} S_{m+l_n} - S_m \leqslant l_n(\alpha-\varepsilon)\}.$$

Let $C_m = \{S_{m+l_n} - S_m \leqslant l_n(\alpha-\varepsilon)\}$ and note that $C_m \in \mathscr{F}_m^{m+l_n-1}$. Then, for any $n - l_n > q \in N$,

$$B_n(\varepsilon) = \bigcap_{m=0}^{n-l_n} C_m \subset \bigcap_{s=0}^{\left[\frac{n-l_n}{l_n+q}\right]} C_{s(l_n+q)}.$$

Using the mixing property we get

$$P(B_n(\varepsilon)) \leqslant P\Big(\bigcap_{s=0}^{\left[\frac{n-l_n}{l_n+q}\right]} C_{s(l_n+q)}\Big) \leqslant \big((1+\psi(q))P(C_0)\big)^{(n-l_n)/(l_n+p)-1}.$$

There exists $\delta > 0$ so that $(I(\alpha-\varepsilon)+\delta)/I(\alpha) < 1-\delta$. By assumption, there exists $\Lambda > 0$ such that for sufficiently large q

$$\psi(q) \leqslant \exp[-\Lambda q];$$

hence we may choose $q = [\gamma \log n]$, where γ satisfies $\gamma\Lambda > 1$. Using the large deviation property for $P((C_0)^c)$ we obtain for large l_n

$$\begin{aligned} 1-P(C_0) &\geqslant \exp[-l_n(I(\alpha-\varepsilon)+\delta)] \\ &\geqslant \exp[-l_n I(\alpha)(1-\delta)] \geqslant \exp[-(1-\delta)\log n]. \end{aligned}$$

Therefore, if n is sufficiently large,

$$\begin{aligned} P(B_n(\varepsilon)) &\leqslant \left((1+e^{-\gamma\Lambda\log n})(1-\exp[-(1-\delta)\log n])\right)^{(n-l_n)/(l_n+\gamma\log n)} \\ &\leqslant [(1+n^{-1})(1-n^{-1+\delta})]^{(n-l_n)/(l_n+\gamma\log n)} = O(\exp[-2n^{\delta/2}]). \end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} P(B_n(\varepsilon)) < \infty;$$

hence by the Borel–Cantelli lemma $B_n(\varepsilon)$ occurs only finitely often, and

$$\liminf_{n\to\infty} \max_{0\leqslant m\leqslant n-l_n} \frac{S_{m+l_n}-S_m}{l_n} \geqslant \alpha-\varepsilon. \blacksquare$$

Remark 4.2. It should be noted that the assumption on the mixing coefficients can be weakened to subexponential decay. The proof can easily (but technically more involved) be adapted. However, polynomial decay is not sufficient to make the present proof go through.

Example 4.3. The theorem applies to mixing Gibbs–Markov dynamics as explained in Section 2. Indeed, by (1) any function g which is measurable with respect to α_0^r for some $r > 0$ generates a ψ-mixing process by setting $X_n = g \circ T^n$, where the ψ-mixing coefficients are given by

$$\psi(n) \leqslant \begin{cases} K'\theta^{n-r} & \text{if } n > r, \\ \infty & \text{elsewhere}. \end{cases}$$

In particular, the continued fraction transformation

$$T\colon [0, 1] \to [0, 1], \quad T(x) = \{1/x\}$$

defines a mixing Gibbs–Markov map under the Gauss measure

$$dm = \frac{1}{\log 2}\frac{dx}{1+x},$$

and any function $g(x) = [1/x]\, 1_a([1/x])$ with $a \in N$ satisfies the assumption of the theorem after centering. Hence the maximal portion of digits in the continued fraction expansion (up to the nth iteration) of a typical x in a string of length l_n approaches a limit, which depends on the parameter $l_n/\log n$.

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